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On the optimal design of IIR filter with its state-space realization

Choudhry Fahad Azim, Syed Faiz Ahmed and Abdul Rehman Memon
Faculty of Engineering Sciences & Technology, Hamdard University, Pakistan

Fahad.azim@hamdard.edu; Faiz.ahmed@hamdard.edu; Abdul.Memon@hamdard.edu

Abstract

The most widely used digital filters are finite impulse response (FIR) filters that are usually implemented with the transversal structures. For FIR filter, the output signal is a linear combination of filter coefficients that produces a quadratic function (mean-square-error) with a distinctive optimal operation point. FIR filter alternatively be realized for obtaining improvements in comparison of transversal filter structure for speed of convergence, computational complexity and finite word length properties. IIR filtering techniques exhibit reliable alternative for traditional FIR filtering. The principal advantage of IIR filters is lesser parameterization to achieve at par performance of FIR filters. In addition, the pole-zero structures ease their modeling in physical systems. On the other hand, the tradeoff for the attributes have few drawbacks inherent to IIR filtering with recursive structure such as filter stability, convergence to bias and/or local minima and for very slow convergence. This research paper proposes a new structure and realization of IIR filter. Special consideration is given to the issue of optimal state-space realization and newpole modulus based stability is derived.

Keywords: IIR filtering, state-space realization, ρ-realization, pole-modulus based stability.

Introduction

The digital filter design is well-studied field (Chaohua *et al.,* 2010; Konopacki & Moscinska, 2010). Digital devices such as micro-processors have been used for implementation of well-designed systems in real-time applications. As a result of which actual implemented circuit differ from well-designed filter in stability terms. Finite world length error is most acute problem in implementation of digital design (Gevers & Li, 1993; Proakis & Manolakis, 2006). Structural design, the second stage of digital design, need to investigate for overcoming the FWL effects. A number of different structures (Zixue Zhaoand & Gang Li, 2006) may be used for realization of digital filters which have equivalent infinite precision implementation. The key parameter among different structures for same realization is the degree of robustness against finite world length error. The optimal design is one which has minimum finite world length effect in certain sense.

The search of optimal finite world length state - space realization for digital filter design has been the active research area from last two decades, where optimality criteria depends on minimizing the sensitivity measure of digital design or rounding off noise gain (Zhao, 2005; Li *et al.,* 2010), in all this optimality analysis stability is rarely investigated. In control system theory, the optimal FWL design thoroughly investigated to overcome the difficulty of setting proper criteria for tractable stability conditions (Dolecek & Mitra, 2002; Carletta *et al.,* 2003). In recent past, a few diversified pole sensitivity based digital filter design has been proposed to strengthen the stability measure (Dolecek & Mitra, 2002; Carletta *et al.,* 2003). (Wu *et al.,* 2001; Jinxin, 2005) had proposed closed loop pole-based modulus designs as an improvement in digital controller design. The (Wu *et al.,* 2001) design was based on ℓ_1- norm. In this paper ℓ_2- norm based alternate pole modulus (Hinamoto *et al.,* 2002; Yamaki *et al.,* 2006) is used to for improvement of sensitivity based stability of digital filters. It has been observed that the speed of convergence decrease drastically because of increased implementation complexity in optimal design because of full parameterization. In this paper a sparse realization is derived based on polynomial operator (Hao, 2005) which is called ρ-realization. The proposed design has high implementation efficiency and can be optimized for finite word length effect reduction. Numerical example has been used to prove that the stability performance of the proposed ρ-realization matches with the optimal parameterized realization.

A stability related measure

The transfer function of discrete linear time-invariant system is given by:

$$\mathrm{H(z)} = \frac{\sum_{k=0}^{K} q_k z^{K-k}}{z^K + \sum_{k=1}^{K} p_k z^{K-k}} \tag{1}$$

The system can be implementable with following state-space equations:

$$\begin{aligned} x(n+1) &= \alpha x(n) + \beta u(n) \\ y(n) &= \Omega x(n) + \Gamma u(n) \end{aligned} \tag{2}$$

In equation (2), $u(n)$ is input to the system and $y(n)$ is output of the system and$x(n) \epsilon\, R^{K-1}$ is the state variable vector. The state-space realization of $H(z)$ is given by$R \triangleq (\alpha, \beta, \Omega, \Gamma\,)$where$\alpha\, \epsilon R^{K\times K}, \beta\, \epsilon R^{K\times 1}$, $\Omega \epsilon R^{1\times K}$ and$\Gamma = q_o \epsilon\, R$, satisfy:

$$\mathrm{H(z)} = \Gamma + \Omega(z\,\mathrm{I} - \alpha)^{-1}\beta \tag{3}$$

Where, *I* represent the identity matrix. If all realization set is represented by S_H, then it is given by:

$$S_H = \{\, (\alpha, \beta, \Omega, \Gamma\,) : H(z) = \Gamma + \Omega(z\, I - \alpha)^{-1} \tag{4}$$

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It is noticeable that this all realization set is characterize by

$$\alpha = \tau^{-1}\alpha_o\tau,\ \beta = \tau^{-1}\beta_o,\quad \Omega = \Omega_o\tau \qquad (5)$$

In equation (5) ,$R_o \triangleq (\alpha_o, \beta_o, \Omega_o, \Gamma)$represents the initial realizations and τ is any non-singular matrix $\epsilon R^{K\times K}$.

$$\bar{\alpha} = \begin{bmatrix} \alpha_p + \Gamma\beta_p\Omega_p & \beta_p\Omega \\ \beta\Omega_p & \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_p & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \beta_p & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \Gamma & \Omega \\ \beta & \alpha \end{bmatrix}\begin{bmatrix} \Omega_p & 0 \\ 0 & I \end{bmatrix}$$

$$= M_o + M_1XM_2 \triangleq \bar{\alpha}(X) \qquad (6)$$

Finite Word Length effect on system stability

For fixed point implementation of B-bits, All parameter of realization is truncated or rounded to ±1. Thus, FWL error is produced by any parameter of X, if it does not belong to the space:

$$S_{fwl} = \{-1, 1\} \cup \{\pm \textstyle\sum_{l=1}^{B} c_l\, 2^{-l},\ \ c_l = 0, 1,\ \forall l\} \qquad (7)$$

$S_{fwl}$is discrete spacing contain $(2^{(B+1)} + 1)$ elements.

Thus the actual implementation of system with finite word length errors is given by$(X + \Delta X)$. Where,

$$\mu(\Delta X) \triangleq \max|\Delta x_{ij}| \leq \frac{\epsilon}{2} \qquad (8)$$

Thus $\bar{\alpha}(X + \Delta X)$ may have Eigen-values outside unit circle that may cause instability. Following robustness measure criteria is used to determine the stability behavior of X.

$$\mu_0(X) \triangleq \inf\{\mu(\Delta X) : \bar{\alpha}(X) + M_1\Delta XM_2 \text{ is unstable}\}$$

An alternative approach is pole modulus sensitivity based robustness. In this approach the stability is measured as:

$$\mu_2(X) = min_k \frac{1-\lambda_k}{\sqrt{N\Phi_k}} \quad (9)$$

Where N represents total number of parameters and

$$\Phi_k \triangleq \textstyle\sum_{i,j} \delta(x_{ij}) \left|\frac{\partial|\lambda_k|}{\partial x_{ij}}\right|^2, \forall k \qquad (10)$$

For optimality, starting with the approach in (Jinxin, 2005),

$$\rho_k(z) \triangleq \frac{z-\gamma_k}{\Delta_k},\quad k = 1,2, \ldots, K \qquad (11)$$

Thus $\{\Delta_k > 0\}$ and $\{\gamma_k\}$ are two step of parameters will be analyzed in due course. Now the numerator $N(z)$ and denominator $D(z)$ of equation (1) can be re-written as:

$$N(z) = \zeta\left[Q_0\textstyle\prod_{k=1}^{K}\rho_k + Q_1\prod_{k=2}^{K}\rho_k + \ldots Q_{K-1}\rho_K + Q_K\right] \quad (12)$$

$$D(z) = \zeta\left[\textstyle\prod_{k=1}^{K}\rho_k + P_1\prod_{k=2}^{K}\rho_k + \ldots P_{K-1}\rho_K + P_K\right] \quad (13)$$

Where$\zeta = \prod_{k=1}^{K}\Delta_k$, polynomial operator is 'ρ_k' and (1) is re-parameterized with $\{P_k, Q_k\}$ and given as:

$$H(Z) = \frac{Q_0 + Q_1\rho_1^{-1} + \ldots + Q_{K-1}\prod_{k=1}^{K-1}\rho_k^{-1} + Q_K\prod_{k=1}^{K}\rho_k^{-1}}{1 + P_1\rho_1^{-1} + \ldots + P_{K-1}\prod_{k=1}^{K-1}\rho_k^{-1} + P_K\prod_{k=1}^{K}\rho_k^{-1}} \qquad (14)$$

Fig. 1. DFIIt structure

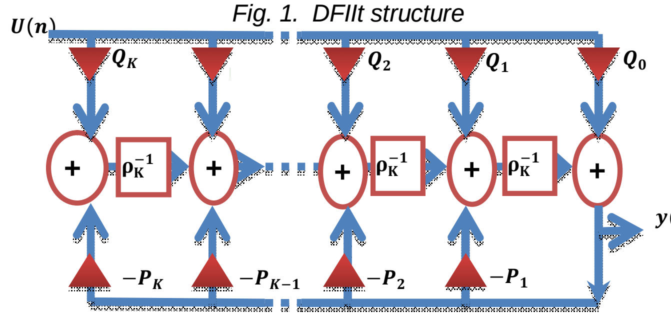

Fig. 2. Operator Realization

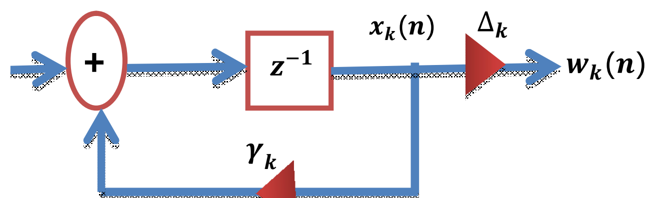

Fig.1. shows DFIIt structure and Fig.2. Operation realization.

Thus the new state-space realization derived from new parameter is termed as "R_ρ^{opt} - realization" and is given by:$(\alpha_\rho, \beta_\rho, \Omega_\rho, \Gamma)$

$$\alpha_\rho = \begin{bmatrix} \gamma_1 - P_1\Delta_1 & \Delta_2 & 0 & \ldots & 0 & 0 \\ -P_2\Delta_1 & \gamma_2 & \Delta_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -P_{k-1}\Delta_1 & 0 & 0 & \ldots & \gamma_{K-1} & \Delta_K \\ -P_k\Delta_1 & 0 & 0 & \ldots & 0 & \gamma_K \end{bmatrix}$$

$$\beta_\rho = [\,Q_1 - Q_0P_1 \quad Q_2 - Q_0P_2\ Q_3 - Q_0P_3 \quad \ldots Q_k - Q_0P_k\,]^T$$

$$\Omega_\rho = [\Delta_1\ 0\ 0\ \ldots\ 0\,]\Gamma = Q_0$$

Numerical Example

Consider a fourth-order low-pass Butter-worth filter using MATLAB command butter (4, 0.05), where '4' is order of filter and '0.05' is cutoff frequency. The resultant poles lie at $0.86 \pm j\,0.05$ and $0.93 \pm j0.14$.The filter realization $R_o \triangleq (\alpha_o, \beta_o, \Omega_o, \Gamma)$can be given as:

$$\alpha_0 = \begin{bmatrix} 3.59 & 1 & 0 & 0 \\ -4.85 & 0 & 1 & 0 \\ 2.92 & 0 & 0 & 1 \\ -0.66 & 0 & 0 & 0 \end{bmatrix},\quad \beta_o = \begin{bmatrix} 0.24 \times 10^{-3} \\ 0.04 \times 10^{-3} \\ 0.22 \times 10^{-3} \\ 0.01 \times 10^{-3} \end{bmatrix},$$

$$\Omega_o = [1 \quad 0 \quad 0 \quad 0], \qquad \Gamma = 3.12 \times 10^{-5}$$

Considering the initial realization (R_o) the fully parameterized normal realization (R_n) can be obtained by the procedure define in (Hao, 2005). By applying extensive search on equation (11), the optimum ρ - realization (R_ρ^{opt}) is achieved by setting$\gamma_1 = 0.50$, $\gamma_2 = \gamma_4 = 1$, $\gamma_3 = 0.75$ and $\Delta_k = 0.25, \forall k$. For

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comparison, shift operatorρ - realization (R_z) (Li & Gevers, 1993) and delta operatorρ - realization (R_δ)(Wong & Ng, 2001)are also obtained by setting $\gamma_k = 0, \forall k$ and $\gamma_k = 1, \forall k$ respectively.

For each structure, μ_0 are estimated using iterative simulations and μ_2 is computed. The estimate of the $\mu_0$are presented in Table 1 that are derived by the procedure define in (Jinxin, 2005).

Table 1. Filter Realization Comparison

Realization	μ2	μ0	Dynamic Range
Initial realization (R_o)	1.09×10^{-4}	1.26×10^{-4}	No
Normal realization (R_n)	2.07×10^{-2}	4.22×10^{-2}	Yes
Shift ρ-realization R_z)	2.08×10^{-4}	3.78×10^{-4}	No
Delta operator ρ-realization (R_δ)	1.27×10^{-2}	1.89×10^{-2}	Yes
Optimized ρ-realization(R_ρ^{opt})	1.32×10^{-2}	2.01×10^{-2}	Yes

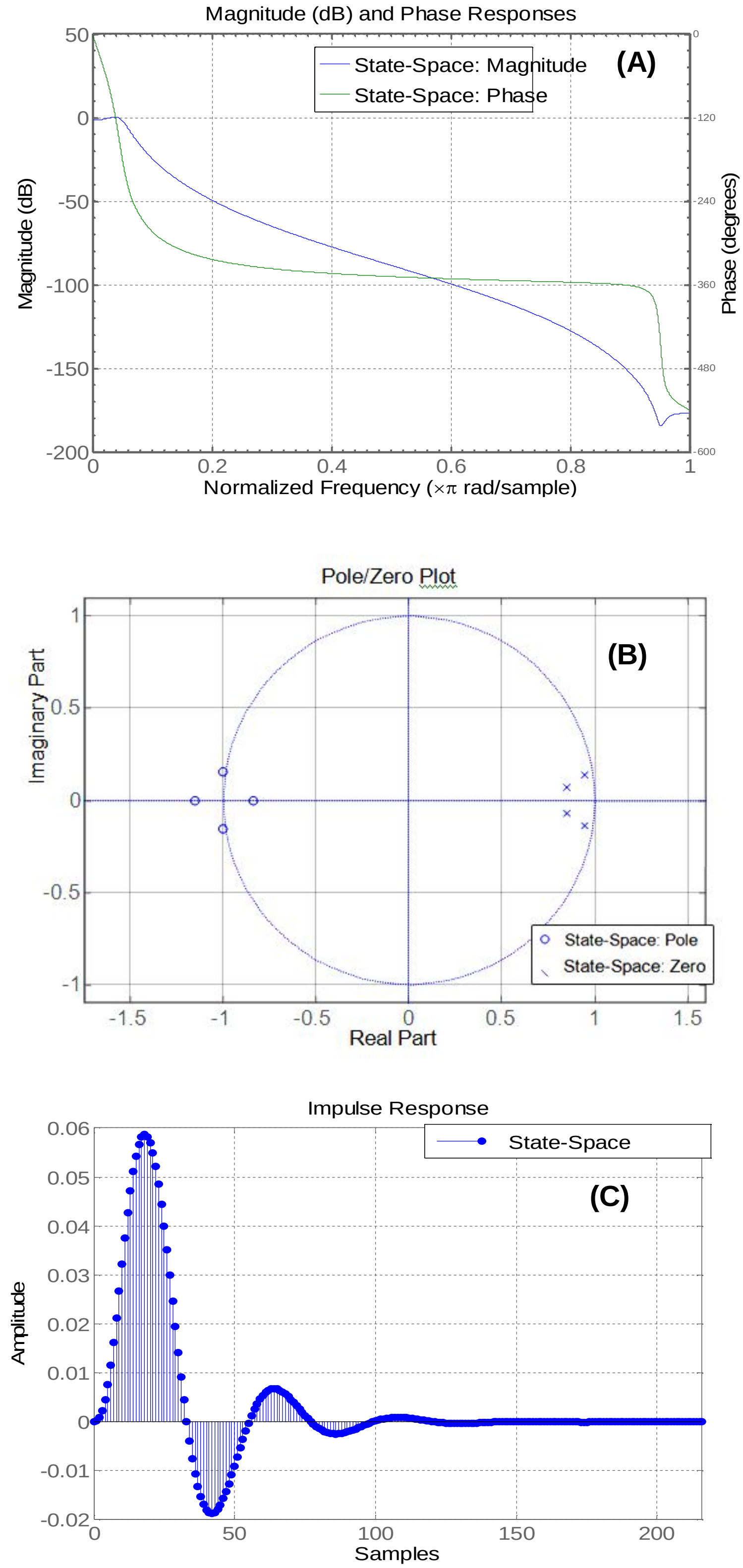
Results and discussion

From Table 1, it has been concluded that shift-operator based structure suffers with stability robustness in comparison with normal, delta and optimized realizations. Normal realization has better stability robustness in comparison with optimal ρ-realization and Delta operator ρ-realization. However, Implementation efficiency of normal realization $\{(K+1)^2$ parameters$\}$ is lesser then optimal ρ-realization $\{K$ parameters$\}$.For Low-pass filter (in which poles are distributed around z = +1) Delta operator ρ-realization has good stability performance. However, for High-pass and Band-pass filters (in which poles are distributed away from z = +1) it suffers in performance.

Conclusion

In this paper a new ρ-realization based IIR structure is derived. Its optimality analysis is investigated with different structure realizations. One design example has been discussed which shows that the proposed structure demonstrates good implementation efficiency and excellent stability performance.

Fig. 3. Fourth order low-pass Butter-worth filter



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References

1. Chaohua Dai, Weirong Chen and Yunfang Zhu (2010) Seeker Optimization Algorithm for Digital IIR Filter Design. *IEEE Tans. Industrial Electronics*. 57(5), 1710-1718.
2. Konopacki J and Moscinska K (2010) Comments on linear-phase IIR filter design using weighted LS optimization. *IEEE Intl. Conf. Signals and Electronic System*, Gliwice, Poland. pp: 241-244.
3. Proakis John G and Dimitris K.Manolakis (2006) Digital Signal processing- principles, Algorithms and Applications. 4th edition, *Prentice-Hall, Inc, NJ*.
4. Zixue Zhao and Gang Li (2006) Round-off noise analysis of two efficient digital filter structures. *IEEE Signal Processing Transactions*. 54(2), 790-795.
5. Gevers M and Li G (1993) Parameterizations in Control, Estimation and Filtering Problems: Accuracy Aspects. *Communication and Control Engg. Series. London, U.K. Springer-Verlag.*
6. Li F, LiminMeng, ZhijiangXu and JingyuHua (2010) A novel digital filter structure with minimum round-off noise. *Digital Signal Processing*. 20(4), 1000-1009.
7. Zixue Zhao (2005) Efficient digital filter structures with minimum round-off noise gain, *IEEE Intl. Symposium Circuits and Sys. Kobe, Japan.* pp: 2587-2590.
8. Carletta, Veillette J, Krach R and Fang F (2003) Determining appropriate precisions for signals in fixed-point IIR filters. *IEEE Intl. Conf. on Design Automation, Anaheim, CA, US.* pp: 656-661.
9. Jovanovic-Dolecek G and Mitra SK (2002), Symbolic sensitivity analysis of digital filter structures using MATLAB. *IEEE Intl. Conf. Electronics, Circuits and Systems, Dubrovnik, Croatia.* pp: 903-906.
10. Wu J, Chen S, Li G, Istepanian RH and J Chu,(2001),An improved closed-loop stability related measure for finite-precision digital controller realizations, *IEEE Trans. Automat. Control.* 46 (7), 1162-1166.
11. JinxinHao (2005) An Efficient Structure for Finite Precision Implementation of Digital Systems. *IEEE Intl. Conf. Information, Communications and Signal Processing, Bangkok, Thailand.* pp: 564-568.
12. Hao J (2005) Polynomial operator based sparse controller structures ith stability consideration, *IEE Proceedings Control Theory and Applications*. 152(5), 521-530.
13. Jiong Zhou, Gang Li and Mingxuan Sun (2006) A structural view of stability in Adaptive IIR filters. *IEEE Transaction on Signal Processing*. 54(12), 4828-4835.
14. Hinamoto T, Yokoyama S, Inoue T, Zeng W and Lu W (2002) Analysis and minimization of L2-sensitivity for linear systems and two-dimensional state-space filters using general controllability and observability gramians. *IEEE Trans. Circuits Syst. I*. 49(9), 1279-128.
15. Yamaki S Abe and Kawamata MM (2006) A Closed Form Solution to L2-Sensitivity Minimization of Second-Order Digital Filters Subject to L2-Scaling Constraints. IEEE Asia Pacific Conf. on Circuits and Sys. Singapore*.* pp: 1216 -1219.
16. Wong N and Ng TS (2001) A generalized direct-form delta operator-basedIIR filter with minimum noise gain and sensitivity. *IEEE Trans. CircuitSyst. II*. 48(4), 425-431.
17. Li G and M Gevers (1993) Comparative study of finite word-length effects inshift and delta operator parameterizations. *IEEE Trans. Automat. Contr*. 38(5), 803-807.