

# Formulating the Torsion Strength of Dry-Stacked Stone Blocks by Comparing Convex and Concave Contact Formulations and Experimental Results

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## Abstract

**Objectives:** Objective of this work is to investigate the reliability of the two formulations most commonly used in modeling the geometric contact of dry-stacked stone blocks, i.e., those based on concave and convex contact assumptions. **Methods:** A comparison between these two formulations for the yield domains of torsion and torsion-shear interaction is carried out; the analytical results are then compared with experimental outcomes existing in the literature and difference percentages are evaluated in terms of shear forces. For both formulations, the reference model of two dry-stacked rigid blocks is adopted with the hypotheses of infinite compressive strength for blocks, absence of tensile strength and frictional behaviour at their contact. **Findings:** The analysis underlines that the convex formulation provides more reliable results in terms of both pure torsion and interaction between torsion and shear. In fact, the percentage difference between experimental and analytical shear forces results to be very small in this case, while a great difference is estimated for concave formulation, both for pure torsion and torsion-shear interaction. Then a possible correction of the torsion capacity in the concavity model is proposed assuming a proper reduction of the contact area. This simple criterion allows obtaining a good agreement of the yield domains with those obtained by convexity model and experimental results. **Application/Improvements:** With the proposed correction the simplicity of the algorithms of the concave formulation can find interesting applications in rigid block limit analysis, especially for non-associative friction solutions of 3D masonry block assemblages, thanks to the very low computational effort compared with all the other existing solution procedures. Additional experimental work is being carried out to fully validate such correction and to investigate its effect on the torsion-bending moment interaction.

**Keywords:** Contact Formulations, Dry-Stacked Stone Blocks, Experimental Yield Domains, Torsion Capacity, Torsion-Shear Interaction

## 1. Introduction

Ancient structures are popularly made of masonry-like material which is an inherently discontinuous material formed by stones/ bricks and mortar. In any numerical modelling approach for structural assessment purposes, the mechanical behaviour of materials and components needs to be elaborated.

Assuming stones and/or bricks as rigid blocks, dry frictional contact among them is a reasonable simplification for analyzing ancient structures that do not have mortar joints or joints that are too weak to significantly

contribute to the structural behaviour. In this case, a reliable mechanical characterization of the frictional contact interfaces requires the definition of the yield domains related to the possible loading conditions taking into account the interaction effects between shear forces, torsion and bending moments. These aspects, poorly studied<sup>1-4</sup> and only recently systematically addressed from both experimental and analytical points of view<sup>5,6</sup>, play a significant role when adopting the discrete element method to simulate the in-plane and out-of-plane behaviour of masonry walls constructed with dry joints<sup>7</sup>.

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Discrete element models generally represent the interface between rigid blocks through a number of point contacts<sup>8</sup>. At each point, the resultant of the interaction stresses has normal and shear components and, depending on the type of model, it can be associated to moment components. The degrees of freedom of each point contact regard the relative displacements between two blocks since the hypothesis of rigid-body elements implies that cracking, crushing or deformation of them are ignored. Instead, some contact rules act as the yield conditions and the failure modes of point friction contacts are assumed to have a rigid perfectly-plastic behaviour<sup>9-11</sup>.

In this framework, one of the aspects which still create some challenging analysis problems yet to be fully resolved is related to the modelling of such assemblages to suit the assumptions of the geometric contacts. In particular, a great difference in torsion capacity, even within the single contact surface, was found in three-dimensional responses of dry assemblages of rigid blocks when adopting the two different models most commonly used, i.e., those based on concave and convex contact formulations<sup>4</sup>. Analogous differences are also expected to be found when torsion moment interacts with shear forces and bending moments.

This paper addresses these crucial aspects first by investigating the main differences between the two contact formulations for the torsion and torsion-shear interaction and then by comparing the analytical results with some experimental outcomes recently published<sup>5,6</sup>. Lastly, a simple correction of the torsion capacity in the concavity model, also affecting the torsion-shear interaction, is proposed in order to obtain a good agreement of the yield domains with those obtained by convexity model and experimental results.

## 2. Analytical Contact Formulations: Concavity Vs Convexity Model

According to discrete element modelling, it is possible to describe masonry walls as systems of rigid blocks interacting through dry frictional interfaces (Figure 1 (a)).

When adopting a convex contact formulation, the stress resultant vector characterizing the interaction between rigid blocks is applied at a single point coincident with the centre of the interface, as sketched in Figure 1(b) with reference to a rectangular contact surface. The six

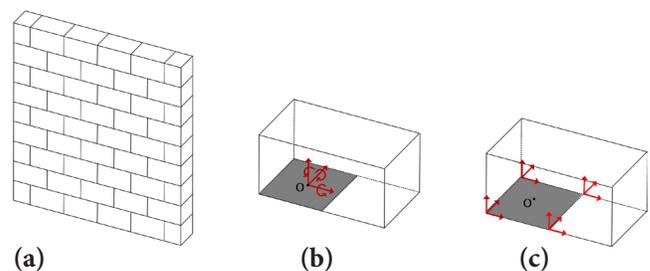
components of this resultant vector (normal and shear forces as well as bending and torsion moments) correspond to the internal degrees of freedom of the contact interface in a virtual work sense. In particular, the bending and torsion components take into account interactions along the entire surface<sup>12</sup>.

According to the concave contact formulation, the interaction between blocks with rectangular interface develops through its four vertexes (Figure 1(c)), as if the two surfaces in contact were slightly concave. The three components of the stress resultant related to each point friction contact are the normal force and two shear force components. In the angular direction, interaction of these elements represents the torsion strength on the interface. However, the geometric arrangement of the point contacts does affect the torsion strength, and also may cause different twisting centres on the interface if shear components and/or normal force are not uniformly distributed along the surface.

In order to critically compare the results from both formulations, the construction of the yield domains in terms of pure torsion and torsion-shear interaction are developed in the following sections, followed by their comparison with experimental outcomes existing in the literature.

### 2.1 Pure Torsion Strength

In the convexity block assemblage model, the pure torsion capacity clearly depends on the geometry of the contact interface. According to the formulation developed by Casapulla<sup>1</sup> and followed by others<sup>3-5,13</sup>, the centre of plastic torsion is coincident with the centroid of the interface and the shear stress vectors at each point of the contact interface are orthogonal to their distance from the centre of torsion and parallel to the relative tangential flows (Coulomb's law). On these assumptions, the torsion moment strength is (Figure 2(a)):



**Figure 1.** (a) 3D masonry blocks assemblage. (b) Convex contact model. (c) Concave contact model.

$$\hat{M}_{T0} = V_0 c_T \tag{1}$$

where the symbol  $\hat{\cap}$  denotes convexity model,  $V_0 = \mu N$  is the shear strength,  $N$  is the normal force assumed uniformly distributed over the contact surface and  $c_T$  is the parameter of the torsion moment capacity expressed as follows:

$$c_T = \frac{1}{12ab} \left[ a^3 \ln \frac{b + \sqrt{a^2 + b^2}}{a} + b^3 \ln \frac{a + \sqrt{a^2 + b^2}}{b} + 2 ab \sqrt{a^2 + b^2} \right] \tag{2}$$

Besides, in the concavity block assemblage model with four point contacts, the torsion strength of the interface strongly depends on the distances between the opposite points and it can be calculated from the combination of moments about the centroid, caused by the sliding resistances at the point contacts. The normal force without any eccentricity from the centroid of the contact interface can be distributed evenly to the number of point contacts involved in sliding. So, with the assumption of the normal force concentrated at the four vertexes of the contact interface (Figure 2(b)), the torsion capacity can be expressed as:

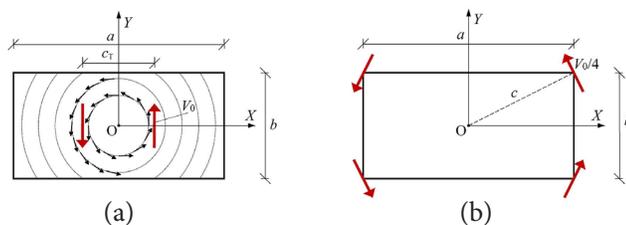
$$\hat{\cup} M_{T0} = V_0 c \tag{3}$$

where the symbol  $\hat{\cup}$  denotes concavity model and  $c$  is the half diagonal of the interface, i.e.:

$$c = \sqrt{(a/2)^2 + (b/2)^2} \tag{4}$$

By comparing the convex and concave contact formulations for pure torsion strength, according to Equations (1) and (3), respectively, it is evident that the result from the concavity model is always higher than that from the convexity assumption, as it is always  $c > c_T$ .

As an example, consider  $b = 0.3\text{m}$ ,  $N = 1\text{kN}$  and  $\mu = 0.7$ . It is evident from Table 1 that the torsion capacity calculated by concave formulation is almost twice the amount



**Figure 2.** Shear stresses and/or forces over the: (a) Convex and (b) Concave contact interface under axial compression and pure torsion moment.

of that by convex formulation, whatever the interface ratio. The difference between them can be estimated as an average percentage of about 88%.

### 2.2 Torsion-Shear Interaction

When the shear force  $V$  ( $V_x, V_y$ ) is applied with some eccentricity  $e$  ( $e_x, e_y$ ) with respect to the centroid  $O$  of the contact interface (Figure 3), its interaction effect with the torsion moment  $M_T$  ( $M_T = Ve = V_x e_y + V_y e_x$ ) must be accounted for. In this case, the convexity model provides highly non-linear relationships among the variables  $V_x, V_y, M_T$  and the coordinates of the torsion centre  $C$  ( $x_c, y_c$ ). Still assuming that the shear stress resultants at each vertex of the interface are orthogonal to their distance from  $C$  (Figure 3(a)), the following relationships define the torsion-shear interaction<sup>1,5</sup>:

$$\hat{\cap} V_x = \tau_0 \int_{x_c}^{x_c+a} dx \int_{y_c}^{y_c+b} \frac{y}{\sqrt{x^2 + y^2}} dy; \quad \hat{\cap} V_y = \tau_0 \int_{y_c}^{y_c+b} dy \int_{x_c}^{x_c+a} \frac{x}{\sqrt{x^2 + y^2}} dx \tag{5}$$

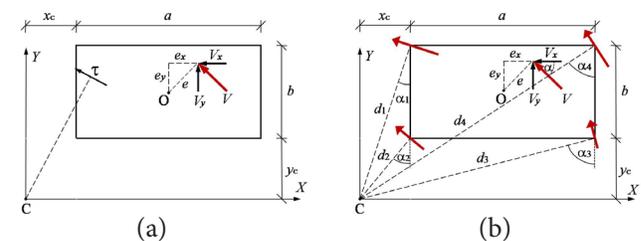
$$\hat{\cap} M_T = \tau_0 \int_{y_c}^{y_c+b} dy \int_{x_c}^{x_c+a} \sqrt{x^2 + y^2} dx - \left[ \hat{\cap} V_x \left( y_c + \frac{b}{2} \right) + \hat{\cap} V_y \left( x_c + \frac{a}{2} \right) \right]$$

where,  $\tau_0 = \mu \sigma$  is the limit shear stress, being  $\sigma$  the normal stress corresponding to a uniform distribution of the normal force  $N$ .

Similarly to the convexity model, the concavity model also assumes that the shear strengths at each point friction

**Table 1.** Torsion capacity by concave and convex formulations vs. interface ratio

$a/b$	$\hat{\cap} M_{T0}$ [kNm] Eq. (1)	$\hat{\cup} M_{T0}$ [kNm] Eq. (3)	%diff
0.25	55.69	108.23	94.34
0.5	62.29	117.39	88.46
1	80.35	148.49	84.82
1.5	101.65	189.29	86.22
2	124.58	234.79	88.46



**Figure 3.** Torsion-shear interaction for: (a) Convex and (b) Concave contact interface.

contact are orthogonal to their distance from C, but these strengths are concentrated in four vertexes whose distances from C are denoted as  $d_i$ , with  $i = 1$  to 4 (Figure 3). Thus the following relations can be derived:

$$\begin{aligned} V_x &= V \cos \alpha = \frac{V_0}{4} \left( \sum_{i=1}^4 \cos \alpha_i \right); & V_y &= V \sin \alpha = \frac{V_0}{4} \left( \sum_{i=1}^4 \sin \alpha_i \right) \\ M_T &= \frac{V_0}{4} \left[ b(\cos \alpha_1 + \cos \alpha_4) + a(\sin \alpha_1 + \sin \alpha_4) + x_c \sum_{i=1}^4 \sin \alpha_i + y_c \sum_{i=1}^4 \cos \alpha_i \right] - \left[ V_x \left( y_c + \frac{b}{2} \right) + V_y \left( x_c + \frac{a}{2} \right) \right] \end{aligned} \quad (6)$$

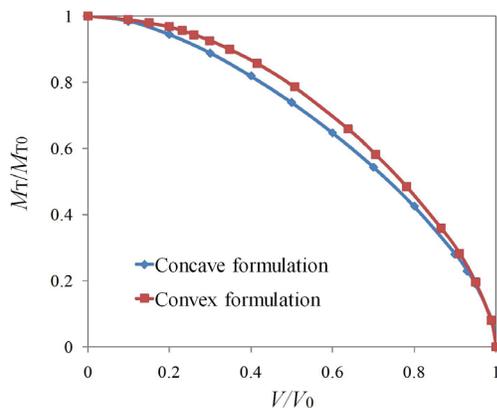
where,  $\cos \alpha_i$  and  $\sin \alpha_i$  are obtainable from geometric relations as functions of the coordinates  $(x_c, y_c)$  of the torsion centre (Figure 3(b)). Assuming  $V_x = 0$ , the torsion centre is on X-axis ( $\alpha = 0$ ) and the following relations result:

$$\begin{aligned} \sin \alpha_1 &= \sin \alpha_2; & \sin \alpha_3 &= \sin \alpha_4; & \cos \alpha_1 &= -\cos \alpha_2 = b/2d_1; & -\cos \alpha_3 &= \cos \alpha_4 = b/2d_4; \\ d_1 &= \sqrt{x_c^2 + (b/2)^2}; & d_4 &= \sqrt{(a+x_c)^2 + (b/2)^2} \end{aligned} \quad (7)$$

Therefore Equation (6) can be expressed as:

$$\begin{aligned} V_y &= V = \frac{V_0}{2} (\sin \alpha_1 + \sin \alpha_4) \\ M_T &= \left[ \frac{b^2(d_1 + d_4) + 4d_1a(a + x_c)}{8d_1d_4} \right] V_0 - \frac{a}{2} V \end{aligned} \quad (8)$$

Equation (5), (6) and (8) clearly depend on the block dimensions and provide a series of interaction curves for different interface aspect ratios. In Figure 4 the curves related to the two formulations for  $V_x = 0$  are represented in non-dimensional form ( $M_T/M_{T0}$  vs.  $V/V_0$ ). It is worth mentioning that some possible approximations of the non-dimensional yield domain related to the convex formulation, regardless of the interface aspect ratio, were proposed in the literature as bi-linear<sup>5</sup> or three-linear<sup>3</sup> piecewise functions, while a more conservative



**Figure 4.** Non-dimensional torsion-shear interaction curves for the two formulations.

approximation could be represented by the parabolic function. However, the comparison between the two non-dimensional yield domains in Figure 4 reveals a very good agreement between the two formulations, meaning that the discrepancy described above only affects the pure torsion capacity. This also means that some difference can be reported if the dimensional relations  $M_T$  vs.  $V$  are taken into account; in particular the difference between the two formulations increases as the shear force decreases and it is maximum in case of pure torsion, as better described in the following section.

### 3. Comparison between Analytical and Experimental Results

In order to investigate the discrepancy observed above in the results from the two contact formulations for the yield domains of pure torsion and, consequently, of torsion-shear interaction, some results of the experimental campaign recently carried out by Casapulla and Portioli<sup>5,6</sup> are here considered for comparison. In these works the frictional contact behaviour of two dry-stacked tuff blocks was investigated through experimental tests considering different loading conditions (pure shear, pure torsion and interactions among shear, torsion and bending moments). The block dimensions used for experiments were 0.3 m × 0.2 m × 0.1 m and the rectangular contact area (bed joint) had dimensions 0.3 m × 0.2 m ( $a \times b$  in Figure 3). For each test, the horizontal shear forces were recorded as the effect of monotonically increasing displacements applied at a constant rate.

In this paper, only four sets of the tests, two for pure torsion and two for torsion-shear interaction, were considered (Table 2), in order to be compared with the results from the convex and concave formulations. For these selected cases, the normal force of 467 N was assumed centrally applied on the bed joint, while the horizontal loading was represented by a couple of shear forces  $V$  (pure torsion) or a single shear force  $V$  (torsion-shear interaction) applied at variable eccentricity in the mid-plane of the upper block and in Y-direction.

In particular, for pure torsion, a pair of horizontal forces in Y-direction (couple) was applied at symmetric points of the upper block with two different level arms, i.e.,  $0.9a$  for Set 2a and  $0.5a$  for Set 2b (the same original name of the sets is here used). The torsion strength  $M_{T0}$  was simply obtained as the moment corresponding to the

applied couple of forces  $V$ . Concerning the experimental testing for the interaction between torsion and shear, a single horizontal force was applied at two different distances from the centroid of the blocks, i.e.,  $0.25a$  (Set 4a) and  $0.45a$  (Set 4b)).

In Table 3 the experimental results for the considered sets of tests were reported together with those derived by the analytical models presented above, in terms of limiting shear force  $V$ . From comparison it is evident that the convex formulation provides more reliable results in terms of both torsion and torsion-shear interaction. In fact, the analytical values of the shear forces are very close to the experimental ones with a percentage difference of about 6% (absolute value); considering the concave formulation, instead, the percentage difference with the experimental results is up to 85% for pure torsion and to 28% for torsion-shear interaction.

Despite these results, it should be remarked that the great advantage of using concave formulation in rigid block limit analysis, especially for non-associative friction solutions of 3D masonry block assemblages, lies on the

simplicity of the algorithms and, as a consequence, on the very low associated computational effort compared with all the other existing solution procedures<sup>13</sup>. Therefore, the great difference in torsion capacity should be treated and further investigated, also taking into account the lacking of studies on this topic. To this purpose, a simple correction of the concave formulation is originally proposed in the following section and validated through the comparison with the convex formulation and the experimental results.

### 4. Proposed Yield Domains by Concave Formulation

As remarked in the previous section, the torsion and torsion-shear yield domains obtained by concave formulation appear to be less reliable in comparison with those by convex formulation, but concave formulation allows enhancing the computational efficiency when it is used in micro-modelling approaches for the limit analysis of masonry walls.

In order to keep the simplification allowed by this latter formulation, possible corrections of the torsion capacity in the concavity model can be defined. One way could be the introduction of partial efficiency factors representing the average conditions of strength while another way could be the definition of a reduced contact area for the concave model, corresponding to the same torsion capacity obtained by convex formulation. The latter solution is here chosen and, keeping the rectangular surface, the dimensions of this reduced area are:

$$a_1 = \frac{c_T}{c} a; \quad b_1 = \frac{c_T}{c} b \tag{9}$$

With these dimensions it results:

$$M_{T0}^{\cup} = V_0 c_1 = M_{T0}^{\cap} = V_0 c_T \tag{10}$$

where  $c_1$  is the half diagonal of the reduced interface, i.e.:

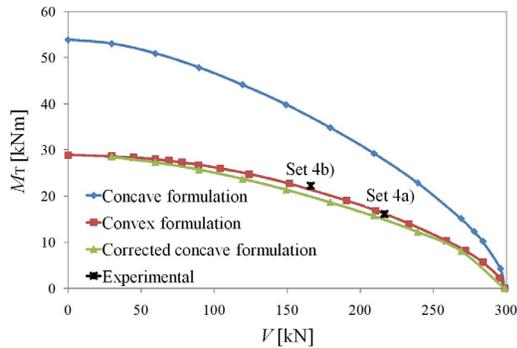
$$c_1 = \sqrt{(a_1 / 2)^2 + (b_1 / 2)^2} \tag{11}$$

**Table 2.** Layouts of the experimental tests for pure torsion and torsion-shear interaction

Yield	Set	
	Set 2a)	Set 2b)
Pure torsion		
Torsion-shear	Set 4a)	Set 4b)

**Table 3.** Experimental results for pure torsion and torsion-shear interaction

Yield	Set	Exp V (N)	Convex formulation V (N)	Concave formulation V (N)	Diff. Convex vs. exp. (%)	Diff. Concave vs. exp. (%)
Pure Torsion	2a)	108	107	200	-1	85
	2b)	205	193	359	-6	75
Torsion-shear	4a)	216	216	254	0	18
	4b)	166	162	212	-2	28



**Figure 5.** Comparison between dimensional yield domains and experimental results in the case of torsion-shear interaction.

As expected, this criterion has also effects on the torsion-shear interaction when it is represented in its dimensional form. In fact, Figure 5 shows that the curve of the corrected concave formulation is in very good agreement with that derived from the convex formulation, and also slightly conservative with reference to the experimental results. On the other hand, this figure also highlights that the difference between the convex and concave formulations without corrections increases as the shear force decreases, while it is null in case of pure shear and maximum in case of pure torsion.

The proposed corrected model appears to be more reliable than the concave model, but, in order to be fully validated, further experimental work is required and the effect of the torsion capacity on the bending moments should also be investigated.

## 5. Conclusions

Limit analysis of masonry walls based on discrete element approaches requires modelling the frictional contact at block interfaces, taking into account the interactions of shear forces, torsion and bending moments. The definition of the yield domains related to such interactions was systematically addressed from both experimental and analytical points of view in recent years, but some uncertainties still arise when modelling the geometric contacts by different contact formulations, e.g., convexity vs. concavity models. In fact, a great difference in torsion capacity and, consequently, in the interactions between torsion and shear or torsion and bending moment can be obtained by using such two formulations.

Since the main difference between the pure torsion capacities was estimated as an average percentage of

about 88%, a useful comparison of these results with some experimental outcomes existing in the literature was also carried out in this paper. Displacement controlled tests were performed on specimens made up with two dry-stacked tuff blocks. Therefore increasing displacements were applied at constant rate to the upper block and force-displacement relations were obtained. Only experimental results on pure torsion and torsion-shear interaction were considered in this paper for comparison.

This study underlined that the convex formulation provides more reliable results in terms of both torsion and interaction between torsion and shear. In fact, the analytical values of the shear forces are very close to the experimental ones with a percentage difference of about 6% (absolute value); considering the concave formulation, instead, the percentage difference with the experimental results is up to 85% for pure torsion and to 28% for torsion-shear interaction.

However, in order to keep the simplification allowed by the concave formulation when it is used in micro-modelling approaches for the limit analysis of 3D rigid block systems, a possible correction of the torsion capacity in such a model was proposed. This was simply based on the reduction of the contact area in the concavity model providing the same torsion strength obtained by the convexity model. This simple correction allowed obtaining a good agreement of the yield domains with those obtained by convexity model and experimental results. Additional experimental work is being carried out to fully validate such correction and to investigate its effect on the torsion-bending moment interaction.

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