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RESEARCH ARTICLE

A note on generalized m-derivations to weakly cancellative semirings

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Abstract

Objectives: Semirings is an important algebraic structure with applications in theory of automata, formal languages and theoretical computer science. The mappings which enforces commutativity in semirings remains attractive for researchers, since commutativity would be helpful in calculations and bring it's applications to ease. Our aim is to enforce commutativity in semirings by generalizing the classical theorem of Martindale [14, Theorem 3] with generalized m-derivation. Further, we discuss that composition of two generalized m-derivations ensure that one of their associated derivation must be trivial. **Method:** We use generalized m-derivations which is associated to multiplicative derivations in certain semirings. **Findings:** We find the conditions of commutativity in semirings through these particular generalized m-derivations. Moreover, we discuss the characteristics of these mappings in weakly cancellative semirings. **Novelty:** The concept of generalized m-derivations is newly introduced by us in ring theory in (1) and here we extend this concept to theory of semirings. We attempt to induce commutativity in weakly cancellative semirings (2) whose concept is unorthodox in the theory of semirings. This article pave new ways to study derivations and its applications on semirings.

Keywords: Derivations; generalized m-derivations; weakly cancellative semiring; commutativity

1 Introduction

The formal definition of semirings was introduced by H. S. Vandiver in 1934 and has since then been studied by many authors. Basic references for semirings are (3; 4). Nowadays semirings have important applications in the theory of automata, formal languages and in theoretical computer science (cf. (4; 5)). In connection with these facts, semirings are studied in various directions. One of the main goals is the study of mappings on semirings and it's relationships with commutativity of semirings (cf. (6; 2; 7; 8)).

An additive mapping d: $R \to R$ is called a derivation of ring R if $d(xy) = xd(y) + d(x)y$ holds for all $x,y \in R$. A map d: $R \to R$ (may not be additive) is called a multiplicative derivation on R if $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in R$. In the year 1991, Bre˘sar (9) introduced the concept of generalized derivation in rings as follows;

an additive mapping G: R → R associated with derivation d: R → R such that G(xy) = G(x)y + xd(y) holds for all x, y ∈ R. It is obvious that any derivation is a generalized derivation, but the converse is not true in general. The notion of derivation has been generalized in several ways by various authors in rings (10; 11; 12; 13). We introduce the concept of generalized m-derivations in (1) as an additive map G : R → R on ring R and there exists a multiplicative derivation d of R such that G(xy) = G(x)y + xd(y) for all x, y ∈ R. Here we extend the concept of generalized m-derivations in semirings. To elaborate completely, we have provided some examples (see Examples). During the last few years, there has been ongoing interest concerning the relationship between the commutativity of a ring R and the existence of derivations of R. In the sequel, Martindale (14) used derivations to enforce commutativity in rings, therefore, it remains notable for many researchers and they attempt to extend it with generalized derivations satisfying certain polynomial constraints., we refer the reader to see (10; 11; 12; 13; 1) for further references. This type of researches provides us motivation to extend some remarkable theorems of commutativity on generalized m-derivations in semirings. Moreover, we study the characteristics of composition of generalized m-derivations in semirings.

An element a ∈ S is said to be additively left (resp. right) cancellable if a + b = a + c (resp. b + a = c + a) yields b = c. In this paper, semiring means additively cancellative semiring. A semiring S is said to be weakly left cancellative (WLC) if axb = axc for all x ∈ S, implies either a = 0 or b = c. This notion is recently introduced by V. De. Fillips (2). In the same pattern, we extend this concept to weakly right cancellative (WRC) semirings if bxa = cxa for all x ∈ S, implies either a = 0 or b = c. A semiring which is weakly left as well as right cancellation is termed as weakly cancellative (WC) semiring. We will denote Z(R) = {z ∈ S : zx = xz, ∀x ∈ S} the center of S, and x ∘ y = xy + yx the Jordan product of x, y ∈ R. Throughout this paper, R will represent a weakly cancellative semiring (WC) with center Z(R). A semiring R is said to be 2-torsion free if 2x = 0, for x ∈ R, implies that x = 0.

The following examples confirm the existence of such maps in semirings.

Examples:

$$\text{(a) Let R} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \quad a,b,c \in D \right\}$$

where D is distributibe lattice, then R is additively inverse semirings. The additive inverse element a'=a, for all a ∈ R. Now define G, g: R→R as follows:

$$g\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c' \\ 0 & 0 & 0 \end{bmatrix}$$

This can be verified that g is multiplicative derivation and G is generalized m-derivations.

$$\text{(b) Let R} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \quad a,b,c \in S \right\}$$

where S is additively inverse semirings, then R is additively inverse semirings. Define g, G: R→R as follows:

$$g\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & b^2 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \quad G\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here g is multiplicative derivation which is non-additive and G is generalized m-derivations.

2 Commutativity of Semirings:

In this section, we extend the idea of H.E. Bell and Martindale (14) to enforce commutativity in semirings by using generalized m-derivations.

Theorem 1: Let G be a generalized m-derivation on R with associated non-zero multiplicative derivation d. If $xG(x) = G(x)x$ for all $x \in R$, then R is commutative.

Proof: We have $xG(x) = G(x)x$ for all $x \in R$. Linearization yields that

$$xG(y) + yG(x) = G(x)y + G(y)x \text{ for all } x, y \in R \tag{1}$$

Replacing y by yx in (1), we have

$$xG(y)x + xyd(x) + yxG(x) = G(x)yx + G(y)x^2 + yd(x)x \text{ for all } x, y \in R \tag{2}$$

Using (1), we have $xyd(x) + yxG(x) = yG(x)x + yd(x)x$. By hypothesis this expression reduces to

$$xyd(x) = yd(x)x \text{ for all } x, y \in R \tag{3}$$

Substituting ry for y in (3), yields that

$$xryd(x) = ryd(x)x \text{ forall } x, y, r \in R \tag{4}$$

Multiplying r from right in (3) gives

$$rxyd(x) = ryd(x)x \text{ for all } x, y, r \in R \tag{5}$$

From (4) and (5), we have

$$rxyd(x) = xryd(x) \text{ for all } x, y, r \in R \tag{6}$$

This implies that either

$$d(x) = 0 \text{ or } x \in Z(R) \tag{7}$$

Let $u \in R$ such that $d(u) = 0$. By replacing x with xu in (6), we have
$xu\, ryd(x)u = rxu\, yd(x)u$ for all $x, y, r \in R$.

By using (6) twice, we get

$$xu\, ryd(x)u = ru\, xyd(x)u = (ru)xyd(x)u = x(ru)yd(x)u \quad \text{for all } x, y, r \in R$$

On replacing r with rz, we obtain

$$xu\, rz\, yd(x)u = xrz\, uyd(x)u \quad \text{for all } x, y, r \in R$$

By the assumption, we have $d(x)u = d(xu)$ this yields

$$\begin{aligned} &(xu)rz\, yd(xu) = xrz\, uyd(x)u \quad \text{for all } x, y, r \in R. \quad o\ r \\ &r(xu) = yd(xu) = xrz\, uyd(x)u \\ &rx(uz)yd(x)u = xrz\, uyd(x)u \\ &rx(uz\, y)d(xu) = xrz\, uyd(x)u \end{aligned}$$

This implies that $xruzyd(xu) = xrz\, uyd(x)u$ for all $x, y, z, r \in R$. Hence by weakly cancellation, we get that either

$$xru\, z = xrz\, uord(x)u = 0 \quad forallx, r, z \in R$$

If $xru\, z = xrz\, u$ for all $x, r, z \in R$ then we get $u \in Z(R)$. On other side, if we take $d(x)u = 0$. By substituting *d(x) = r, x = y and y = u* in (6) we have $yd(x)u\, d(y) = d(x)yu\, d(y)$ for all $x, y \in R$. and using the assumption implies that $0 = d(x)yu\, d(y)$ for all $y \in R$ Due to weakly cancellation we get that $d(x) = 0 \quad or \quad u\, d(y) = 0$ for all $y \in R$. By replacing y with ry and using weakly cancellation we get that $u = 0ord = 0$, whence in both cases we obtain that $u \in Z(R)$. On implication of (7), we have $x \in Z(R)$ for all $x \in R$, hence, is commutative.

Theorem 2. Let G be a generalized m-derivation on R with associated non-zero multiplicative derivation d. If $xG(x) + G(x)x = 0,$ for all $x \in R$, then R is commutative.

Proof. We have $xG(x) + G(x)x = 0,$ for all $x \in R$. Linearization of this expression gives

$$G(x)y + G(y)x + yG(x) + xG(y) = 0 \text{ for all } x, y \in R \tag{1}$$

Replacing y by yx in (1), we get

$$(G(x)y + G(y)x + xG(y))x + yd(x)x + xyd(x) + yxG(x) = 0 \tag{2}$$

for all $x, y \in R$. From (1) and (2), we have

$$yxG(x) + yd(x)x + xyd(x) = yG(x)x \text{ for all } x, y \in R \tag{3}$$

By adding *yxG(x)* on both sides and using the hypothesis we get

$$yd(x)x + xyd(x) = 2yG(x)x \text{ for all } x, y \in R \tag{4}$$

Replacing y by ry in (4) yields that

$$ryd(x)x + xryd(x) = 2ryG(x)x \quad \text{ for all } x, y, r \in R \tag{5}$$

Further, (4) also gives that

$$ryd(x)x + rxyd(x) = 2ryG(x)x \quad \text{ for all } x, y, r \in R$$

The last two expressions yields that

$$xryd(x) = rxyd(x) \quad \text{ for all } x, y, r \in R \tag{6}$$

Hence, we get that either $d(x) = 0$ or $x \in Z(R)$. Arguing in the similar manner as we have done in the proof of previous theorem, we achieve the required result.

From above two theorems, we get the following corollary as their immediate consequence:

Corollary 3. Let R be a prime ring. Let G be a generalized m-derivation on R with associated derivation d. If either F is commuting or skew commuting on R, then R is commutative.

3 Composition of two generalized m-derivations

In this section, we find the result that if composition of two generalized m-derivations are also generalized m-derivation then one of their associated derivation must be zero. For this result the following lemma is crucial.

Lemma 4

Let G be a generalized m-derivation with associated multiplicative derivation d of a WC semiring R and $a \in R$. If $aG(x) = 0$, for all $x \in R$ then either $a = 0$ or $d = 0$.

Proof: By hypothesis, we have

$$aG(x) = 0, \quad \text{ for all } x \in R \tag{1}$$

On replacing x with xy, we get $aG(xy) = 0$

$$aG(x)y + axd(y) = 0$$

By using (1), we get $axd(y) = 0, forallx, y \in R$

or $aRd(y) = 0$

By using the weakly cancellation property of *R*, the last expression axd$(y) = ax0$ for all $x \in R$ implies that either $a = 0ord(y) = 0, forally \in R$. This completes the proof.

Theorem 5

Let be a 2-torsion free prime ring and G_1, G_2 are generalized multiplicative derivations of with associated multiplicative derivations d_1, d_2 respectively. If the iterate (composition) G_1G_2 s also a generalized m-derivation with associated multiplicative derivation d_1d_2 that is $G_1G_2(xy) = G_1G_2(x)y + xd_1d_2(y)$ then either $d_1 = 0$ or $d_2 = 0$.

Proof: By hypothesis, G_1G_2 is generalized m-derivation with associated multiplicative derivation d_1d_2 then we have

$$G_1G_2(xy) = G_1G_2(x)y + xd_1d_2(y) \tag{1}$$

Moreover, G_1, G_2 are generalized m-derivations associated with multiplicative derivations d_1, d_2 respectively, this implies that

$$\begin{aligned} G_1G_2(xy) = G_1(G_2(xy)) = G_1(G_2(x)y + xd_2(y)) \\ = G_1G_2(x)y + G_2(x)d_1(y) + G_1(x)d_2(y) + xd_1d_2(y) \end{aligned} \tag{2}$$

From (1) and (2) and R is WC semirings therefore we get

$$G_2(x)d_1(y)+G_1(x)d_2(y)=0,\ \text{for all } x,y\in R \tag{3}$$

On substituting ry in place of y yields

$$G_2(x)d_1(r)y+G_2(x)rd_1(y)+G_1(x)d_2(r)y+G_1(x)rd_2(y)=0$$

while (3) implies that

$$G_2(x)rd_1(y)+G_1(x)rd_2(y)=0,\ \text{for all } x,y,r\in R \tag{4}$$

Now replace x by rx in (3), we have

$$\begin{aligned}G_2(rx)d_1(y)+G_1(rx)d_2(y)&=0\\ G_2(r)xd_1(y)+rd_2(x)d_1(y)+G_1(r)xd_2(y)+rd_1(x)d_2(y)&=0,\ \text{for all } r,x,y\in R\end{aligned}$$

By using (4), we get $rd_2(x)d_1(y)+rd_1(x)d_2(y)=0$
or $r\left(d_2(x)d_1(y)+d_1(x)d_2(y)\right)=0$

By using the primeness of R, we obtain

$$d_2(x)d_1(y)+d_1(x)d_2(y)=0 \tag{5}$$

Now replace x by $rd_1(z)$ in (3), we have

$$\begin{aligned}G_2\left(rd_1(z)\right)d_1(y)+G_1\left(rd_1(z)\right)d_2(y)&=0\\ G_2(r)d_1(z)d_1(y)+rd_2\left(d_1(z)\right)d_1(y)+G_1(r)d_1(z)d_2(y)+rd_1\left(d_1(z)\right)d_2(y)&=0\end{aligned} \tag{6}$$

By taking $x=d_1(z)$ in (5) and using in (6), we obtain

$$G_2(r)d_1(z)d_1(y)+G_1(r)d_1(z)d_2(y)=0 \tag{7}$$

or $G_1(r)d_1(z)d_1(y)+G_2(r)d_1(z)d_1(y)+G_1(r)d_1(z)d_2(y)=G_1(r)d_1(z)d_1(y)$

By using (3), we get

$$G_1(r)d_1(z)d_2(y)=G_1(r)d_2(z)d_1(y),\qquad \text{for all } r,y,z\ \in R$$

Now adding $G_1(r)d_1(z)d_2(y)$ on both sides and using (5), we obtain $2d_1(z)d_2(y)=0$, since R is 2-torsion free, therefore $d_1(z)d_2(y)=0$. On replacing $d_1(z)$ with a and applying Lemma (1), we get $d_2=0 or d_1(z)=0 for all z\ \in R$ i.e. $d_2=0$ or $d_1=0$.

4 Conclusion

The generalized m-derivations, have great potential to play an important role in the theory of semiring as well as other fields of mathematics (like theory of semirings with Involutions, functional analysis, C^*- algebras, B^*- algebras and linear differential equations etc). This article explores new way to study the commutativity of semirings through generalized m-derivations. The questions arise here is interesting to discuss that how the generalized m-derivations induce commutativity in semirings with involution. This article invites the researcher's to explore the results, discussed here, in the canvas of ideals in semirings.

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