Soft h-ideals of Soft BCI-algebras

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Abstract

The notion of soft *h*-ideals and *h*-idealistic soft BCI-algebras is introduced and their basic properties are discussed. Relations between soft ideals and soft *h*-ideals of soft BCI-algebras are provided. Also idealistic soft BCI-algebras and *h*-idealistic soft BCI-algebras are being related. The intersection, union, "AND" operation and "OR" operation of soft *h*-ideals and *h*-idealistic soft BCI-algebras are established. Using soft sets, characterizations of (fuzzy) *h*-ideals in BCI-algebras are given. Relations between fuzzy *h*-ideals and *h*-idealistic soft BCI-algebras are discussed.

Keywords: (*h*-idealistic) Soft BCI-algebra, Soft Ideal, Soft Set,,Soft *h*-ideal

1. Introduction

Soft set is a parameterized general mathematical tool which deals with a collection of approximate descriptions of objects. Each approximate description has two parts, a predicate and an approximate value set. In classical mathematics, a mathematical model of an object is constructed and the notion of exact solution of this model is defined. The mathematical model is usually complicated and the exact solution is difficult to obtain. So the notion of approximate solution is introduced and the solution is calculated. In the soft set theory, we adopt an opposite approach to this problem. The initial description of the object has an approximate nature and there is no need to introduce the notion of exact solution. The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. Any parametrization we prefer can be used with the help of words and sentences, real numbers, functions, mappings and so on. Soft set theory was firstly proposed by Molodtsov¹⁶ in 1999 as a new mathematical tool for dealing with uncertainties. This theory is free from difficulties that have troubled the usual theoretical approaches. Soft set theory has wide applications in many different fields which include, the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory. Applications of soft set theory in other disciplines and real life problems are now catching momentum.

Maji et al¹⁴ in 2002, gave first practical application of soft sets in decision making problems. Celik and Yamak² discussed various applications of fuzzy soft set theory in medical diagnosis using fuzzy arithmetic operations. Ali et al¹ proved that De Morgan's Laws hold in soft set theory for their newly defined relative complement, restricted union and restricted intersection. Jun et al6 applied the soft set theory to BCC-algebras and introduced the notion of soft BCC-algebras and soft BCC-subalgebras. Yang and Guo¹⁹ introduced the notions of anti-reflexive kernel, symmetric kernel, reflexive closure and symmetric closure of a soft set relation. They also discussed soft set relation mappings and inverse soft set relation mappings. Xu et al18 introduced vague soft sets as an extension of the soft sets. Zou and Xiao²⁰ presented data analysis approaches of soft sets under incomplete information, in

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view of the particularity of the value domains of mapping functions in soft sets. We refer the readers to^{3,17} for further information regarding development of soft set theory.

Jun⁵ in 2008, applied the soft set theory to BCK/BCIalgebras and introduced the notion of soft BCK/BCIalgebras and soft subalgebras. Jun and Park9, introduce the notion of soft ideals and idealistic soft BCK/BCIalgebras and investigated relations between soft BCK/ BCI-algebras and idealistic soft BCK/BCI-algebras. Jun et al⁸, introduced the notion of soft *p*-ideals and *p*-idealistic soft BCI-algebras and derived their basic properties. Later, Jun et al⁷ introduced the notions of fuzzy soft BCK/ BCI-algebras, (closed) fuzzy soft ideals and fuzzy soft *p*-ideals and investigated related properties. In this paper, we apply the notion of soft sets by Molodtsov to *h*-ideals in BCI-algebras. We introduce the notion of soft *h*-ideals and *h*-idealistic soft BCI-algebras and discuss their basic properties. Using soft sets, we give characterizations of (fuzzy) h-ideals in BCI-algebras. We provide relations between fuzzy *h*-ideals and *h*-idealistic soft BCI-algebras.

2. Basic Results on BCI-algebras

BCK/BCI-algebras are important classes of logical algebras introduced by Y. Imai and K. Iseki⁴ and were extensively investigated by several researchers.

An algebra (X, *, 0) of type (2,0) is called a BCI-algebra if it satisfies the following conditions:

(I) $((x^*y)^*(x^*z))^*(z^*y)=0$

(II) $(x^*(x^*y))^*y=0$

(III) $x^*x=0$

(IV) $x^*y=0$ and $y^*x=0$ imply x=y

for all $x, y, z \in X$. In a BCI-algebra X, we can define a partial ordering " \leq " by putting $x \leq y$ if and only if $x^*y=0$.

If a BCI-algebra *X* satisfies the identity:

(V) $0^*x=0$, for all $x \in X$, then *X* is called a BCK-algebra.

In any BCI-algebra the following hold:

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(VI) (x^*y)^*z=(x^*z)^*y
(VII) x^*0=x
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(VIII) $x \le y$ implies $x^* z \le y^* z$ and $z^* y \le z^* x$ (IX) $0^* (x^* y) = (0^* x)^* (0^* y)$

- (X) $x^*(x^*(x^*y)) = (x^*y)$
- (XI) $(x^*z)^*(y^*z) \le x^*y$

for all $x, y, z \in X$.

A non-empty subset *S* of a BCI-algebras *X* is called a subalgebra of *X* if $x^*y \in S$ for all $x,y \in S$. A non-empty subset *I* of a BCI-algebra *X* is called an ideal of *X* if for any $x \in X$

(I1) $0 \in I$

(I2) $x^*y \in I$ and $y \in I$ implies $x \in I$

Any ideal *I* of a BCI-algebra *X* satisfies the following implication:

 $x \le y$ and $y \in I \Rightarrow x \in I, \forall x \in X$

A non-empty subset I of a BCI-algebra X is called an h-ideal (see Khalid and Ahmad¹⁰) of X if it satisfies (I1) and

(I3) $x^*(y^*z) \in I$ and $y \in I \Rightarrow x^*z \in I$ for all $x,z \in X$.

We know that every *h*-ideal of a BCI-algebra *X* is also an ideal of *X*.

We refer the readers to¹²,¹⁵ for further study about ideals in BCK/BCI-algebras.

3. Basic Results on Soft Sets

In¹⁶ the soft set is defined in the following way: Let *U* be an initial universe set and *E* be a set of parameters. Let P(U) denotes the power set of *U* and $A \subseteq E$.

Definition 3.1 (Molodtsov¹⁶)

A pair (F, A) is called a soft set over U, where F is a mapping given by

In other words, a soft set over *U* is a parameterized family of subsets of the universe *U*. For $a \in A$, F(a) may be considered as the set of *a*-approximate elements of the soft set (*F*, *A*).

Definition 3.2 (Maji et al¹³)

Let (F, A) and (G, B) be two soft sets over a common

universe *U*. The intersection of (F, A) and (G, B) is defined

to be the soft set (H, C) satisfying the following conditions: (i) $C=A\cap B$

(ii) H(x)=F(x) or G(x) for all $x \in C$, (as both are same sets)

In this case, we write (F, A) (G, B)=(H, C).

Definition 3.3 (Maji et al¹³)

Let (F, A) and (G, B) be two soft sets over a common universe U. The union of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:

(i) $C=A\cup B$

(ii) for all $x \in C$,

In this case, we write (F, A) (G, B)=(H, C).

Definition 3.4 (Maji et al¹³)

Let (F, A) and (G, B) be two soft sets over a common universe U. Then "(F, A) AND (G, B)" denoted by (F, A) (G, B) is defined as (F, A) $(G, B)=(H, A \times B)$, where $H(x,y)=F(x) \cap G(y)$ for all $(x,y)\in A \times B$.

Definition 3.5 (Maji et al¹³)

Let (F, A) and (G, B) be two soft sets over a common universe *U*. Then "(F, A) *OR* (G, B)" denoted by (F, A) (G, B) is defined as (F, A) $(G, B)=(H, A \times B)$, where H(x,y)=F(x) \cup G(y) for all $(x,y) \in A \times B$.

Definition 3.6 (Maji et al¹³)

For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B), denoted by (F, A) (G, B), if it satisfies:

(i) $A \subseteq B$

(ii) For every $a \in A$, F(a) and G(a) are identical approximations.

4. Soft h-ideals

In what follows let *X* and *A* be a BCI-algebra and a nonempty set, respectively and *R* will refer to an arbitrary binary relation between an element of *A* and an element of *X*, that is, *R* is a subset of $A \times X$ without otherwise specified. A set valued function can be defined as $F(x)=\{y \in X | xRy\}$ for all $x \in A$. The pair (*F*, *A*) is then a soft set over *X*.

Definition 4.1 (Jun and Park⁹)

Let *S* be a subalgebra of *X*. A subset *I* of *X* is called an ideal of *X* related to *S* (briefly, *S*-ideal of *X*), denoted by $I \triangleright S$, if it satisfies:

(i) $0 \in I$ (ii) $x^*y \in I$ and $y \in I \Rightarrow x \in I$ for all $x \in S$

Definition 4.2

Let S be a subalgebra of X. A subset I of X is called an *h*-ideal of X related to S (briefly, *S*-*h*-ideal of X), denoted by, if it satisfies:

(i) 0∈*I*

(ii) $x^*(y^*z) \in I$ and $y \in I \Rightarrow x^*z \in I$ for all $x,z \in S$

Example 4.3

Let $X = \{0, a, b, c, d\}$ be a BCK-algebra and hence a BCIalgebra, with the following Cayley table:

Then $S=\{0,a,b\}$ is a subalgebra of X and $I=\{0,a,b,d\}$ is an S-h-ideal of X.

Note that every *S*-*h*-ideal of *X* is an *S*-ideal of *X*.

Definition 4.4 (Jun⁵)

Let (F, A) be a soft set over *X*. Then (F, A) is called a soft BCI-algebra over *X* if F(x) is a subalgebra of *X* for all $x \in A$.

Definition 4.5 (Jun and Park⁹)

Let (F, A) be a soft BCI-algebra over X. A soft set (G, I) over X is called a soft ideal of (F, A), denoted (G, I) (F, A), if it satisfies:

(i) *I*⊂*A*(ii) *G*(*x*) ▷ *F*(*x*) for all *x*∈*I*

Definition 4.6

Let (F, A) be a soft BCI-algebra over X. A soft set (G, I) over X is called a soft *h*-ideal of (F, A), denoted, if it satisfies:

(i) *I*⊂*A*

(ii) for all $x \in I$

Let us illustrate this definition using the following example.

Example 4.7

Consider a BCI-algebra $X = \{0, a, b, c, d\}$ which is given in

Example 4.3. Let (F, A) be a soft set over *X*, where A=X and is a set-valued function defined by:

 $F(x) = \{y \in X | y^*(y^*x) \in \{0,a\}\}$

for all $x \in A$. Then F(0)=F(a)=X, $F(b)=F(c)=\{0,a,d\}$, $F(d)=\{0,a,b,c\}$, which are subalgebras of *X*. Hence (*F*, *A*) is a soft BCI-algebra over *X*. Let $I=\{0,a,b\}\subset A$ and be a set-valued function defined by:

 $\mathbf{G}(x) = \{ y \in X | y^*(y^*x) \in \{0,c\} \}$

for all $x \in I$. Then . Hence (G, I) is a soft *h*-ideal of (F, A). Note that every soft *h*-ideal is a soft ideal but the converse is not true as seen in the following example.

Example 4.8

Let $X=\{0,a,b,c\}$ be a BCI-algebra with the following Cayley table:

Let (F, A) be a soft set over *X*, where A=X and is a setvalued function defined by:

 $F(x) = \{0\} \cup \{y \in X | y^*(y^*x) \in \{0,a\}\}$

for all $x \in A$. Then F(0)=F(a)=X and $F(b)=F(c)=\{0\}$, which are subalgebras of X. Hence (F, A) is a soft BCI-algebra over X.

Let (*G*, *I*) be a soft set over *X*, where $I=\{0,a\} \subset A$ and is a set-valued function defined by:

 $G(x) = \{0\} \cup \{y \in X \mid x \le y\}$

for all $x \in I$. Then $G(0) = \{0,a\} \triangleright X = F(0)$ and $G(a) = \{0,a\} \triangleright X = F(a)$. Hence (G, I) is a soft ideal of (F, A) but it is not a soft *h*-ideal of (F, A) because G(a) is not an F(a)-*h*-ideal of *X* since $b^*(0^*c) = b^*b = 0 \in G(a)$ and $0 \in G(a)$ but $b^*c = c \notin G(a)$.

Theorem 4.9

Let (F, A) be a soft BCI-algebra over X. For any soft sets and over X where we have

Proof. Using Definition 3.2, we can write

where, and $e \in I$ or Obviously, I \subset A and is a mapping. Hence (*G*, *I*) is a soft set over *X*. Since and, it follows that or for all *e*∈*I*. Hence

This completes the proof.

Corollary 4.10

Let (F, A) be a soft BCI-algebra over *X*. For any soft sets (G, I) and (H, I) over *X*, we have

Proof. Straightforward.

Theorem 4.11

Let (F, A) be a soft BCI-algebra over *X*. For any soft sets (G, I) and (H, J) over *X* in which *I* and *J* are disjoint, we have

Proof. Assume that and . By means of Definition 3.3, we can write (G, I) (H, J)=(R, U), where $U=I\cup J$ and for every $e \in U$,

Since $I \cap J = \emptyset$, either $e \in I \setminus J$ or $e \in J \setminus I$ for all $e \in U$. If $e \in I \setminus J$, then since. If $e \in J \setminus I$, then since . Thus for all $e \in U$ and so

It *I* and *J* are not disjoint in Theorem 4.11, then Theorem 4.11 is not true in general as seen in the following example.

Example 4.12

Let $X=\{0,1,a,b,c\}$ be a BCI-algebra with the following Cayley table:

Let (F, A) be a soft set over *X*, where $A = \{0,1\}$ and is a setvalued function defined by:

$$F(x) = \{y \in X | y^*x = y\}$$

for all $x \in A$. Then F(0)=X and $F(1)=\{0,a,b,c\}$, which are subalgebras of *X*. Hence (*F*, *A*) is a soft BCI-algebra over *X*.

If we take *I*=*A* and define a set valued function by:

 $G(x) = \{y \in X | x^*(x^*y) \in \{0, b\}\}$

for all $x \in I$. Then and . Hence (G, I) is a soft *h*-ideal of (F, A).

Now consider $J=\{0\}$ which is not disjoint with I and let be a set valued function by:

$$H(x) = \{y \in X | x^*(x^*y) \in \{0,c\}\}$$

for all $x \in J$. Then . Hence (H, J) is a soft *h*-ideal of (F, A). But if (R, U)=(G, I) (H, J), then $R(0)=G(0)\cup H(0)=\{0,1,b,c\}$, which is not an *h*-ideal of X related to F(0) since $a^*(b^*0)=c \in (0)$ and $b \in R(0)$ but $a^*0=a \notin R(0)$. Hence (R, U)=(G, I) (H, J) is not a soft *h*-ideal of (F, A).

5. h-idealistic Soft BCI-algebras

Definition 5.1 (Jun and Park⁹)

Let (F, A) be soft set over X. Then (F, A) is called an idealistic soft BCI-algebra over X if F(x) is an ideal of X for all $x \in A$.

Definition 5.2

Let (F, A) be soft set over X. Then (F, A) is called an *h*-idealistic soft BCI-algebra over X if F(x) is an *h*-ideal of X for all $x \in A$.

Example 5.3

Consider a BCI-algebra $X=\{0,1,a,b,c\}$ which is given in Example 4.12. Let (*F*, *A*) be a soft set over *X*, where A=X and is a set-valued function defined by:

where, $Z(\{0,1\}) = \{x \in X | 0^*(0^*x) \in \{0,1\}\}$. Then (*F*, *A*) is an *h*-idealistic soft BCI-algebra over *X*.

For any element *x* of a BCI-algebra *X*, we define the order of *x*, denoted by o(x), as

where, in which *x* appears n-times.

Example 5.4

Let $X=\{0,a,b,c,d,e,f,g\}$ be a BCI-algebra defined by the following Cayley table:

Let (*F*, *A*) be a soft set over *X*, where $A = \{a, b, c\} \subset X$ and is a set-valued function defined by:

 $F(x) = \{y \in X | o(x) = o(y)\}$

for all $x \in A$. Then $F(a)=F(b)=F(c)=\{0,a,b,c\}$ is an *h*-ideal of *X*. Hence (*F*, *A*) is an *h*-idealistic soft BCI-algebra over *X*. But if we take $B=\{a,b,f,g\}\subset X$ and defined a set-valued function by:

 $G(x) = \{0\} \cup \{y \in X | o(x) = o(y)\}$

for all $x \in B$, then (G, B) is not an *h*-idealistic soft BCIalgebra over *X*, since $G(f) = \{0, d, e, f, g\}$ is not an *h*-ideal of *X* because $g^*(f^*d) = g^*b = e \in G(f)$ and $f \in G(f)$ but $g^*d = c \notin G(f)$.

Example 5.5

Consider a BCI-algebra $X = \{0,a,b,c\}$ with the following cayley table:

Let (F, A) be a soft set over *X*, where A=X and is a setvalued function defined by:

for all $x \in A$. Then $F(0)=\{0\}$, $F(a)=\{0,a\}$, $F(b)=\{0,b\}$, $F(c)=\{0,c\}$, which are *h*-ideals of *X*. Hence (*F*, *A*) is an *h*-idealistic soft BCI-algebra over *X*.

Obviously, every *h*-idealistic soft BCI-algebra over *X* is an idealistic soft BCI-algebra over *X*, but the converse is not true in general as seen in the following example.

Example 5.6

Consider a BCI-algebra $X:=Y\times Z$, where (Y,*,0) is a BCIalgebra and (Z,-,0) is the adjoint BCI-algebra of the additive group (Z,+,0) of integers. Let be a set-valued function defined as follows:

for all $(y,n) \in X$, where is the set of all non-negative integers. Then (F, X) is an idealistic soft BCI-algebra over *X* but it is not an h-idealistic soft BCI-algebra over *X* since $\{(0,0)\}$ may not be an *h*-ideal of *X*.

Proposition 5.7

Let (F, A) and (F, B) be soft sets over X where $B \subseteq A \subseteq X$. If (F, A) is an h-idealistic soft BCI-algebra over X, then so is (F, B).

Proof. Straightforward.

The converse of Proposition 5.7 is not true in general as seen in the following example.

Example 5.8

Consider an *h*-idealistic soft BCI-algebra over *X* which is described in Example 5.4. If we take $B = \{a, b, c, d\} \supseteq A$, then (*F*, *B*) is not an *h*-idealistic soft BCI-algebra over *X* since $F(d) = \{d, e, f, g\}$ is not an *h*-ideal of *X*.

Theorem 5.9

Let (F, A) and (G, B) be two *h*-idealistic soft BCI-algebras over *X*. If $A \cap B \neq \emptyset$, then the intersection (F, A) (G, B) is an *h*-idealistic soft BCI-algebra over *X*. **Proof.** Using Definition 3.2, we can write

(F, A) (G, B) = (H, C)

where, $C=A\cap B$ and H(e)=F(e) or G(e) for all $e\in C$. Note that is a mapping, therefore (H, C) is a soft set over X. Since (F, A) and (G, B) are *h*-idealistic soft BCI-algebras over X, it follows that H(e)=F(e) is an *h*-ideal of X or H(e)=G(e) is an *h*-ideal of X for all $e\in C$. Hence (H, C)=(F, A) (G, B) is an *h*-idealistic soft BCI-algebra over X.

Corollary 5.10

Let (F, A) and (G, A) be two *h*-idealistic soft BCIalgebras over *X*. Then their intersection (F, A) (G, A) is an *h*-idealistic soft BCI-algebra over *X*. **Proof.** Straightforward.

Theorem 5.11

Let (F, A) and (G, B) be two *h*-idealistic soft BCI-algebras over *X*. If *A* and *B* are disjoint, then the union (F, A) (G, B) is an *h*-idealistic soft BCI-algebra over *X*.

Proof. By means of Definition 3.3, we can write $(F, A) \sim (G, B)=(H, C)$, where $C=A\cup B$ and for every $e\in C$,

Since $A \cap B = \emptyset$, either $e \in A \setminus B$ or $e \in B \setminus A$ for all $e \in C$. If $e \in A \setminus B$, then H(e) = F(e) is an *h*-ideal of *X* since (F, A) is an *h*-idealistic soft BCI-algebra over *X*. If $e \in B \setminus A$, then H(e) = G(e) is an *h*-ideal of *X* since (G, B) is an *h*-idealistic soft BCI-algebra over *X*. Hence (H, C) = (F, A) (G, B) is an *h*-idealistic soft BCI-algebra over *X*.

Theorem 5.12

Let (F, A) and (G, B) be two *h*-idealistic soft BCI-algebras over *X*, then (F, A) (G, B) is an *h*-idealistic soft BCI-algebra over *X*.

Proof. By means of Definition 3.4, we know that

 $(F, A) (G, B) = (H, A \times B),$

where, $H(x,y)=F(x) \cap G(y)$ for all $(x,y) \in A \times B$. Since F(x) and G(y) are *h*-ideals of *X*, the intersection $F(x) \cap G(y)$ is also an *h*-ideal of *X*. Hence H(x,y) is an *h*-ideal of *X* for all $(x,y)\in A\times B$.

Hence (F, A) $(G, B)=(H, A \times B)$ is an *h*-idealistic soft BCIalgebra over *X*.

Definition 5.13

An *h*-idealistic soft BCI-algebra (*F*, *A*) over *X* is said to be trivial (resp., whole) if F(x)=0 (resp., F(x)=X) for all $x \in A$.

Example 5.14

Let *X* be a BCI-algebra which is given in Example 5.5 and let be a set-valued function defined by

 $F(x) = \{0\} \cup \{y \in X | o(x) = o(y)\}$

for all $x \in X$. Then $F(0)=\{0\}$ and F(a)=F(b)=F(c)=X, which are *h*-ideals of *X*. Hence (*F*, $\{0\}$) is a trivial *h*-idealistic soft BCI-algebra over *X* and (*F*, $X \setminus \{0\}$) is a whole *h*-idealistic soft BCI-algebra over *X*.

The proofs of the following three lemmas are straightforward, so they are omitted.

Lemma 5.15

Let be an onto homomorphism of BCI-algebras. If I is an ideal of X, then f(I) is an ideal of Y.

Lemma 5.16

Let be an isomorphism of BCI-algebras. If I is an h-ideal of X, then f(I) is an h-ideal of Y.

Let be a mapping of BCI-algebras. For a soft set (F, A) over X, (f(F), A) is soft set over Y, where is defined by f(F) (x)=f(F(x)) for all $x \in A$.

Lemma 5.17

Let be an isomorphism of BCI-algebras. If (F, A) is an h-idealistic soft BCI-algebra over X, then (f(F), A) is an h-idealistic soft BCI-algebra over Y.

Theorem 5.18

Let f:X $\mathbb{B}Y$ be an isomorphism of BCI-algebras and let (*F*, *A*) be an *h*-idealistic soft BCI-algebra over *X*.

- (1) If F(x)=ker(f) for all $x \in A$, then (f(F), A) is a trivial *h*-idealistic soft BCI-algebra over *Y*.
- (2) If (*F*, *A*) is whole, then (*f*(*F*), *A*) is a whole *h*-idealistic soft BCI-algebra over *Y*.

Proof.

- Assume that *F*(*x*)=*ker*(*f*) for all *x*∈A. Then for all *x*∈A. Hence (*F*, *A*) is a trivial *h*-idealistic soft BCI-algebra over *Y* by Lemma 5.17 and Definition 5.13.
- (2) Suppose that (F, A) is whole. Then F(x)=X for all

 $x \in A$ and so f(F)(x)=f(F(x))=f(X)=Y for all $x \in A$. It follows from Lemma 5.17 and Definition 5.13 that (f(F), A) is a whole *h*-idealistic soft BCI-algebra over *Y*.

Definition 5.19 (Khalid and Ahmad¹⁰)

A fuzzy set μ in *X* is called a fuzzy *h*-ideal of *X*, if for all $x,y,z \in X$,

(i) $\mu(0) \ge \mu(x)$

(ii) $\mu(x^*z) \ge \min\{\mu(x^*(y^*z)), \mu(y)\}$

The transfer principle for fuzzy sets described in¹¹ suggest the following theorem.

Lemma 5.20 (Khalid and Ahmad¹⁰)

A fuzzy set μ in *X* is a fuzzy *h*-ideal of *X* if and only if for any $t \in [0,1]$, the level subset $U(\mu;t):=\{x \in X | \mu(x) \ge t\}$ is either empty or an *h*-ideal of *X*.

Theorem 5.21

For every fuzzy *h*-ideal μ of *X*, there exists an *h*-idealistic soft BCI-algebra (*F*, *A*) over *X*.

Proof. Let μ be a fuzzy *h*-ideal of *X*. Then $U(\mu;t):=\{x \in X | \mu(x) \ge t\}$ is an *h*-ideal of *X* for all $t \in Im(\mu)$. If we take $A=Im(\mu)$ and consider a set valued function given by $F(t)=U(\mu;t)$ for all $t \in A$, then (F, A) is an *h*-idealistic soft BCI-algebra over *X*.

Conversely, the following theorem is straightforward.

Theorem 5.22

For any fuzzy set μ in *X*, if an *h*-idealistic soft BCI-algebra (*F*, *A*) over *X* is given by $A=Im(\mu)$ and $F(t)=U(\mu;t)$ for all $t \in A$, then μ is a fuzzy *h*-ideal of *X*.

Let μ be a fuzzy set in *X* and let (*F*, *A*) be a soft set over *X* in which $A=Im(\mu)$ and is a set-valued function defined by

$$F(t) = \{x \in X | \mu(x) + t > 1\}$$

$$(5.2)$$

for all $t \in A$. Then there exists $t \in A$ such that F(t) is not an h-ideal of X as seen in the following example.

Example 5.23

For any BCI-algebra X, define a fuzzy set μ in X by and for all $x\neq 0$. Let $A=Im(\mu)$ and be a set-valued function defined by (5.2). Then, which is not an *h*-ideal of *X*.

Theorem 5.24

Let μ be a fuzzy set in *X* and let (*F*, *A*) be a soft set over *X* in which *A*=[0,1] and is given by (5.2). Then the following assertions are equivalent:

(1) μ is a fuzzy *h*-ideal of *X*.

(2) for every $t \in A$ with $F(t) \neq \emptyset$, F(t) is an *h*-ideal of *X*.

Proof. Assume that μ is a fuzzy *h*-ideal of *X*. Let $t \in A$ be such that $F(t) \neq \emptyset$. Then for any $x \in F(t)$, we have $\mu(0)+t \ge \mu(x)+t>1$, that is, $0 \in F(t)$. Let $x^*(y^*z) \in F(t)$ and $y \in F(t)$ for any $t \in A$ and $x, y, z \in X$. Then $\mu(x^*(y^*z))+t>1$ and $\mu(y)+t>1$. Since μ is a fuzzy *h*-ideal of *X*, it follows that

 $\mu(x^*z) + t \ge \min\{\mu(x^*(y^*z)), \mu(y)\} + t$ = min{\u(x^*(y^*z)) + t, \u(y) + t} > 1

so that $x^*z \in F(t)$. Hence F(t) is an *h*-ideal of *X* for all $t \in A$ such that $F(t) \neq \emptyset$.

Conversely, suppose that (2) is valid. If there exists such that , then there exists to such that . It follows that and , which is a contradiction. Hence $\mu(0) \ge \mu(x)$ for all $x \in X$. Now assume that

for some . Then there exists some soÎA such that

which implies that and so but . This is a contradiction. Therefore

 $\mu(x^*z) \ge \min\{\mu(x^*(y^*z)), \mu(y)\}$

for all $x, y, z \in X$ and thus μ is fuzzy *h*-ideal of *X*.

Corollary 5.25

Let μ be a fuzzy set in *X* such that $\mu(x)>0.5$ for all $x \in X$ and let (*F*, *A*) be a soft set over *X* in which

 $A:=\{t\in Im(\mu)|t>0.5\}$

and is given by (5.2). If μ is a fuzzy *h*-ideal of *X*, then (*F*, *A*) is an *h*-idealistic soft BCI-algebra over *X*. **Proof.** Straightforward.

Theorem 5.26

Let μ be a fuzzy set in *X* and let (*F*, *A*) be a soft set over *X* in which *A*=(0.5,1] and is defined by

 $F(t)=U(\mu;t)$ for all $t \in A$

Then F(t) is an *h*-ideal of *X* for all $t \in A$ with $F(t) \neq \emptyset$ if and only if the following assertions are valid:

- (1) $max\{\mu(0), 0.5\} \ge \mu(x)$ for all $x \in X$.
- (2) $max\{\mu(x^*z), 0.5\} \ge min\{\mu(x^*(y^*z)), \mu(y)\}$ for all $x, y, z \in X$.

Proof. Assume that F(t) is an h-ideal of X for all $t \in A$ with $F(t) \neq \emptyset$. If there exists such that , then there exists to such that . It follows that , so that and . This is a contradiction. Therefore (1) is valid. Suppose that there exist $a,b,c \in X$ such that

 $max\{\mu(a^*c), 0.5\} < min\{\mu(a^*(b^*c)), \mu(b)\}$ Then there exists such that

which implies and , but . This is a contradiction. Hence (2) is valid.

Conversely, suppose that (1) and (2) are valid. Let $t \in A$ with $F(t) \neq \emptyset$. Then for any $x \in F(t)$, we have

 $max\{\mu(0), 0.5\} \ge \mu(x) \ge t > 0.5$

which implies $\mu(0) \ge t$ and thus $0 \in F(t)$. Let $x^*(y^*z) \in F(t)$ and $y \in F(t)$, for any $x, y, z \in X$. Then $\mu(x^*(y^*z)) \ge t$ and $\mu(y) \ge t$. It follows from the second condition that

 $max\{\mu(x^*z), 0.5\} \ge min\{\mu(x^*(y^*z)), \mu(y)\} \ge t > 0.5$

So that $\mu(x^*z) \ge t$, i.e., $x^*z \in F(t)$. Therefore F(t) is an *h*-ideal of *X* for all $t \in A$ with $F(t) \ne \emptyset$.

6. Conclusion

The concept of soft set, which is introduced by Molodtsov¹⁶, is a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Soft sets are deeply related to fuzzy sets and rough sets. We introduced the notion of soft *h*-ideals and *h*-idealistic soft BCI-algebras and discussed related properties. We established the intersection, union, "AND" operation and "OR" operation of soft *h*-ideals and *h*-idealistic soft BCI-algebras. From above discussion it can be observed that fuzzy *h*-ideals can be characterized using the concept of soft sets. For a soft set (*F*, *A*) over X, a fuzzy set μ in X is a fuzzy *h*-ideal of X if and only if for every *t* $\in A$ with

 $F(t) = \{x \in X | \mu(x) + t > 1\} \neq \emptyset$, F(t) is an *h*-ideal of *X*. Finally we have discussed the relations between fuzzy *h*-ideals and *h*-idealistic soft BCI-algebras.

7. References

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