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The Dynamics of 1-Step Shifts of Finite Type Over Two Symbols

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Abstract

A 1-step shift of finite type over two symbols is a collection of sequences over symbols 0 and 1 with some constrains. The constrains are identified by a set of forbidden blocks which are not allowed to appear in any sequences in the space. The space is of finite type since the number of forbidden blocks is finite and it is of 1-step type since the forbidden blocks are of length of 2. The aim of this paper is to look at the chaotic behaviour of 1-step shift of finite type by considering all spaces of it type. We found that there are six different 1-step shifts of finite type which exhibits totally different dynamics behaviour. We explain the dynamics of each space and then discuss on the difference of the dynamic properties between these spaces. Two of them are chaotic in the sense of Devaney. However the two chaotic shift spaces have totally different behaviour where one of them has trivial dynamics. The other four spaces are not chaotic but, they have some interesting behaviour to be highlighted. It turns out that some of the non-chaotic shift spaces satisfy some chaotic properties.

Keywords: Blending, Devaney Chaos, Locally Everywhere Onto, Mixing, Shift of Finite Type

1. Introduction

Dynamics is the subject that deals with change in systems that evolve in time. For a space X and a continuous function f acting on it, the dynamical system of (X, f) describes how each point, $x \in X$ moves to another place or state, $f(x) \in X$ and so forth. So whether the system is settles down to equilibrium, keeps repeating in cycles, or does something more complicated, we use it is dynamics to analyze its behaviour.

Mathematicians interested in studying chaos because chaos can be found in trivial system as well as complex one. For example the tent map has simple equation, but has a very complex behaviour[6], moreover, there is a complex system which has trivial behaviour as the complex biological system which has been described as "anti-chaotic"[12]. For that the interest in the chaos phenomenon has been increased and the Mathematicians tried to give this phenomenon a precise meaning, but tell now there is no standard definition of chaos.

As a result of the different efforts to give chaos a precise meaning, chaos has been defined in different ways, and the first used of the mathematical notion "chaos" was in 1975 by Li and Yorke in their paper[9] and has been called Li-Yorke chaos. After that in 1986 the widely used definition of chaos was by Devaney[5]. There exist different definitions of chaotic systems; most of them are based on the appropriate formalization of the stability concept and the notion of attractor. In this work, we put our emphasis on Devaney chaos.

Definition 1. A continuous function f acting on a metric space X with a metric d is topologically transitive[6]

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if $\exists\, n > 0$ such that $f^n(U) \cap V \neq \emptyset$, where U, V are any two non-empty open subsets of X.

Definition 2. A continuous function f acting on a metric space X with a metric d has sensitive dependence on initial conditions4 if $\exists\, \delta > 0$ such that for any $x \in X$ and neigborhood N of x, $\exists\ y \in N$ and $n > 0$ such that $d(f^n(x), f^n(y)) > \delta$, briefly, we will write (SDIC).

Definition 3. Let $f: X \rightarrow X$ be a function on a metric space X with metric d. The dynamical system (X, f) is said to be Devaney chaotic[6] if:

i) The periodic points of f are dense in X.
ii) f is topologically transitive.
iii) f depends sensitively on initial conditions.

After this definition of chaotic dynamical system by Devaney, Banks[2] et al. showed that SDIC is redundant since the condition is implied by the other two conditions. Assaf and Gadbois showed that[1] SDIC is the only redundant condition in definition 3 for general maps and spaces. However, on some spaces such as intervals and SFT, transitive map implies dense periodic points and the converse is not necessarily true (the identity map is the counter example).

In this paper, we also consider some other chaotic concepts that relate closely to the three chaotic ingredients; totally transitive, mixing, blending, and locally everywhere onto.

Definition 4. Let $f: X \rightarrow X$ be continuous, then f is said to be totally transitive[11] if f^n is transitive for all $n \geq 1$.

Definition 5. Let $f: X \rightarrow X$ be continuous, then f is said to be is topologically mixing[8] if for any non-empty open subsets U, V of V, $\exists\ N > 0$ such that $f^k(U) \cap V \neq \emptyset$ for all $k > N$.

It is obvious that totally transitive and topologically mixing are stronger then transitivity

Definition 6. A dynamical system (X, f) is weakly3 blending if for any non-empty open subsets U and V of $X, \exists$ such that $f^n(U) \cap f^n(V) \neq \emptyset$. A dynamical system (X, f) is strongly3 blending if for any non-empty open subsets U and V of $X, \exists\ n > 0$ such that $f^n(U) \cap f^n(V)$ contains an open set.

It is clear that functions which are blending are not necessarily transitive, and transitive functions are not necessarily blending. The counter examples can be found in Cranell[3].

Definition 7. Let $f: X \rightarrow X$ be a continuous function on a compact metric space X, then f is said to be locally everywhere onto7 ($l.e.o$) if for every open set $W \subseteq X$, $\exists n > 0$ such that $f^n(W) = X$.

$l.e.o$ is stronger than totally transitive, mixing and (weakly and strongly) blending.

2. Shift of Finite Type

Full-2-Shift, Σ_2 is the collection of all infinite sequences of symbols 0 and 1. Therefore the elements of Σ_2 is in a form of $\mathbf{s} = s_0 s_1 s_2 \cdots$, where $s_i \in \{0,1\}$ for every $i \in \mathbf{N}$. The space is a metric space which equipped with metric

$$d(\mathbf{s}, \mathbf{t}) = \begin{cases} 0 & if \quad \mathbf{s} = \mathbf{t} \\ 2^{-i} & if \quad \mathbf{s} \neq \mathbf{t} \end{cases}$$

where i is the smallest integer such that $s_i \neq t_i$, for every pair $\mathbf{s}, \mathbf{t} \in \Sigma_2$. Therefore Σ_2 is a topological space induced by the metric d and the basic open ball is a any subset of the full-2-shift of the form $X = X_w = \{\mathbf{s} \in \Sigma_2 | s_0 s_1 \cdots = w_0 w_1 \cdots w_{n-1} = w\}$ for any block (or sometime called word, i.e. a finite sequence) w of length n. We now define a continuous map on the full-2-shift. The shift map, $\sigma: \Sigma_2 \rightarrow \Sigma_2$ is defined as

$$\sigma(s_0 s_1 s_2 \cdots) = s_1 s_2 s_3 \cdots$$

The map simply shifts every element in Σ_2 one step to the left and deletes the first entry of the element (sequence).

Definition 8. A shift space $X \subset \Sigma_2$ is a shift of finite type (SFT) if there exists a finite number of blocks from symbols 0 and 1 such that the blocks do not occur in any element of X. The blocks are called forbidden blocks in X.

Golden Mean Shift is an example of SFT. This is the set of all binary sequences with no two consecutive 1's. It is mean that the only forbidden block is $\{11\}$.

2.1 1-Step Shift of Finite Type

A SFT is an M-step[10] (or have memory M, for some integer $M \geq 1$) if it can be described by a set of forbidden blocks all of which have length $M + 1$. Indeed Lind and Marcus show that[10] every SFT share some common dynamical property with 1-step SFT, which give every SFT a simple representation.

In this section we aim to describe all 1-step SFT over two symbols. To do so, we list all possible sets $\mathcal{F}_i$ of blocks of length two and then generate 1-step SFT defined by the forbidden blocks in each set $\mathcal{F}_i$. Since we only have four possible different blocks of length two i.e.

$\mathcal{F}_1 = \emptyset$	$\mathcal{F}_2 = \{00\}$	$\mathcal{F}_3 = \{01\}$	$\mathcal{F}_4 = \{10\}$
$\mathcal{F}_5 = \{11\}$	$\mathcal{F}_6 = \{00,01\}$	$\mathcal{F}_7 = \{00,10\}$	$\mathcal{F}_8 = \{00,11\}$
$\mathcal{F}_9 = \{01,10\}$	$\mathcal{F}_{10} = \{01,11\}$	$\mathcal{F}_{11} = \{10,11\}$	$\mathcal{F}_{12} = \{00,01,10\}$
$\mathcal{F}_{13} = \{00,01,11\}$	$\mathcal{F}_{14} = \{00,10,11\}$	$\mathcal{F}_{15} = \{01,10,11\}$	$\mathcal{F}_{15} = \{00,01,10,11\}$

00,01,10 and 11, then we have 16 set of forbidden blocks, as follows;

For each $i = \{1,2,\cdots,16\}$, $X_i \subset \Sigma_2$ is the 1-step SFT with set of forbidden blocks $\mathcal{F}_i$. However, there are some of them are singletons, empty set or the whole Σ_2, which has trivial dynamics and are not in our interest. There are also some of them that are equal, and some are topologically conjugate. We describe the elements in each 1-step SFT in the propositions;

Proposition 9. $X_1 = \Sigma_2$.
Proof. This obvious since X_1 does not have any forbidden block.¨

Proposition 10. $X_{13} = X_{14} = X_{16} = \emptyset$.
Proof. If $s \in X_{13}$ $s \in X_{13}$, then $s_0 = 1$ $s_0 = 1$ (since 00 and 01 are forbidden). Since 11 is forbidden then $s_1 = 0$ $s_1 = 0$. However 00 and 01 are forbidden. Therefore $s \notin X_{13}$ $s \notin X_{13}$ and $X_{13} = \emptyset$.

The same argument goes for X_{14}, and $X_{16} = \emptyset$ because every 2-block is forbidden. ¨

Proposition 11. X_6, X_{11}, X_{12} and X_{15} are singletons where $X_6 = X_{12} = \{\overline{111}\}$ and $X_{11} = X_{15} = \{\overline{000}\}$. Moreover X_m and X_n are topologically conjugate for $m = 6,12$ and $n = 11,15$.

Proof. Since 00 and 01 are forbidden, then for every $s \in X_6$, $s_i \neq 0$ for every $i \in \mathbb{N}$. Since 11 is allowed then, $s_i = 1$ for every $i \in \mathbb{N}$ and therefore $s = \overline{111}$.

We omitted the same proof for the other spaces. The topological conjugacy between X_6 and X_{11} can be shown by using conjugacy $h: X_6 \to X_{11}$ where $h(s) = t$ and $t_i \neq s_i$ for every i. ¨

Proposition 12. X_3 and X_4 are topoligically conjugate.
Proof. The topological conjugacy between X_3 and X_4 can be shown by using conjugacy, $h: X_3 \to X_4$ where $h(s) = t$ and $t_i \neq s_i$ for every i. ¨

Proposition 13. X_7 and X_{10} are set of two where $X_7 = \{\overline{111}, 0\overline{111}\}$ and $X_{10} = \{\overline{000}, 1\overline{000}\}$. Therefore X_7 and X_{10} are topoligically conjugate.
Proof. Let $s \in X_7$ If $s_0 = 1$, then $s_i = 1$ for all $i > 1$. Therefore $s = (\overline{111})$. If $s_0 = 0$, then $s_i = 1$ for all $i > 1$. Therefore $s = (0\overline{111})$. We omitted the same proof for X_{10}. The topological conjugacy can be shown by using the same conjugacy h in the previous propositions. ¨

Proposition 14. X_2 and X_5 are topoligically conjugate.

Proof. The topological conjugacy can be shown by using the same conjugacy h in the previous propositions.

Proposition 15. $X_8 = \{\overline{01}, \overline{10}\}$

Proof. For every $s \in X_8$, $s_i \neq s_{i+1}$ for every i. Therefore $X_8 = \{\overline{01}, \overline{10}\}$.

Proposition 16. $X_9 = \{\overline{00}, \overline{11}\}$

Proof. For every $s \in X_9$, $s_i = s_{i+1}$ for every i. Therefore $X_9 = \{\overline{00}, \overline{11}\}$.

From what we have proven in the above propositions, we can conclude that there are only six different non-empty 1-step SFT. There are shifts of finite type with set of forbidden blocks $\mathcal{F}_2 = \{00\}$, $\mathcal{F}_3 = \{01\}$, $\mathcal{F}_6 = \{00,01\}$, $\mathcal{F}_7 = \{00,10\}$, $\mathcal{F}_8 = \{00,11\}$, $\mathcal{F}_9 = \{01,10\}$. In the next section we will look at the dynamics of each of these shift spaces, $X_2, X_3, X_6, X_7, X_8,$ and X_9.

3. The Dynamics of 1-Step Shifts of Finite Type

In this section, we exhibit the dynamical property of six different 1-step SFT as mentioned in the previous section.

Theorem 17. The 1-step SFT $X_2 \subset \Sigma_2$ which has set of forbidden blocks $\mathcal{F}_2 = \{00\}$ is Devaney chaotic. Moreover, it is mixing, locally everywhere onto, totally transitive, and (strongly and weakly) blending.

Proof. It is sufficient to show that the periodic points of X_2 are dense, and X_2 is $l.e.o.$

Let $\varepsilon > 0$ and $s = (s_0\ s_1 s_2 \dots)$ be any point in X_2. Choose n such that $1/_{2^n} < \varepsilon$, now let $t = (t_0\ t_1 t_2 \dots)$ be another point such that $t_i = s_i$ for $i = 0,1,2,\dots,n$. Then $d(s,t) < 1/_{2^n}$, so in order for the set of periodic points to be dense in X_2 we need to construct a periodic point within ε of s. Let $t = (\overline{s_0\ s_1 s_2 \dots s_n 1})$. It is clear that t is periodic within ε of s. Hence the periodic points are dense in X_2.

To prove that X_2 $l.e.o.$ let U be any nonempty open ball in X_2, where $s = (s_0\ s_1 s_2 \dots s_n \dots) \in U$, then we have two cases; case 1: if $s_n = 1$, since (10) and (11) are allowed, then $\sigma^n(U) = X_2$. Case 2: if $s_n = 0$, since (00) is forbidden then $\forall s \in U,\ s_{n+1} = 1$ therefor $\sigma^{n+1}(U) = X_2$. Since for every open set $U \subseteq X_2$ there exists a positive integer n such that $\sigma^n(U) = X_2$, hence X_2 is $l.e.o.$

Since X_2 is $l.e.o.$ then it is transitive, topologically mixing, totally transitive, strongly blending, and weakly blending. Also since X_2 has dense periodic points and transitive, then it is SDIN, and hence X_2 is Devaney chaotic.

Therefore X_2 is Devaney chaotic, and satisfied all other chaotic properties.

Theorem 18. The 1-step SFT $X_3 \subset \Sigma_2$ which has set of forbidden blocks $\mathcal{F}_3 = \{01\}$ is neither transitive, has dense periodic points, posses SDIC nor weakly blending.

Proof. It is sufficient to show that X_3 is neither has dense periodic points, transitive, weakly belnding nor posses SDIN.

Since 01 is not allowed then X_3 does not contain any periodic point. Therefore the periodic points are not dense. To show that X_3 is not transitive, consider two open balls X_{00} and X_{11}. Since 01 is not allowed, then for any point $s \in X_{00}$, $s_i \neq 1$ for any integer i. Therefore $\sigma^n(s) \notin X_{11}$ for all integer n. To show that X_3 is not weakly blending, we use the same open balls X_{00} and X_{11}. By the same argument, $\sigma^n(X_{00}) \cap \sigma^n(X_{11}) = \emptyset$ for any integer n. To see that X_3 is not SDIC, let $s = (\overline{00})$ and $t = (1\overline{00})$. Now $d(s,t) = 1$, $\sigma^n(s) = (\overline{00})$ and $\sigma^n(t) = (\overline{00})$, therefore $d(\sigma^n(s), \sigma^n(t)) = 0$. Hence there exists no $\delta > 0$ such that $(d(\sigma^n(s), \sigma^n(t)) > \delta$, so X_3 is not SDIC.

Therefore X_3 is neither transitive, has dense periodic points, posses sensitive dependence on initial conditions nor weakly blending.

Theorem 19. The 1-step SFT $X_7 \subset \Sigma_2$ which has set of forbidden blocks $\mathcal{F}_7 = \{00,10\}$ is neither transitive, has dense periodic points nor posses SDIC. However it is (strongly and weakly) blending.

Proof. It is sufficient to show that X_7 is strongly blending but neither has dense periodic points, transitive, nor SDIN.

From Proposition 13, $X_7 = \{\overline{111}, 0\overline{111}\}$, then the only open sets are $X_7, \emptyset, \{\overline{111}\}$ and $\{0\overline{111}\}$.

Since, $\sigma(\overline{111}) \cap \sigma(0\overline{111}) = \{\overline{111}\}$ then X_7 is strongly blending. The periodic points is not dense since $\{0\overline{111}\}$ does not contain any periodic points.

Since $0\overline{111} \notin \sigma^n(\overline{111})$ for all integer n, then X_7 is not transitive. Now if we Let $s = (0\overline{11})$, $t = (\overline{11})$ then $d(s,t) = 1$, $\sigma^n(s) = (\overline{11})$, $\sigma^n(t) = (\overline{11})$, therefore $d(\sigma^n(s), \sigma^n(t)) = 0$. Hence there exists no $\delta > 0$ such that $d(\sigma^n(s), \sigma^n(t)) > \delta$, so X_7 is not SDIC.

Therefore $X_7 X_7$ is (strongly and weakly) blending but is neither transitive, has dense periodic points nor posses SDIC.

Theorem 20. The 1-step SFT $X_8 \subset \Sigma_2$ which has set of forbidden blocks $\mathcal{F}_8 = \{00,11\}$ is Devaney chaotic. However it is not $l.e.o$, mixing, totally transitive, nor (weakly and strongly) blending.

Proof. It is sufficient to show that the periodic points of X_8 are dense, and X_8 is transitive but not mixing, totally transitive nor weakly blending.

By Proposition 15, $X_8 = \{\overline{01}, \overline{10}\}$. Therefore the only open sets in X_8 are $X_8, \emptyset, \{\overline{01}\}$ and $\{\overline{10}\}$. Since every point of $X_8 = \{\overline{01}, \overline{10}\}$ is periodic point, then the periodic points of X_8 are dense. To prove that X_8 is transitive let $U = \{\overline{01}\}$ and $V = \{\overline{10}\}$. Then for $n > 1$, it is either $\sigma^n(U) = U$ or $\sigma^n(U) = V$. If $\sigma^n(U) = U$, then $\sigma^{n+1}(U) = V$, and $\sigma^{n+1}(U) \cap V \neq \emptyset$, also if $\sigma^n(U) = V$, then $\sigma^n(U) \cap V \neq \emptyset$. So X_8 is transitive, and hence X_8 X_8 is Devaney chaotic.

If we let $r = 2$ and $U = \{\overline{01}\}$, $V = \{\overline{10}\}$. Then $(\sigma^2)^n(U) = \{\overline{01}\}$, for $n > 0$. So $(\sigma^2)^n(U) \cap V = \emptyset$ which means that σ^n is not transitive and hence X_8 is not totally transitive. Since X_8 is not totally transitive, then it is not $l.e.o.$ Now since either

Table 1. The Dynamics of 1-step shift of finite type

Shift space	Dense periodic point	Transitive	SDIN	l.e.o	Mixing	Totally transitive	Strongly blending	Weakly blending
X_2	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
X_3	No	No	No	No	No	No	No	No
X_6	No	No	No	No	No	No	No	No
X_7	No	No	No	No	No	No	Yes	Yes
X_8	Yes	Yes	Yes	No	No	No	No	No
X_9	Yes	No	Yes	No	No	No	No	No

$\sigma^n(U) \cap V = \emptyset$, for n is even or $\sigma^n(U) \cap V \neq \emptyset$, for n is odd., then there is no $N > 0$ such that $\sigma^n(U) \cap V \neq \emptyset$, for each $n > N$, hence X_8 is not topologically mixing. Finally let $U = \{01\}$ and $V = \{10\}$. Then for every $n > 1, n > 1,$ it is either $\sigma^n(U) = U$ and $\sigma^n(V) = V$ or $\sigma^n(U) = V$ and $\sigma^n(V) = U.$ So for any case $\sigma^n(U) \cap \sigma^n(V) = \emptyset$, hence X_8 is not weakly blending, and then is not strongly blending. Therefore X_8 is Devaney chaotic but is neither $l.e.o$, mixing, totally transitive, nor (weakly and strongly) blending. ¨

Theorem 21. The 1-step SFT $X_9 \subset \Sigma_2$ which has set of forbidden blocks $\mathcal{F}_9 = \{01, 10\}$ is not Devaney chaotic since it is not transitive. However the periodic points of X_9 are dense, and it posses SDIC. X_9 is also not l.e.o., mixing, totally transitive nor (weakly and strongly) blending.

Proof. It is sufficient to show that the periodic points of X_9 are dense, and X_9 has SDIC, but neither transitive nor weakly blending.

By Proposition 16, $X_9 = \{\overline{00}, \overline{11}\}$. Therefore the only open sets in X_9 are $X_9, \emptyset, \{\overline{00}\}$ and $\{\overline{11}\}$. Since every point of X_9 is periodic, then the periodic points of X_9 are dense. Also it is not transitive, for that let $U = \{\overline{00}\}$ and $V = \{\overline{11}\}$, then $\sigma^n(U) = \{\overline{00}\}$, and $\sigma^n(U) \cap V = \emptyset$, for each $n > 0$. Hence X_9 is not transitive, and then is neither $l.e.o$, topologically mixing, nor totally transitive. Now take $c = 1$, and let $s = (\overline{00})$, $t = (\overline{11})$. Then for all $n > 0$ we have $\sigma^n(s) = s$, $\sigma^n(t) = t$, and $d(\sigma^n(s), \sigma^n(t)) \geq 1$. Hence X_9 is SDIN.

Finally let $U = \{\overline{00}\}$ and $V = \{\overline{11}\}$. Since $\sigma^n(U) = U$, and $\sigma^n(V) = V$, then $\sigma^n(U) \cap \sigma^n(V) = \emptyset$, for each n > 0. Hence X_9 is neither weakly blending nor strongly blending.

Therefore X_9 has dense periodic points and SDIC, but is neither Devaney chaotic, totally transitive, $l.e.o$, topologically mixing nor blending.¨

We list the results we shown from Theorem 17 to Theorem 21 above in Table 1.

4. Conclusion

Refer to the Table 1, out of six spaces, there are only two spaces that satisfied the properties of Devaney Chaos. They are X_2 and X_8. Although these two spaces are both Devaney Chaotic but they exhibit totally different dynamical behaviour. X_2 is not just Devaney chaotic but also satisfied all others chaotic properties, but on the other hand the dynamics of X_8 is trivial since it contains only two points and does not satisfy any of highlighted chaotic properties. This is interesting because Devaney chaos does not mean that the system has a complicated behaviour as we expected. The other four spaces are not Devaney chaotic because they are all not transitive. The dynamics of X_3 and X_6 are similar as they both do not satisfy other chaos properties. However, it is interesting that non-chaotic (in sense of Devaney) X_7 satisfies one of the chaos properties i.e. weakly and strongly blending. Therefore we suggest that blending property is insignificant and meaningless for chaos notion.

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