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Approximate Solution of Real Definite Integrals in Adaptive Routine

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Abstract

A mixed quadrature rule of higher precision has been formulated by taking two constituent rules each of lower degree of precision. Mixed quadrature rule in adaptive environment is used for evaluation of real definite integrals over a circle or triangle which is best fit to the Fracture Mechanics. Mixed quadrature rules have been applied in various fields of Science and Technology **Objectives:** Mixed quadrature rule has become a milestone in the field of Science and Technology. **Methods/Statistical Analysis:** A mixed quadrature rule of degree of precision nine has been formed by taking two constituent rules each of degree of precision seven. **Findings:** The mixed quadrature rule has been tested in adaptive routine and it has found to be more effective than that of Clenshaw-curtis seven-point rule. **Application:** Mixed quadrature rule in adaptive environment is used for evaluation of real definite integral over a circle or triangle which is best fit to the fracture mechanics.

Keywords: Adaptive Quadrature Method, Degree of Precision, Maclaurin's Series, Mixed Quadrature Rule

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1. Introduction

The basic method for adaptive quadrature is the additive property of a definite integral. If $h \in [c, d]$ then

$$\int_c^h p(x)dx + \int_h^d p(x)dx = \int_c^d p(x)dx$$

The motivation to compute a real integrable function f, an interval [c, d] for a prescribed tolerance ε, the integral $\int_c^d p(x)\,dx = I$ so that $|A - I| \le \varepsilon$ .This can be done in following adaptive integration schemes[1,2,3]. In adaptive integration, the points at which the integrand is evaluated are so chosen in such a way that depends on the nature of the integrand. The fundamental principle to get the sum which produces the appropriate result with approximation of two integrals for a specified tolerance. If not, we can recursively apply the additive property to each of the intervals [c, h] and [h, d]. Adaptive subdivision of course has geometrical appeal. It seems intuitive that points should be concentrated in regions where the integrand is badly behaved. The whole interval rules can take no direct account of this.

Keeping the fact in mind[5–10] in this paper, Newton Cotes n-point open type rule and Gaussian type of rules have been taken into account. Here we have mixed Gauss-Legendre four-point rule and Clenshaw-Curtis seven-point rule as proposed by Oliver[4] each of precision seven to find a quadrature rule of precision nine. The result has numerically verified and gives a good agreement on different integrals which is superior to Clenshaw-Curtis quadrature rule in adaptive routine and also the error bound for the said rule has been determined.

2. Clenshaw-Curtis Quadrature Rule

To approximate the integral $p(v)$ over the interval $[\alpha - h, \alpha + h]$ by $I(p)$ Clenshaw-Curtis method and replacing v by $\alpha + hx$ we get,

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$$h\int_{-1}^{1} p(\alpha+hx)dx=\int_{a}^{b} p(v)dv=I(p)$$

Assuming $I = I_n$ where

$$I_n=h\int_{-1}^{1}\sum_{q=0}^{n} a_q T_q(x)\,dx=h\sum_{q=0}^{n}a_q\int_{-1}^{1}T_q(x)\,dx \qquad (1)$$

The Chebyshev polynomial $T_q(x)=\cos(q\times\arccos(x)),\ q\ge 0$ of degree n and $T_q(x_i)=\cos(q\times\arccos(x_i))$

where, $x_i=\cos\left(\frac{i\pi}{n}\right), \qquad i=0,1,\ldots\ldots,n$

So, $T_q(x_i)=\cos\left(\frac{qi\pi}{n}\right)$

$$a_q=\begin{cases}\frac{2}{n}\sum_{i=0}^{n}p(\alpha+hx_i)T_q(x_i) & q=0,1,\ldots,n-1\\ \frac{1}{n}\sum_{i=0}^{n}p(\alpha+hx_i)T_q(x_i) & q=n\end{cases}$$

Substituting a_q and $T_q(x)$ in Eq. (2.1)

$$I_n=h\sum_{q=0}^{n}\frac{2}{n}\sum_{i=0}^{n}p(\alpha+hx_i)T_q(x_i)\int_{-1}^{1}T_q(x)dx$$

Since $\int_{-1}^{1}T_q(x)dx=-\frac{2}{q^2-1},\ (q=even)$

$I_n=h\sum_{i=0}^{n}w_i\,p(\alpha+hx_i)$ where $w_i=-\frac{4}{n}\sum_{r=0}^{n}\frac{1}{q^2-1}T_q(x_i)$, $q=even$ $(i=0,1,\ldots\ldots,n)$

With $n = 6$,

$$I_6=h\left[\frac{9}{315}\{p(\alpha-h)+p(\alpha+h)\}+\frac{80}{315}\left\{p\left(\alpha-\frac{\sqrt{3}}{2}h\right)+p\left(\alpha+\frac{\sqrt{3}}{2}h\right)\right\}+\frac{144}{315}\left\{p\left(\alpha-\frac{h}{2}\right)+p\left(\alpha+\frac{h}{2}\right)\right\}+\frac{164}{315}p(\alpha)\right] \quad (2)$$

3. Quadrature Rule of Degree of Precision Nine

Here two constituent rules each of precision seven have been mixed to get higher rule of precision nine. Here the interval $a\le x\le b$ transform to $-1\le x\le 1$ taking the monomial transformation $2x=(b-a)t+(b+a)$.

$$\int_{-1}^{1}p(x)\,dx=I(p)\cong R_{CC7}(p)$$

$$=\left[\frac{9}{315}\{p(-1)+p(1)\}+\frac{80}{315}\left\{p\left(-\frac{\sqrt{3}}{2}\right)+p\left(\frac{\sqrt{3}}{2}\right)\right\}+\frac{144}{315}\left\{p\left(-\frac{1}{2}\right)+p\left(\frac{1}{2}\right)\right\}+\frac{164}{315}p(0)\right] \qquad (3)$$

Gauss Legendre four point rule (R_{GL4})

$$I(p)=\int_{-1}^{1}p(x)dx\cong R_{GL4}(p)=\frac{1}{36}\left[\left(18+\sqrt{30}\right)\{p(\beta)+p(-\beta)\}+\left(18-\sqrt{30}\right)\{p(\gamma)+p(-\gamma)\}\right] \qquad (4)$$

where, $\beta=\sqrt{\frac{3-2\sqrt{\frac{6}{5}}}{7}}$ and $\gamma=\sqrt{\frac{3+2\sqrt{\frac{6}{5}}}{7}}$.

Theorem 3.1
Let $E_{cc7}(p)$ and $E_{GL4}(p)$ be the error in evaluating the integral $I(p)$ for $R_{cc7}(p)$ and $R_{GL4}(p)$ respectively. Then,

$$I(p)=R_{CC7GL4}(p)+E_{CC7GL4}(p)$$

where,

$$R_{CC7GL4}(p)=\frac{1}{477}\left[512R_{CC7}(p)-35R_{GL4}(p)\right]$$

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Proof- Using Maclaurin's expansion of Eq. (3) and Eq. (4) we get,

$$I(p)=R_{CC7}(p)+E_{CC7}(p)\text{ and} \qquad (5)$$

$$I(p)=R_{GL4}(p)+E_{GL4}(p) \qquad (6)$$

where,

$$E_{CC7}(p)=I(p)-R_{CC7}(p)=\frac{1}{1260\times 8!}p^{viii}(0)+\frac{1}{308\times 10!}p^{x}(0)$$

$$E_{GL4}(p)=I(p)-R_{GL4}(p)=\frac{6272}{540225\times 8!}p^{viii}(0)+\frac{119168}{4621925\times 10!}p^{x}(0)$$

Now multiplying $\left(\frac{128}{35}\right)$ in Eq. (5) and $\left(\frac{1}{4}\right)$ in Eq. (6) and subtracting Eq. (6) from Eq. (5)

$$I(p) = R_{CC7GL4}(p) + E_{CC7GL4}(p) \quad (7)$$

where, $R_{CC7GL4}(p) = \frac{1}{477}\left[512R_{CC7}(p) - 35R_{GL4}(p)\right]$ (8)

which is the quadrature rule of degree of precision nine.

The truncation error is given by,

$$E_{CC7GL4}(p) = \frac{1}{477}\left[512E_{CC7}(p) - 35E_{GL4}(p)\right] \quad (9)$$

4. Error Estimation

The error estimate and error bound for Eq. (8) are given by Theorem 1 and Theorem 2 respectively.

Theorem 1:

Let $p(x)$ is sufficiently differentiable in the interval $[-1,1]$. Then the error $E_{CC7GL4}(p)$ for the rule $R_{CC7GL4}(p)$ is

$$\left|E_{CC7GL4}(p)\right| \cong \frac{2048}{1285515 \times 10!}\left|p^{x}(0)\right|$$

Proof- From Eq. (7)

$$I(p) = R_{CC7GL4}(p) + E_{CC7GL4}(p)$$

where, $R_{CC7GL4}(p) = \frac{1}{477}\left[512R_{CC7}(p) - 35R_{GL4}(p)\right]$

$$E_{CC7GL4}(p) = \frac{1}{477}\left[512E_{CC7}(p) - 35E_{GL4}(p)\right]$$

Hence $\left|E_{CC7GL4}(p)\right| \cong \frac{2048}{1285515}\left|p^{x}(0)\right|$

Theorem 2:

The bound for the truncation error $E_{CC7GL4}(p) = I(p) - R_{CC7GL4}(p)$ is given by, $\left|E_{CC7GL4}(p)\right| \le \frac{128M}{150255 \times 8!}\left|\eta_2 - \eta_1\right|$, where $\eta_1, \eta_2 \in [-1,1]$ and $M = \max_{-1 \le x \le 1}\left|p^{ix}(x)\right|$.

Proof: We have $E_{CC7}(p) = \frac{1}{1260 \times 8!}p^{viii}(\eta_2)$

$E_{GL4}(p) = \frac{6272}{540225 \times 8!}p^{viii}(\eta_1)$ where $\eta_1, \eta_2 \in [-1, 1]$

Hence, $E_{CC7GL4}(p) = \frac{1}{477}\left[512E_{CC7}(p) - 35E_{GL4}(p)\right]$

$$= \frac{128}{150255 \times 8!}\left[p^{viii}(\eta_2) - p^{viii}(\eta_1)\right] \quad (\text{assuming } \eta_1 < \eta_2)$$

$$= \frac{128}{150255 \times 8!}\int_{\eta_1}^{\eta_2} p^{ix}(x)dx$$

$$\left|E_{CC7GL4}(p)\right| = \left|\frac{128}{150255 \times 8!}\int_{\eta_1}^{\eta_2} p^{ix}(x)dx\right|$$

$$\le \frac{128}{150255 \times 8!}\left|\int_{\eta_1}^{\eta_2} p^{ix}(x)dx\right|$$

$\left|E_{CC7GL4}(p)\right| \le \frac{128M}{150255 \times 8!}\left|\eta_2 - \eta_1\right|$ where $M = \max_{-1 \le x \le 1}\left|p^{ix}(x)\right|$

It is obvious that error will be less if η_1 and η_2 are closer to each other.

Corollary

The error bound for the truncation error $E_{CC7GL4}(p)$ is given by $\left|E_{CC7GL4}(f)\right| \le \frac{256M}{150255 \times 8!}$ where, $M = \max_{-1 \le x \le 1}\left|p^{ix}(x)\right|$.

Proof- We know from Theorem 2

$$\left|E_{CC7GL4}(p)\right| \le \frac{128M}{150255 \times 8!}\left|\eta_2 - \eta_1\right|$$

where, $\eta_1, \eta_2 \in [-1, 1]$ and $M = \max_{-1 \le x \le 1}\left|p^{ix}(x)\right|$

Choosing $\left|\eta_2 - \eta_1\right| \le 2$;[11,12]

$$\left|E_{CC7GL4}(p)\right| \le \frac{256M}{150255 \times 8!}$$

5. Numerical verification

In this section there is a comparison of $R_{CC7GL4}(p)$ with $R_{CC7}(p)$ for evaluation of some real definite integrals in adaptive routine. The integrals under considerations are

$$I_1 = \int_0^1 e^{-x^2}\cos x\,dx, \quad I_2 = \int_1^2 \frac{1}{1+x^3}dx, \quad I_3 = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx,$$

$$I_4 = \int_2^3 \frac{\cos 2x}{1+\sin x}dx, \quad I_5 = \int_1^2 \frac{\ln x}{x}dx, \quad I_6 = \int_{-1}^1 e^{-x^2}dx, \quad I_7 = \int_0^1 \frac{1}{1+25x^2}dx$$

Table 1. Comparison of the rule $R_{CC7GL4}(p)$ with $R_{CC7}(p)$ in adaptive method

Exact value of the integrals	$R_{CC7}(p)$ *by adaptive method*	No of Intervals for $R_{CC7}(p)$	$R_{CC7GL4}(p)$ by adaptive method	No of Intervals for $R_{CC7GL4}(p)$	Maximum admissible absolute error or error tolerance (ε)
I_1=0.656174362731507	0.656174362731949	3	0.656174362727461	1	$\varepsilon_1 = 0.000000000004$
I_2=0.254352881963739	0.254352882007175	2	0.254352881941658	1	$\varepsilon_2 = 0.00000000002$
I_3=0.785398163397448	0.785398163397448	3	0.785398163397449	1	$\varepsilon_3 = 0.000000000005$
I_4=0.202704655520540	0.202704655528059	2	0.202704655523394	1	$\varepsilon_4 = 0.000000000002$
I_5=0.240226506959100	0.240226507012065	2	0.240226507058752	1	$\varepsilon_5 = 0.00000000009$
I_6=1.493648265624854	1.493648265686538	3	1.493648258924214	1	$\varepsilon_6 = 0.000000006$
I_7=0.274680153389003	0.274680386445199	3	0.274680320235561	2	$\varepsilon_7 = 0.0000001$

6. Conclusion

The given examples give better result towards the effectiveness of $R_{CC7GL4}(p)$ in adaptive scheme. The number of steps will be less for the rule $R_{CC7GL4}(p)$ required to approximate an integral in adaptive quadrature method in comparison to its constituent Clenshaw-Curtis quadrature rule.

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