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*Corresponding author.
Muhammad Aamir Javed
Department of Mathematics, GC University, Katchery Road, 54000, Lahore, Pakistan
aamirmath@live.com

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Properties of circuits in coset diagrams by modular group

Muhammad Aamir Javed[1]*, Muhammad Aslam Malik[2]

1 Department of Mathematics, GC University, Katchery Road, 54000, Lahore, Pakistan
2 Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, 54590, Lahore, Pakistan

Abstract

Background/Objectives: Graham Higman gave the idea of coset diagrams for the action of modular group $PSL(2,Z)$ on real quadratic irrationals. These special types of graphical figures are composed of closed paths known as Circuits. These circuits can be classified into certain types of even length with respect to the number of inside\outside triangles. This study is to discuss different properties of reduced numbers in coset diagrams of the type (p,q). **Methods:** In this study, we have investigated different properties of type $(p,\ q)$ using reduced quadratic irrationals and continued fractions. We have categorized reduced numbers in accordance with their position in the real line. Distance between two ambiguous numbers and reduced numbers is introduced in this article which will help the reader to understand the structural significance of reduced numbers in a circuit. We have explored different conditions under which certain reduced numbers have the same circuit. Moreover, continued fractions have been used to assist the foundation laid by modular group action and different general results have been derived in this context. **Findings:** It was possible to define new notions of equivalent, cyclically equivalent and similar circuits using partitions of n and discuss various properties of reduced numbers included in coset diagrams of circuits with length up to four.
Applications: This study helps us in classifying PSL (2, Z)-orbits of $Q(\sqrt{m})\backslash Q = \bigcup_{k\in N} Q^*(\sqrt{k^2m})$, where $Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : \left(a,\ \frac{a^2-n}{c},\ c\right) = 1\}$.

Keywords: Modular group; Coset diagram; reduced numbers; equivalence classes.

1 Introduction

Our universe is full of unexplored beauties of nature which hide different symmetries and patterns in it. Scientists are continuously defining new bounds to the existing knowledge of numbers and figures and their relationships. Graphical methods were first used in the theory of groups in [(1)]. Graphical approach is considered as an outstanding way of visualizing any abstract idea. Modular group also known as $PSL(2,\ Z)$ is an eminent group which is generated by two linear fractional transformations $x : r \to -1/r$ and $y : r \to 1-1/r$ and satisfy the relations $x^2 = y^3 = 1$. A real quadratic irrational number is of the form $\chi+\phi\sqrt{n}$ where $\chi\neq 0$ and $Q(\sqrt{n})=\{\chi+\phi\sqrt{n}\,;\,\chi,\phi\in Q\}$ is known as real quadratic field. Graphical representation of action of modular group is known as coset diagram and it was introduced by[2]. A number $\xi=(\chi+\phi\sqrt{n})/\vartheta$ is said to be an ambiguous number if $\xi\bar{\xi}=-1$. It is proved in[3] and[4] that there are only finite number of such ambiguous numbers for a particular n and these numbers are connected on a unique closed path known as circuit and that each orbit contains a unique circuit, thus the number of disjoint orbits α^G where $\alpha\in Q^*(\sqrt{n})$ is equal to the number of closed paths in the coset diagram under the action of−. This action of modular group on different subsets has been discussed by Malik in[5] and calculated the exact cardinality of ambiguous numbers in a coset diagram. In 2018, Malik & Sajjad in[6] and[7] discussed different types of even lengths in coset diagram. Distinct homomorphic images are obtained by contraction of vertices in coset diagram in[8]. Reader may look[9] a book on number theory for in depth knowledge of the work. Reduced number is defined as a real quadratic irrational number ξ with $\xi>1$

and $-1<\bar{\xi}<0$. A circuit has type $(l_1,l_2,l_3,\ldots,l_{2p})$ if there are l_1 outside triangles and l_2 inside triangles in the circuit and so on l_{2p} triangles inside the circuit. We use reduced numbers as initial points in the construction of a circuit in the coset diagram which later help us to accurately decide the type of that circuit. We use above mentioned technique to make coset diagrams and corresponding circuits of length two and four that is also in line with repeated part of continued fraction expression of a reduced number. It is pertinent to mention here that the length of a G-circuit in coset diagram is a different phenomenon to the length of a path in a graph. Transitive G-subsets (G-orbits) are fundamental to the study of G-subsets, thus classification of G-orbits is actually the classification of circuits. In this paper, an attempt has been made to attack this long standing problem and to address in detail all the circuits of length four. It is easy to see that there are precisely two classes of equivalent circuits of length 2 namely $[l_1,l_1]$ and $[l_1,l_2]$. Throughout this paper, a circuit means a G-circuit unless stated otherwise. We have developed results about the relationship of type of circuit and quadratic irrationals. Throughout this paper, a closed circuit of type $(p,\ q)$ means that there are p number of triangles outside the circuit and q number of triangles inside the circuit. Moreover the behaviors of different ambiguous numbers and reduced numbers, in this particular type (p,q) will be discussed as real numbers and in the context of continued fractions.

2 Continued fractions and the type (p, q)

A circuit of type (p,q) is generated by reduced number $\xi=(pq+\sqrt{p^2q^2+4pq})/2p$ which is in fact fixed by $(xy^2)^p(xy)^q$ as every outside triangle is actually due to the transformation xy^2 and every inside triangle is due to the transformation xy moving in counter clockwise direction. It can be verified easily by definitions of x and y that $\xi(xy^2)^p=\xi/(p\xi+1)$ and $\xi(xy)^q=\xi+q$. Thus reduced number in the circuit of type (p,q) is determined by working out $\xi(xy^2)^p(xy)^q=\xi$. Continued fractions are used to authenticate the type of a circuit using ξ as recurrent part of continued fraction of ξ is every time $\overline{pq}$, that can be checked from[10].

We now divide all the circuits of type (p,q) into three genres.

2.1 Genre A

Let p and q be different positive integers then all circuits of type (p,q) in $Q^*(\sqrt{p^2q^2+4pq})$ are said to be of genre A, we say $\xi_1=\frac{pq+\sqrt{p^2q^2+4pq}}{2p}$ which is the reduced number of this genre.

2.2 Genre B

Let p and q be different positive integers then all circuits of type (q,p) in $Q^*(\sqrt{p^2q^2+4pq})$ are said to be of genre B, we say $\xi_2=\frac{pq+\sqrt{p^2q^2+4pq}}{2q}$ which is the reduced number of this genre.

2.3 Genre C

Let p be a positive integer then all the circuits of type (p,p) in $Q^*(\sqrt{p^2+4})$ are said to be of genre C, we say $\xi_3=\frac{p+\sqrt{p^2+4}}{2}$ which is the reduced number of this genre.

2.4 Ambiguous distance

The Shortest number of edges linking two ambiguous numbers, say a and b, in a circuit is called ambiguous distance which is denoted by $d(a,b)$.

It has been able to establish the connection between reduced numbers of above three genres.

Theorem 2.5. Let ξ_1 and ξ_2 be reduced numbers of genre A and B then $-\bar{\xi}_1$and $-\bar{\xi}_2$ occur in the same circuit. Mathematically,

$$(\xi_1)^G = (-\bar{\xi}_1)^G \tag{1}$$

$$(\xi_2)^G = (-\bar{\xi}_2)^G \tag{2}$$

Proof: We move p outside triangles from ξ_1 in anticlockwise direction. Consider $\xi_1(xy^2)^p = \frac{\xi_1}{p\xi_1+1} = \frac{pq-\sqrt{p^2q^2+4pq}}{-2p} = -\bar{\xi}_1$ Hence the result.

Similar can be done for ξ_2.

Theorem 2.6. Ambiguousdistance between ξ_1 and $\bar{\xi}_2$ as well as ξ_2 and $\bar{\xi}_1$ is one. Precisely,

$$(\xi_1)x = \bar{\xi}_2 \tag{1}$$

$$(\xi_2)x = \bar{\xi}_1 \tag{2}$$

Proof: By definition, $(\xi)x = \frac{-1}{\xi} \Longrightarrow (\xi_1)x = \frac{-2p}{pq+\sqrt{p^2q^2+4pq}} = \frac{pq-\sqrt{p^2q^2+4pq}}{2q} = \bar{\xi}_2$

ξ_1 and $\bar{\xi}_2$ are linked by only one transformation therefore $d(\xi_1, \bar{\xi}_2) = 1$.

Similarly $(\xi_2)x = \bar{\xi}_1$ can be proved easily.

Remark 2.7. It is straightforward from theorem 2.6.

$$(\xi_1)^G = (\bar{\xi}_2)^G \tag{1}$$

$$(\xi_2)^G = (\bar{\xi}_1)^G \tag{2}$$

Theorem 2.8 . Ambiguousdistance between $-\xi_1$ and $-\bar{\xi}_2$ as well as $-\xi_2$ and $-\bar{\xi}_1$ is one. Hence

$$(-\xi_1)x = -\bar{\xi}_2 \tag{1}$$

$$(-\xi_2)x = -\bar{\xi}_1 \tag{2}$$

Proof: By definition, $-\xi_1 = \frac{pq+\sqrt{p^2q^2+4pq}}{-2p} \Longrightarrow (-\xi_1)x = \frac{2p}{pq+\sqrt{p^2q^2+4pq}} = \frac{-pq+\sqrt{p^2q^2+4pq}}{2q} = -\bar{\xi}_2$.

It is clear that (p,q) and (q,p) are only circuits of length two for different positive integers p and q. The exact structures of both circuits are clear from [Figure 1] and [Figure 2].

Theorem 2.9. For $p > q$, in Genre A and Genre B,

$$d(-\xi_2,\ \xi_1) = 2q+1 \tag{1}$$

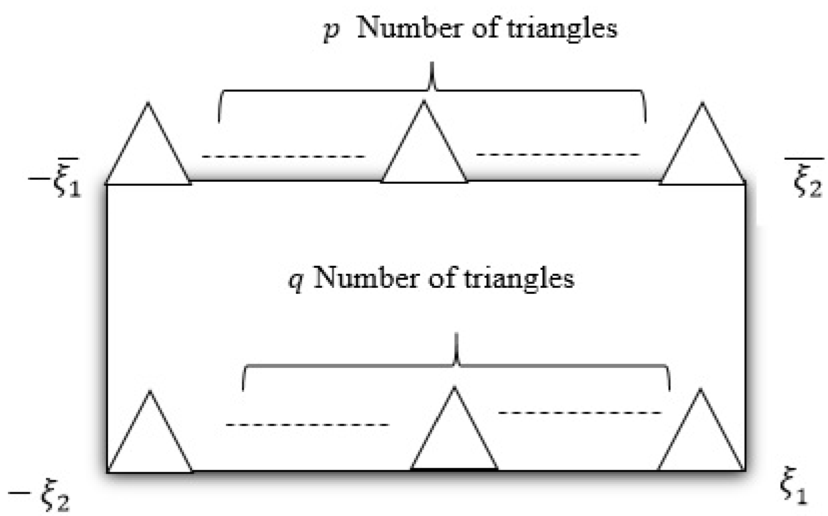

Fig 1. Coset diagram of type (p,q)

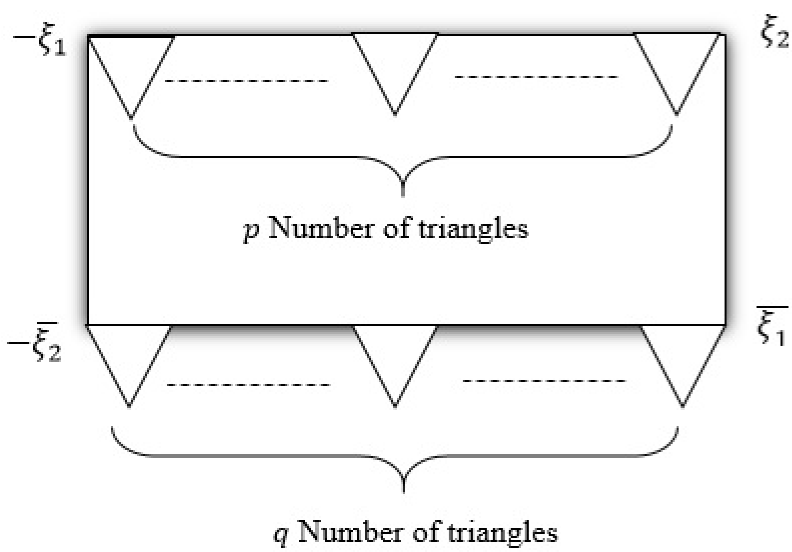

Fig 2. Coset diagram of type (q,p)

$$d(-\xi_1,\ \xi_2) = \ 2q-1 \tag{2}$$

$$(-\xi_2 \bar{\xi_2}) = d(\xi_1 - \bar{\xi_1}) = 2q. \tag{3}$$

Proof: (i) It is clear from [Figure 1] that q triangles are $2q-1$ distance apart given that we measure distance between the corners. Since $p > q$ therefore the shortest distance is measured from the side of q triangles (either inside or outside), which includes two more x edges to complete the path between $-\xi_2$ and ξ_1.

(ii) Straightforward from proof of (i

(iii) Again from [Figure 1] and [Figure 2], for $p > q$ the shortest path includes q triangles and one x edge.
Corollary 2.10. For $p < q$, in Genre A and Genre B
(i) $d(-\xi_2, \xi_1) = 2p + 1$.
(ii) $(-\bar{\xi}_1, \bar{\xi}_2) = 2p - 1$
(iii) $(-\xi_2, \bar{\xi}_2) = d(\xi_1, -\bar{\xi}_1) = 2p$.

Theorem 2.11. For reduced number ξ_3 of Genre C we have $(\xi_3)^G = (\bar{\xi}_3)^G = (-\xi_3)^G = (-\bar{\xi}_3)^G$.
Proof: By definition,
$x(\xi_3) = \bar{\xi}_3$ And $x(-\xi_3) = -\bar{\xi}_3$
$\bar{\xi}_3(xy^2)^p = \frac{\xi_3}{1+p\xi_3} = -\xi_3$
also
$-\bar{\xi}_3(xy)^p = p - \bar{\xi}_3 = \xi_3$
Corollary 2.12. In Genre C,
(i) $d(\xi_3, -\xi_3) = d(-\xi_3, -\bar{\xi}_3) = 1$
(ii) $d(\bar{\xi}_3, -\xi_3) = d(\xi_3, -\bar{\xi}_3) = 2p - 1$
Proof: It is obvious from [Figure 3] and proof of theorem 2.11 and theorem 2.9.

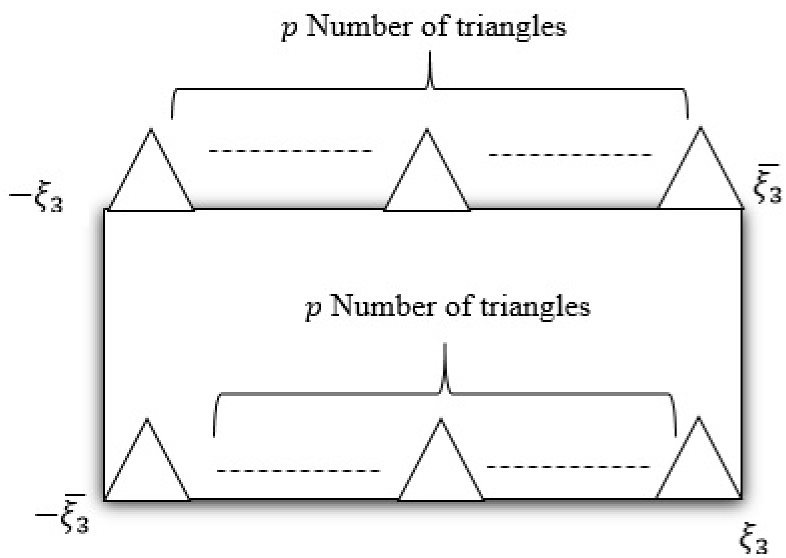

Fig 3. Coset diagram of type (p, p)

Following theorem will elaborate some properties of genre C.
Theorem 2.13. For ξ_3 and non-zero positive integer p the following statements are equivalent.
(a) As a real number the whole part of ξ_3 is p.
(b) The numeric part of continued fraction of ξ_3 is p
(c) $p < \xi_3 = \frac{p+\sqrt{p^2+4}}{2} < p + 1$
(d) For any non-zero positive integer p
$p < \sqrt{p^2+4} < p + 2$.
Proof: We will only prove (c) by contradiction others are obvious.
First suppose $p > \sqrt{p^2+4}$ squaring both sides
$p^2 > p^2 + 4$ Which is false.
Secondly suppose $\sqrt{p^2+4} > p + 2$ again squaring both sides $p^2 + 4 > p^2 + 4 + 4p$ which is also not true for positive p.

Remark 2.14. Since the repeated part of continued fraction shows the type of corresponding closed-circuit, from the above theorem it can be concluded that the continued fraction of ξ_3 is $[p;\bar{p}]$.

[Table 1] shows continued fractions of some numbers of genre C for $p=3$.

Table 1. Continued fractions of Genre C

Sr. No	Number in Coset diagram	Continued fraction
1	$\frac{1+\sqrt{13}}{2}$	$[2;\bar{3}]$
2	$\xi_3=\frac{3+\sqrt{13}}{2}$	$[3;\bar{3}]$
3	$\frac{5+\sqrt{13}}{2}$	$[4;\bar{3}]$
4	$\frac{1+\sqrt{13}}{6}$	$[0;1,\bar{3}]$
5	$\frac{7+\sqrt{13}}{6}$	$[1;1,\bar{3}]$
6	$\frac{5+\sqrt{13}}{6}$	$[1;2,\bar{3}]$

Theorem 2.15. Let q be a non-zero positive integer and $(1,q)$ be the type of circuit of genre A then $q<\xi_1<q+1$ for all q.

Proof: $\xi_1=\frac{q+\sqrt{q^2+4q}}{2}$ for the type $(1,q)$

On contrary suppose that $\xi_1>q+1$

Then $\frac{q+\sqrt{q^2+4q}}{2}>q+1$

$\sqrt{q^2+4q}>q+2$

$q^2+4q>q^2+4+4q$

Gives $4<0$ which is false.

Similarly suppose on contrary $q>\xi_1$

Gives $4q<0$ which is false for non-zero positive q. Hence, the result.

This shows that for any particular type $(1,q)$ the corresponding reduced number has an upper and lower bound depending on q.

Corollary 2.16. Let p be a non-zero positive integer and $(p,1)$ be the type of circuit of genre A then $1<\xi_1<2$ for all p.

Proof: On contrary suppose $\xi_1>2$ then

$\frac{p+\sqrt{p^2+4p}}{2p}>2$ gives $4p(2p-1)<0$ which is false for non-zero positive p.

Similarly suppose $\xi_1<1$ then $p<0$ which is again not true hence the result.

Now it can be generalized that our result regarding the location of reduced quadratic irrational number occurring in the closed-circuit of length 2 on real line for genre A.

Now it will be possible to derive a general result.

Theorem 2.17. Let p *and* q be two non-zero positive integers in the type (p,q) of genre A then $q<\xi_1<q+1$ for all q.

Proof: Suppose on contrary $\xi_1>q+1$

Then $pq+\sqrt{p^2q^2+4pq}>2pq+2p$

Gives $q>p+pq$ which in not possible for non-zero positive integral values of p and q.

Also, if $q>\xi_1$ then $pq>\sqrt{p^2q^2+4pq}$ then $4pq<0$ which is not true for non-zero positive integral values of p and q.

Corollary 2.18. Let p *and* q be two non-zero positive integers in the type (q,p) of genre B then $p<\xi_2<p+1$ for all q.

Proof: It is straightforward from the proof of above theorem.

Therefore, conclusion can be derived that for any type of length two the reduced number has specific limits as a real number.

Now we take a specific interval of length, one on real line, it becomes interesting to know how many distinct reduced numbers can have in this particular unit the interval.

Theorem 2.19. For any non-zero positive integer $\alpha>1$, there are infinite reduced numbers in closed interval $[\alpha,\alpha+1]$.

Proof: It is clear from theorem 2.17 that taking $q=\alpha$ the corresponding reduced number will be always between α and $\alpha+1$ regardless any value p. For infinite values of p there will be infinite reduced numbers in the close interval $(\alpha,\alpha+1]$.

Remark 2.20. There will be a unique reduced number in each unit interval of real line $[\alpha,\alpha+1]$ with $\alpha>1$ of genre C, namely ξ_3 with $p=\alpha$

It is clear from the definition of reduced number that every reduced number lies between $(1,+\infty)$ and using above remark it can be classified by these reduced numbers with respect to their position on real line by considering the reduced number of genre C in type (p, p) and generalize the results by taking types $(p, p+1), (p, p+2), \ldots, (p, p+r)$ where $q = p+r$ for any positive integer r.

Theorem 2.21. For fixed positive integers r and p there is a unique reduced number in the interval $(p+r, p+r+1]$for the coset diagram of type $(p, p+r)$.

Proof: combining the above remark with theorem 2.19, results can be obtained.

3 Equivalence classes and classification of g-circuits of length four

In this section, group of permutation is used to classify G-circuits therefore, square brackets represent the type of G-circuit and round brackets to represent permutations. $P(n)$ denotes the number of partitions of n whereas D_n stands for dihedral group of order $2n$ and S_n stands for symmetric group of order $n!$. It is easy to see that $(S_n| = n! = \frac{(n-1)!}{2} \times 2n = (A_{n-1}| \times (D_n|$. Reader may see any book on number theory and statistics for in depth knowledge of the work.

We now formally define the notion of equivalent, cyclically equivalent and similar circuits in G-orbits of $Q(\sqrt{m})\backslash Q$.

Definition 3.1.1 : Two circuits are said to be equivalent if they differ only by the order of arrangement. i.e.

If $\{a_1,\ a_2, \ldots, a_{2m}\}$ are positive integers then circuit $[a_1, a_2, \ldots, a_{2m}] \sim [a_{\theta(1)}, a_{\theta(2)}, \ldots, a_{\theta(2m)}]$ if and only if $\theta \in S_{2m}$. In other words permuting entries $a_1, a_2, \ldots, a_{2m}$ of a circuit gives us equivalent circuits.

Definition 3.1.2 : Let $[a_1, a_2, \ldots, a_{2m}]$ be a circuit then regarding circular order of the positive integers $a_1, a_2, \ldots, a_{2m}$ if we start from any of these integers and adopt clockwise or counter clockwise direction, the $4m$ circuits so obtained are all said to be cyclically equivalent. In particular

$[a_1, a_2, \ldots, a_{2m}] \sim_c [a_{\theta(1)}, a_{\theta(2)}, \ldots, a_{\theta(2m)}]$ if and only if $\theta \in D_{2m}$. Mathematically, $(a_1, a_2, \ldots, a_{2m}] \sim_c [a_2, a_3, \ldots, a_{2m}, a_1] \cdots \sim_c [a_{2m}, a_1, a_2, \ldots, a_{2m-1}] \sim_c [a_{2m}, \ldots, a_2, a_1]$

$\sim_c [a_{2m-1}, \ldots, a_1, a_{2m}] \cdots \sim_c [a_1, a_{2m}, \ldots, a_2]$.

Definition 3.1.3 : Two equivalent circuits are said to be similar if they correspond to the same circuit. Mathematically we write $[a_1, a_2, \ldots, a_{2m}] \sim_s [a_3, a_4, \ldots, a_{2m}, a_1, a_2] \cdots \sim_s [a_{2m-1}, a_{2m}, \ldots, a_{2m-2}]$.

We have proved that there are exactly four classes of equivalent circuits of length 4 namely $[a_1, a_2, a_3, a_4]$, $[a_1, a_1, a_2, a_3]$, $[a_1, a_1, a_1, a_2]$ and $[a_1, a_1, a_2, a_2]$ where $a_1,\ a_2,\ a_3\ and\ a_4$ are different positive integers and four equivalent circuits $[a_1, a_2, a_3, a_4]$, $[a_2, a_1, a_3, a_4]$ and $[a_1, a_3, a_2, a_4]$ corresponds to the orbits contained in $Q^*\left(\sqrt{n_2}\right)$, $Q^*(\sqrt{n_3})$ and $Q^*\left(\sqrt{n_1}\right)$ respectively.

Now reversion towards the application and physical interpretation in this area of research with a consideration of finding non-equivalent circuits of length q.

Given the integer q, we say the sequence of positive integers $q_1,\ q_2, \ldots, q_r,\ q_1 \le q_2 \le \cdots \le q_r$ constitute a partition of q if $q = q_1 + q_2 + \cdots + q_r$. Let $P(q)$ denote the number of partitions of q. Let's determine $P(q)$ for small values of q:

$P(1) = 1$ since 1=1 is the only partition of 1,

$P(2) = 2$ since $2 = 2$ and $2 = 1+1$,

$P(3) = 3$ since $3 = 3,\ 3 = 2+1,\ 3 = 1+1+1$,

$P(4) = 5$ since $4 = 4,\ 4 = 3+1,\ 4 = 2+2,\ 4 = 2+1+1,\ 4 = 1+1+1+1$,

Some others are $P(5) = 7,\ P(6) = 11,\ P(61) = 1121505$. There are large mathematical literature on $P(q)$. Following is the crucial and elementary result.

The number of non-equivalent circuits of length two are only 2. For the circuit of length 2, $P(2) = 2$ where $2 = 1+1$ and $2 = 2$ corresponds to the circuits $[a_1, a_2]$ and $[a_1, a_1]$ respectively.

It is interesting to note that a circuit of length 2 corresponding to the partition $2 = 2$ whereas, circuit with this pattern with length greater than or equal to four is not possible. see[4]

Lemma 3.1.4. The number of classes E_q of equivalent circuits with length q, $q \ge 4$ are precisely $P(q) - 1$.

Proof: Every time in determining all non-equivalent circuits of length q, a partition can be obtained for q in the sense that if q entries are appearing in the circuit of length $q = q_1 + q_2 + \cdots + q_k$ for which q_1 are alike, q_2 are alike,..., q_k are alike. We shall say that a circuit of length q has the circuit decomposition $\{q_1,\ q_2, \ldots,\ q_k\}$ if it involves q entries for which q_1 are alike, q_2 are alike,..., q_k are alike. Thus, the number of non-equivalent circuits are equal to $P(q) - 1$ as $(a_1, a_1, a_1, \ldots, a_1)$ is not possible for circuits of length $q \ge 4$, see [4].

Since we have such an explicit description of determining the non-equivalent circuits of length q. Therefore, in this paper we define all the classes of circuits of the length 4 with $a_1,\ a_2,\ a_3,\ a_4$ as different positive integers as mentioned below.

1. $E_{[a_1, a_2, a_3, a_4]}$
2. $E_{(a_1, a_1, a_2, a_3]}$
3. $E_{(a_1, a_1, a_1, a_2]}$
4. $E_{[a_1, a_1, a_2, a_2]}$

Hence, all the circuits of length 4 are $E = E_{[a_1,a_2,a_3,a_4]} \bigcup E_{(a_1,a_1,a_2,a_3]} \bigcup E_{(a_1,a_1,a_1,a_2]} \bigcup E_{[a_1,a_1,a_2,a_2]}$. It is possible to find all the equivalent classes of length four against each above defined class.

3.2. Equivalent circuits of $E_{[a_1,a_2,a_3,a_4]}$

Theorem 3.2.1. Consider a_1,a_2,a_3,a_4, be different positive integers then we have 24 equivalent circuits corresponding to $E_{[a_1,a_2,a_3,a_4]}$.

Proof: We use permutations to prove that there are exactly 24 circuits corresponding to a_1,a_2,a_3,a_4.

For the number of possibilities of selecting r distinct objects from n objects, where the order of arrangements is considered, are ${}^nP_r = \frac{n!}{(n-r)!}$.

The number of different arrangements of n objects of which n_1 are alike, n_2 are alike,..., n_k are alike is $\frac{n!}{n_1!n_2!...n_k!}$ where $n_1+n_2+\cdots+n_k = n$

Therefore, for $[a_1,a_2,a_3,a_4]$ we have ${}^4P_4 = 4! = 24$ equivalent circuits.

We claim that all above twenty-four circuits are divided in three sub classes of equivalent circuits called classes of cyclically equivalent circuits. This claim is proved in the following theorem.

Theorem 3.2.2. There are three classes ${E^c}_{[a_1,a_2,a_3,a_4]}, {E^c}_{[a_2,a_1,a_3,a_4]}$ and ${E^c}_{[a_1,a_3,a_2,a_4]}$ of cyclically equivalent circuits of length 4 in $E_{[a_1,a_2,a_3,a_4]}$.

Proof: Let a_1,a_2,a_3,a_4 be different positive integers. It is well-known that group of rotational symmetries of a cube is isomorphic to the Symmetric group S_4 and the circuits which are cyclically equivalent are obtained by $[a_{\theta(1)},a_{\theta(2)},a_{\theta(3)},a_{\theta(4)}]$ for each $\theta \in A_3$, since the order of A_3 is 3 therefore for each θ there are three classes of cyclically equivalent circuits.

Corollary 3.2.3. For each class of cyclically equivalent circuits there is a unique n in $Q^*(\sqrt{n})$. Thus there are 3 orbits of (say) n_2 which corresponds to ${E^c}_{[a_1,a_2,a_3,a_4]}$, n_1 which corresponds to ${E^c}_{[a_1,a_3,a_2,a_4]}$ and n_3 which corresponds to ${E^c}_{[a_2,a_1,a_3,a_4]}$.

Proof: It is proved in [6] that all cyclically equivalent circuits belong to the same orbit of $Q^*(\sqrt{n})$.

We prove the relationship between n_1,n_2 and n_3 in the next sections of this paper.

Corollary 3.2.4. Each class ${E^c}_{[a_1,a_2,a_3,a_4]}, {E^c}_{[a_2,a_1,a_3,a_4]}$ and ${E^c}_{[a_1,a_3,a_2,a_4]}$ of cyclically equivalent circuits contain 8 cyclically equivalent circuits.

Proof: Let a_1,a_2,a_3,a_4 be different numbers and so these circuits are $[a_{\theta(1)},a_{\theta(2)},a_{\theta(3)},a_{\theta(4)}]$ for each $\theta \in S_4$ as shown in Tables 2 and 3.

Table 2. First cyclically equivalent class ${E^c}_{[a_1,a_2,a_3,a_4]}$

$\theta \in D_4$	$[a_{\theta(1)},a_{\theta(2)},a_{\theta(3)},a_{\theta(4)}]$
(1)(2)(3)(4)	$[a_1,a_2,a_3,a_4]$
(1234)	$[a_2,a_3,a_4,a_1]$
(13)(24)	$[a_3,a_4,a_1,a_2]$
(1432)	$[a_4,a_1,a_2,a_3]$
(12)(34)	$[a_2,a_1,a_4,a_3]$
(14)(23)	$[a_4,a_3,a_2,a_1]$
(24)	$[a_1,a_4,a_3,a_2]$
(13)	$[a_3,a_2,a_1,a_4]$

Table 3. Second and third cyclically equivalent class ${E^c}_{[a_2,a_1,a_3,a_4]}$ & ${E^c}_{[a_1,a_3,a_2,a_4]}$

$\theta \in (23)D_4$	$[a_{\theta(1)},a_{\theta(2)},a_{\theta(3)},a_{\theta(4)}]$	$\theta \in (12)D_4$	$[a_{\theta(1)},a_{\theta(2)},a_{\theta(3)},a_{\theta(4)}]$
(23)	$(a_1,a_3,a_2,a_4]$	(12)	$(a_2,a_1,a_3,a_4]$
(134)	$(a_3,a_2,a_4,a_1]$	(234)	$(a_1,a_3,a_4,a_2]$
(1243)	$(a_2,a_4,a_1,a_3]$	(1324)	$(a_3,a_4,a_2,a_1]$
(142)	$(a_4,a_1,a_3,a_2]$	(143)	$(a_4,a_2,a_1,a_3]$
(14)	$(a_4,a_2,a_3,a_1]$	(1423)	$(a_4,a_3,a_1,a_2]$
(123)	$(a_2,a_3,a_1,a_4]$	(132)	$(a_3,a_1,a_2,a_4]$
(1342)	$(a_3,a_1,a_4,a_2]$	(34)	$(a_1,a_2,a_4,a_3]$
(243)	$[a_1,a_4,a_2,a_3]$	(124)	$[a_2,a_4,a_3,a_1]$

It is interesting to see that in this case $S_4 = \{D_4,\ (12)\,D_4,\ (23)D_4\}$

These twenty-four circuits in 3 classes can be written depicted by

$$E^c{}_{[a_1,a_2,a_3,a_4]} = (a_1,a_2,a_3,a_4] \sim_c (a_2,a_3,a_4,a_1] \sim_c (a_3,a_4,a_1,a_2] \sim_c (a_4,a_1,a_2,a_3] \sim_c$$

$$(a_4,a_3,a_2,a_1] \sim_c (a_3,a_2,a_1,a_4] \sim_c (a_2,a_1,a_4,a_3] \sim_c [a_1,a_4,a_3,a_2]$$

$$E^c{}_{[a_2,a_1,a_3,a_4]} = (a_2,a_1,a_3,a_4] \sim_c (a_1,a_3,a_4,a_2] \sim_c (a_3,a_4,a_2,a_1] \sim_c (a_4,a_2,a_1,a_3] \sim_c$$

$$(a_4,a_3,a_1,a_2] \sim_c (a_3,a_1,a_2,a_4] \sim_c (a_1,a_2,a_4,a_3] \sim_c [a_2,a_4,a_3,a_1]$$

$$E^c{}_{[a_1,a_3,a_2,a_4]} = (a_1,a_3,a_2,a_4] \sim_c (a_3,a_2,a_4,a_1] \sim_c (a_2,a_4,a_1,a_3] \sim_c (a_4,a_1,a_3,a_2] \sim_c$$

$(a_4,a_2,a_3,a_1] \sim_c (a_2,a_3,a_1,a_4] \sim_c (a_3,a_1,a_4,a_2] \sim_c [a_1,a_4,a_2,a_3]$.

Such that $E_{[a_1,a_2,a_3,a_4]} = E^c{}_{[a_1,a_2,a_3,a_4]} \bigcup E^c{}_{[a_2,a_1,a_3,a_4]} \bigcup E^c{}_{[a_1,a_3,a_2,a_4]}$.

Remark 3.2.5. Let D_4 be the dihedral group of order 8 and $\alpha D_4, \alpha \in S_4$ be a right coset of D_4 then

$$E^c{}_{(a_1,a_2,a_3,a_4]} = \left([a_{\theta(1)},a_{\theta(2)},a_{\theta(3)},a_{\theta(4)}\right],\ \theta \in D_4\}$$

$$E^c{}_{[a_2,a_1,a_3,a_4]} = \left([a_{\theta(1)},a_{\theta(2)},a_{\theta(3)},a_{\theta(4)}\right],\ \theta \in (12)D_4\}$$

$E^c{}_{[a_1,a_3,a_2,a_4]} = \left([a_{\theta(1)},a_{\theta(2)},a_{\theta(3)},a_{\theta(4)}\right],\ \theta \in (23)D_4\}$.

Next corollary will classify G-orbits of $Q(\sqrt{m})\backslash Q$ having circuits of length 4.

Corollary 3.2.6.

All the circuits in $E^c{}_{[a_1,a_2,a_3,a_4]}, E^c{}_{[a_2,a_1,a_3,a_4]}$ and $E^c{}_{[a_1,a_3,a_2,a_4]}$ corresponds to the orbits contained in $Q^*(\sqrt{n_2})$, $Q^*(\sqrt{n_3})$ and $Q^*\left(\sqrt{n_1}\right)$ respectively, where

$n_1 = (a_1a_3)^2+(a_1a_4)^2+(a_2a_3)^2+(a_2a_4)^2+(a_1a_2a_3a_4)^2+2a_1^2a_3a_4+2a_1a_2^2a_3^2a_4+2a_1^2a_3^2a_2a_4+2a_1a_2a_4^2+2a_1^2a_4^2a_2a_3+2a_1a_3a_2^2a_4^2+2a_3a_2^2a_4+4a_1a_4+4a_2a_3+4a_1a_3+4a_2a_4+8a_1a_2a_3a_4$

$n_2 = (a_1a_2)^2+(a_1a_4)^2+(a_2a_3)^2+(a_3a_4)^2+(a_1a_2a_3a_4)^2+2a_1^2a_2a_4+2a_1a_2^2a_3^2a_4+2a_1^2a_2^2a_3a_4+2a_1a_3a_4^2+2a_1^2a_4^2a_2a_3+2a_1a_2a_3^2a_4^2+2a_2a_3^2a_4+4a_1a_4+4a_2a_3+4a_1a_2+4a_3a_4+8a_1a_2a_3a_4$

$n_3 = (a_1a_2)^2+(a_2a_4)^2+(a_1a_3)^2+(a_3a_4)^2+(a_1a_2a_3a_4)^2+2a_2^2a_1a_4+2a_2a_1^2a_3^2a_4+2a_1^2a_2^2a_3a_4+2a_2a_3a_4^2+2a_2^2a_4^2a_1a_3+2a_1a_2a_3^2a_4^2+2a_1a_3^2a_4+4a_2a_4+4a_1a_3+4a_1a_2+4a_3a_4+8a_1a_2a_3a_4$

Proof: It is proved in [6] that a circuit $[p,q,r,s]$ of length four corresponds to the orbits contained in $(pq+ps+rs-qr+pqrs)^2+4(p+r+pqr)(q+s+qrs)$.

We put $p=a_1$, $q=a_2$, $r=a_3$ and $s=a_4$ above and after simplification we get the result for n_2.

Similarly, we can prove it for n_1 and n_3 with the corresponding change of variables.

Moreover, all the circuits in $E^c{}_{(a_1,a_2,a_3,a_4]}, E^c{}_{[a_2,a_1,a_3,a_4]}$ and $E^c{}_{[a_1,a_3,a_2,a_4]}$, n_2, n_3 and n_1 remains invariant under all the $\theta \in D_4$, $\theta \in (12)D_4$ and $\theta \in (23)D_4$ respectively.

Now we explore three more possible circuits of length four in $Q^*(\sqrt{n})$ in which all the four positive integers need not to be distinct.

3.3. Equivalent circuits of $E_{[a_1,a_3,a_2,a_4]}$

Following results can be easily deduced from subsection 3.2.

Corollary 3.3.1. Consider the circuit $[a_1,a_1,a_2,a_3]$ then we have the following two classes of 12 equivalent circuits namely

$E^c_{(a_1,a_1,a_2,a_3]} = (a_1,a_1,a_2,a_3] \sim_c (a_3,a_2,a_1,a_1] \sim_c (a_1,a_2,a_3,a_1] \sim_c (a_1,a_3,a_2,a_1]$

$\sim_c (a_2,a_3,a_1,a_1] \sim_c (a_1,a_1,a_3,a_2] \sim_c (a_3,a_1,a_1,a_2] \sim_c (a_2,a_1,a_1,a_3]$

$E^c_{(a_1,a_2,a_1,a_3]} = (a_1,a_2,a_1,a_3] \sim_c (a_3,a_1,a_2,a_1] \sim_c (a_2,a_1,a_3,a_1] \sim_c [a_1,a_3,a_1,a_2]$

Proof: We have $\frac{4!}{2!} = 12$ different circuits of length 4.

Corollary 3.3.2. All the circuits in $E^c_{(a_1,a_1,a_2,a_3]}$ and $E^c_{(a_1,a_2,a_1,a_3]}$ corresponds to the orbits contained in $Q^*(\sqrt{n_4})$ and $Q^*(\sqrt{n_5})$ respectively where

$n_4 = a_1^4 + (a_1a_3)^2 + (a_1a_2)^2 + (a_2a_3)^2 + (a_1^2a_2a_3)^2 + 2a_1^3a_3 + 2a_1^3a_2^2a_3 + 2a_1^4a_2a_3 + 2a_3^2a_1a_2 + 2a_1^3a_2a_3^2 + 2a_1^2a_2^2a_3^2 + 2a_2^2a_1a_3 + 4a_1a_3 + 4a_1a_2 + 4a_1^2 + 4a_2a_3 + 8a_1^2a_2a_3$

$n_5 = 2(a_1a_2)^2 + 4(a_1a_3)^2 + (a_1^2a_2a_3)^2 + 12a_1^2a_2a_3 + 4a_2^2a_1^3a_3 + 4a_1^3a_3^2a_2 + 8a_1a_3 + 8a_2a_1$

Proof: Replacement of a_2 by a_1, a_3 by a_2 and a_4 by a_3 in the expression of n_2 and get the required result. Similar can be done for other 7 circuits in the $E^c_{(a_1,a_1,a_2,a_3]}$ by making suitable substitutions in the expression of n_2, getting the same result every time, i.e. n_4. For second part of the result, replacement can be done for a_3 by a_1 and a_4 by a_3 in the expression of n_2 which on simplification gives n_5. Similar can be done for other 3 cyclically equivalent circuits of $E^c_{(a_1,a_2,a_1,a_3]}$ by making suitable substitutions in the expression of n_2, getting the same result every time i.e. n_5.

3.4. Equivalent circuits of $E_{[a_1,a_3,a_2,a_4]}$

Corollary 3.4.1. Consider the circuit $[a_1,a_1,a_1,a_2]$ then we have the following class of 4 equivalent circuits

$E^c_{[a_1,a_1,a_1,a_2]} = (a_1,a_1,a_1,a_2] \sim_c (a_1,a_1,a_2,a_1] \sim_c (a_1,a_2,a_1,a_1] \sim_c [a_2,a_1,a_1,a_1]$.

Proof: From the permutation theory we have $\frac{4!}{3!} = 4$ circuits which are 4 possible permutations of four elements in which 3 are same.

Corollary 3.4.2. All the circuits in $E^c_{[a_1,a_1,a_1,a_2]}$ corresponds to the orbits contained in $Q^*(\sqrt{n_6})$ where

$n_6 = 2a_1^4 + 4a_1^2a_2^2 + 4a_1^4a_2^2 + 4a_1^5a_2 + a_1^6a_2^2 + 12a_1^3a_2 + 8a_1^2 + 8a_1a_2$

Proof: We replace a_2 and a_3 by a_1 and replace a_4 by a_2 in the expression of n_2 to get the desired result. Similar can be proved for other equivalent circuits of this class.

3.5. Equivalent circuits of $E_{[a_1,a_3,a_2,a_4]}$

Corollary 3.5.1. There is only one class of cyclically equivalent circuits $E^c_{[a_1,a_1,a_2,a_2]}$ in $E_{[a_1,a_1,a_2,a_2]}$ containing 4 circuits.

Proof: we have $\frac{4!}{2!2!} = 6$ but the circuits $[a_1,a_2,a_1,a_2]$ and $[a_2,a_1,a_2,a_1]$ failed to exist in length 4.

Hence $E^c_{[a_1,a_1,a_2,a_2]} = (a_1,a_1,a_2,a_2] \sim_c (a_1,a_2,a_2,a_1] \sim_c (a_2,a_2,a_1,a_1] \sim_c [a_2,a_1,a_1,a_2]$.

Corollary 3.5.2. All the circuits in $E^c_{[a_1,a_1,a_2,a_2]}$ corresponds to the orbits contained in $Q^*(\sqrt{n_7})$ where

$n_7 = a_1^4 + 10a_1^2a_2^2 + a_2^4 + a_1^4a_2^4 + 2a_1^3a_2 + 2a_1^4a_2^2 + 2a_1^2a_2^4 + 4a_1a_2^3 + 4a_1^3a_2^3 + 4a_1^2 + 8a_1a_2 + 4a_2^2$.

Proof: Replacement of a_2 by a_1 and a_3, a_4 by a_2 in n_2 to get the required formation of n_7. It can be proved that n_7 is same for all the four circuits in $E^c_{[a_1,a_1,a_2,a_2]}$. For substitute a_3 by a_2 and a_4 by a_1 in the expression for n_2, we have n_7. Similar result can be proved by making suitable substations in the expression involving n_2 to get the desired results.

[Table 4] shows the complete classification of circuits of length four.

Remark 3.5.3. Twelve equivalent circuits in corollary 3.3.1 are isomorphic to 12 elements of Alternating group A_4.

This is now clear from [Table 4] that for all 44 circuits of length 4, we obtain only 7 corresponding values of n.

4 Classification of n in $Q^*(\sqrt{\mathbf{n}})$ in circuits of length four

It is clear from [Table 3] that there are only two classes of equivalent circuits which corresponds to more than one orbits of $Q^*(\sqrt{n})$ namely $E_{[a_1,a_2,a_3,a_4]}$ and $E_{[a_1,a_1,a_2,a_3]}$ which raises an obvious question about the order of these numbers. Next two theorems will answer this question.

Theorem 4.1. Let a_1, a_2, a_3, a_4 be positive integers such that $0 < a_1 < a_2 < a_3 < a_4$. Then the three equivalent but not similar classes $E^c{}_{[a_1,a_2,a_3,a_4]}$, $E^c{}_{[a_1,a_3,a_2,a_4]}$ and $E^c{}_{[a_2,a_1,a_3,a_4]}$ correspond to the orbits contained in $Q^*(\sqrt{n_2})$, $Q^*(\sqrt{n_1})$ and $Q^*(\sqrt{n_3})$ respectively with

$n_1 < n_2 < n_3$

Proof: To prove the above result we suppose that $a_2 = a_1 + \in_1 a_3 = a_1 + \in_2$ and $a_4 = a_1 + \in_3$ where $\in_1 \geq 1$,$\in_2 \geq 2$ and $\in_3 \geq 3$ respectively,

From Corollary 3.2.6 we have,

$n_1 = (a_1a_3)^2 + (a_1a_4)^2 + (a_2a_3)^2 + (a_2a_4)^2 + (a_1a_2a_3a_4)^2 + 2a_1^2a_3a_4 + 2a_1a_2^2a_3^2a_4 + 2a_1^2a_3^2a_2a_4 + 2a_1a_2a_4^2 + 2a_1^2a_4^2a_2a_3 + 2a_1a_3a_2^2a_4^2 + 2a_3a_2^2a_4 + 4a_1a_4 + 4a_2a_3 + 4a_1a_3 + 4a_2a_4 + 8a_1a_2a_3a_4$

$n_2 = (a_1a_2)^2 + (a_1a_4)^2 + (a_2a_3)^2 + (a_3a_4)^2 + (a_1a_2a_3a_4)^2 + 2a_1^2a_2a_4 + 2a_1a_2^2a_3^2a_4 + 2a_1^2a_2^2a_3a_4 + 2a_1a_3a_4^2 + 2a_1^2a_4^2a_2a_3 + 2a_1a_2a_3^2a_4^2 + 2a_2a_3^2a_4 + 4a_1a_4 + 4a_2a_3 + 4a_1a_2 + 4a_3a_4 + 8a_1a_2a_3a_4$

Now consider

$n_2 - n_1 = a_1^2\left(a_2^2 - a_3^2\right) + a_4^2\left(a_3^2 - a_2^2\right) + 2a_1^2a_4(a_2 - a_3) + 2a_1^2a_2a_4a_3(a_2 - a_3) + 2a_1a_4^2(a_3 - a_2) + 2a_1a_2a_3a_4^2(a_3 - a_2) + 4a_2a_3a_4(a_3 - a_2) + 4a_1(a_2 - a_3) + 4a_4(a_3 - a_2)$

Table 4. All classes of circuits of length four

Equivalence Class $E_{[\_,\_,\_,\_]}$	$\lvert E_{[\_,\_,\_,\_]}\rvert$	**Cyclically Equivalence class** ${E^c}_{[\_,\_,\_,\_]}$	$\lvert {E^c}_{[\_,\_,\_,\_]}\rvert$	**Distinction of orbits** $\in Q^*(\sqrt{\_})$
$E_{[a_1,a_2,a_3,a_4]}$	24	${E^c}_{[a_1,a_2,a_3,a_4]}$	8	$\gamma^G, \bar{\gamma}^G, -\gamma^G, -\bar{\gamma}^G \in Q^*(\sqrt{n_2})$
		${E^c}_{[a_1,a_3,a_2,a_4]}$	8	$\gamma^G, \bar{\gamma}^G, -\gamma^G, -\bar{\gamma}^G \in Q^*(\sqrt{n_1})$
		${E^c}_{[a_2,a_1,a_3,a_4]}$	8	$\gamma^G, \bar{\gamma}^G, -\gamma^G, -\bar{\gamma}^G \in Q^*(\sqrt{n_3})$
$E_{[a_1,a_1,a_2,a_3]}$	12	${E^c}_{[a_1,a_1,a_2,a_3]}$	8	$\gamma^G, \bar{\gamma}^G, -\gamma^G, -\bar{\gamma}^G \in Q^*(\sqrt{n_4})$
		${E^c}_{[a_1,a_2,a_1,a_3]}$	4	$\gamma^G = -\bar{\gamma}^G,\ -\gamma^G = \bar{\gamma}^G \in Q^*(\sqrt{n_5})$
$E_{[a_1,a_1,a_1,a_2]}$	4	${E^c}_{[a_1,a_1,a_1,a_2]}$	4	$\gamma^G = \bar{\gamma}^G, -\gamma^G = -\bar{\gamma}^G \in Q^*(\sqrt{n_6})$
$E_{[a_1,a_1,a_2,a_2]}$	4	${E^c}_{[a_1,a_1,a_2,a_2]}$	4	$-\gamma^G = \bar{\gamma}^G, -\bar{\gamma}^G = \gamma^G \in Q^*(\sqrt{n_7})$

Now using above substitutions we have,

$n_2 - n_1 = -4 \in_1\in_3 + 4 \in_2\in_3 - 4 \in_1^2\in_2\in_3 + 4 \in_1\in_2^2\in_3 - \in_1^2\in_3^2 + \in_2^2\in_3^2 - 4 \in_1^2\in_2 a_1 + 4 \in_1\in_2^2 a_1 - 6 \in_1^2\in_3 a_1 + 6 \in_2^2\in_3 a_1 - 4 \in_1\in_3^2 a_1 + 4 \in_2\in_3^2 a_1 - 2 \in_1^2\in_2\in_3^2 a_1 + 2 \in_1\in_2^2\in_3^2 a_1 - 4 \in_1^2 a_1^2 + 4 \in_2^2 a_1^2 - 10 \in_1\in_3 a_1^2 + 10 \in_2\in_3 a_1^2 - 2 \in_1^2\in_2\in_3 a_1^2 + 2 \in_1\in_2^2\in_3 a_1^2 - 2 \in_1^2\in_3^2 a_1^2 + 2 \in_2^2\in_3^2 a_1^2 - 4 \in_1 a_1^3 + 4 \in_2 a_1^3 - 2 \in_1^2\in_3 a_1^3 + 2 \in_2^2\in_3 a_1^3 - 2 \in_1\in_3^2 a_1^3 + 2 \in_2\in_3^2 a_1^3 - 2 \in_1\in_3 a_1^4 + 2 \in_2\in_3 a_1^4$

after simplification, we have,

$n_2 - n_1 = (\in_2 - \in_1)\,(4 \in_3 + 4 \in_1\in_2\in_3 + \in_1\in_3^2 + \in_2\in_3^2 + 4 \in_1\in_2 a_1 + 6 \in_1\in_3 a_1 + 6 \in_2\in_3 a_1 + 4 \in_3^2 a_1 + 2 \in_1\in_2\in_3^2 a_1 + 4 \in_1 a_1^2 + 4 \in_2 a_1^2 + 10 \in_3 a_1^2 + 2 \in_1\in_2\in_3 a_1^2 + 2 \in_1\in_3^2 a_1^2 + 2 \in_2\in_3^2 a_1^2 + 4a_1^3 + 2 \in_1\in_3 a_1^3 + 2 \in_2\in_3 a_1^3 + 2 \in_3^2 a_1^3 + 2 \in_3 a_1^4 > 0$

Since $\in_2 - \in_1 \geq 1$ and all other terms in bracket are positive therefore the above expression is positive.

Similarly, we can prove $n_3 - n_2 > 0$, Thus the result.

Theorem 4.2. Let a_1, a_2, a_3, a_4 be positive integers such that $0 < a_1 < a_2 < a_3$. Then the two equivalent but not similar classes ${E^c}_{[a_1,a_1,a_2,a_3]}$ and ${E^c}_{[a_1,a_2,a_1,a_3]}$ correspond to the orbits contained in $Q^*(\sqrt{n_4})$ and $Q^*(\sqrt{n_5})$ respectively with the condition $n_5 < n_4$.

Proof: From Corollary 3.3.2, we have

Since $a_1 < a_2 < a_3$ we may assume $a_2 = a_1 + \in_1$ and $a_3 = a_1 + \in_2$ where $\in_1$ and $\in_2$ are positive integers with $\in_1 \geq 1$ and $\in_2 \geq 2$.

Thus $n_5 - n_4 = -\in_1 (4 \in_2 + \in_1\in_2^2 + 4 \in_1\in_2 a_1 + 4 \in_2^2 a_1 + 2 \in_1 a_1^2 + 8 \in_2 a_1^2 + 2 \in_1\in_2^2 a_1^2 + 2a_1^3 + 2 \in_1\in_2 a_1^3 + 2 \in_2^2 a_1^3 + 2 \in_2 a_1^4) < 0$. Hence, the result.

5 Conclusion

Discussion about the different properties of reduced numbers and distributed types of length 2 in three different categories leads to the reader to understand more deeply the role of reduced numbers in the coset diagram with the help of continued fractions. Distance in coset diagram is defined and new results are derived in this sense. General results are obtained by considering these reduced numbers on real line which makes easy for us to understand their behavior. This work can be used to generalize the properties of types of length 4 and so on. Reduced numbers have upper and lower bounds as real numbers in a specific type of length 2. Classification has been done for all the circuits of length four under the action of modular group into different classes and sub classes of circuits. These provide answer to the question that for given four positive integers which need not to be distinct, how many circuits of length four exist? Moreover, how many to these circuits are contained in the same orbits of $Q^*(\sqrt{n})$. In this paper we have calculated all possible numbers i.e. n_i, $i = 1, 2, \ldots, 7$ and proved that $n_1 < n_2 < n_3$ and $n_5 < n_4$. This research motivates to explore other circuits of length six and higher in the similar manner.

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