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RESEARCH ARTICLE

On circuit structure of PSL (2, Z)-orbits of length eight

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Abstract

Background/Objectives: Modular group or PSL(2,Z) is a well-known group of the non-singular square matrices of order two by two with unit determinant. An Action of this group on real quadratic fields is represented by coset diagrams consisting of closed paths known as circuits. For a particular value of n , one or more than one circuits combine to form an orbit. The length of a circuit identifies all the circuits of an orbit. The main objective of this study is to explore all the circuits of length eight and their corresponding orbits. **Methods**: Some circuit equivalent properties, the group theoretical approach, along with the statistical methodology, is adopted to classify the orbits containing circuits of length eight. We also use already discovered results related to circuits of lengths two, four, and six to formulate the basis for our new results of length eight. **Findings:** We have discovered all the equivalence classes for the circuits of length eight, which are twenty-one in number. For a particular reduced number α, circuits of length eight can have all four; $\alpha, -\alpha$, its algebraic conjugate$\bar{\alpha}$ and $-\bar{\alpha}$, either in one circuit or $(\alpha)^G = (-\alpha)^G$ with $(-\alpha)^G = (-\bar{\alpha})^G$ or $(\alpha)^G = (\bar{\alpha})^G$ with $(-\alpha)^G = (-\overline{\alpha})^G$ or $(\alpha)^G = (-\overline{\alpha})^G$ with $(-\alpha)^G = (\overline{\alpha})^G$, depending upon the equivalent class of the circuit. Moreover, we have introduced reduced positions and G-midway and discovered that for any reduced numbers α_i starting from α_1 , reduced positions have a recurring pattern $\overline{\alpha_2}, -\overline{\alpha_3}, -\alpha_4, \alpha_5$, and so on. **Applications:** PSL(2, Z) orbits are entirely classified and drawn, along with the cyclically equivalent circuits of length eight.

Keywords: Modular group; coset diagram; reduced numbers; G-midway; reduced positions

1 Introduction

Group theory research is traditional for many years, as its vast applications can be represented by group theory in a variety of fields such as physics, chemistry, computer science, and even puzzles like Rubik's Cube. In the past few decades, the idea of visualizing abstract theory using geometric figures and diagrams was one of the most popular ideas in mathematical books. In this article, we focus on the action of the modular group on $Q^*(\sqrt{n})$. This field is maturing, with a wealth of well under-stood methods and algorithms. The modular group is isomorphic to $G = \langle \omega, \sigma : \omega^2 = \sigma^3 = 1 \rangle$, where ω and σ are transformations defined as $\omega : z \to -1/z$ and $\sigma : z \to 1 - 1/z$.

Let m be a square-free positive integer, take $n = k^2m$ where $k \in N$ then

$$Q^*(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : a, b = \frac{a^2-n}{c}, c \in \mathbf{Z}, c \neq 0, (a,b,c) = 1 \right\}$$

If $\alpha \in Q^*(\sqrt{n})$ and its algebraic conjugate $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$ both have opposite signs, then α is called an ambiguous number. An ambiguous number α is said to be reduced number if $\alpha > 1$ and $-1 < \overline{\alpha} < 0$. A coset diagram is a particular type of graph consisting of vertices and edges. It illustrates a permutation representation of $PSL(2,\mathbf{Z})$, the three cycles of σ are denoted by three vertices of a triangle permuted counter-clockwise by σ and two vertices that are interchanged by ω re joined by an edge. A circuit of length $2m$ denoted as $[l_1, l_2, l_3, \ldots, l_{2m}]$ is a closed path where l_1 triangles are outside, l_2 triangles are inside, and so on. Two circuits are said to be equivalent if they differ only by order of arrangement. Group action, along with their graphical properties, has also been explored in prior studies by the authors in (1). This idea is then applied in the action of the modular group with the concept of coset diagrams. We have a large number of existing studies in the broader literature that have examined this in (2; 3; 4; 5). In (6; 7), authors have proved the importance of reduced numbers in the literature of circuits of length four and six and related coset diagrams of the action of the modular group. The idea of contraction of vertices was given in (8), which helped to understand more deeply about these circuits. The Modular group is not the only group used for this group-theoretic action. Bianchi group is also used to study the evolution of ambiguous numbers in (9). The circuits of length four and their properties are studied in detail in (10) Graham Higman made remarkable contributions in this field, mainly his conjecture, related to coset diagrams, which was later partially proved in (11). Although coset diagrams are particular kinds of graphical figures, different from graph-theoretical aspects, authors in (12; 13)have discussed the graph-theoretic approach towards this particular area of research. Most of the theories of action of the modular group are focused on explaining different classifications of the elements of $Q^*(\sqrt{n})$ for a particular length of a circuit. However, a circuit generating approach has rarely been studied directly. The problem with such implementations is that there is no previous research based on the study of circuits with the same structural properties and different lengths. Thus the aims of this study are twofold: First, we define circuit generating sets and reduced positions. Secondly, we develop some techniques for the particular circuits of length eight based on previously defined notions. We follow the techniques and results from (14) to find all the equivalent and cyclically equivalent classes of circuits of length eight. In (15; 16)the authors have discussed all the G-subsets and G-orbits for the action of modular group on $Q^*(\sqrt{n}) \approx Q^*(\sqrt{k^2m})$.

2 Circuit Generating Sets

Let $X_j = \{l_1, l_2, l_3, \ldots, l_j\}$ be a set of j distinct non-zero positive integers. We call X_j "A circuit generating set" or only "generating set" as we generate circuits of even length by using all j elements from this set. It is clear from [14] that the number of non-equivalent circuits of length 2m is equal to the number of partitions of 2m. For $X_1 = \{l_1\}$ can only generate one circuit $[l_1, l_1]$ of length two, where $X_2 = \{l_1, l_2\}$ generates $[l_1, l_1]$ of length two and $[l_1, l_1, \ldots, l_1, l_2]$ of length 2m. For a circuit of length eight, X_8 is the most extensive generating set and will be discussed in detail in the next section. The following two definitions provide a basis for our main findings.

Definition 2.1: A reduced position is defined as the position of an ambiguous number in the circuit where inside triangles are becoming outside triangles and vice versa. It is represented by a dot in the coset diagram.

Definition 2.2: It is clear from the definition of the length of a circuit that a circuit of length 2m has 2m triangles. We draw a line between m^{th} and $(m+1)^{th}$ triangle of the circuit and call that line a G-midway.

G-midway divides the circuit into two segments, and these two segments have two different symmetries or patterns of circuit construction depending upon the type of circuit. Before moving on to our main result, here are some essential Lemmas which help to understand the inevitable main results.

Lemma 2.3: A circuit of length 2m has a total of 4m reduced positions.

Lemma 2.4: On a reduced position, there will be either a reduced number, a negative of a reduced number, conjugate of a reduced number, or minus conjugate of a reduced number.

Lemma 2.5: Let α be a reduced number in the reduced position in a circuit, then the next reduced position will have conjugate of some other reduced number, moving in an anti-clockwise direction. In fact, for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in Q^*(\sqrt{n})$ being distinct, reduced numbers then $\alpha_1, \overline{\alpha_2}, -\overline{\alpha_3}, -\alpha_4$ lie on four consecutive reduced positions of a circuit.

Let $X_2 = \{l_1, l_2\}$ be a circuit generating set of length two. Then we generate circuits of length 2m by finding all the possible partial sums of 2m by writing $2m = A + B$ where A and B represent the repetition of l_1 and l_2 respectively. Thus all the possible combinations of length two are $2m = (2m-1)+1 = (2m-2)+2 = \ldots = (m+1)+(m+1)$ each representing a circuit that is generated by X_2 . In general, to find the exact number of possibilities, we write $X_j = \{l_1, l_2, l_3, \ldots, l_j\}$, we can generate the following circuit of length 2m depending upon m,

$$2m = \begin{cases} 0(\mod j) \to \left[\frac{2m}{j}\right] + \left[\frac{2m}{j}\right] + \left[\frac{2m}{j}\right] + \cdots + \left[\frac{2m}{j}\right] \\ 1(\mod j) \to \left\lceil\frac{2m}{j}\right\rceil + \left\lfloor\frac{2m}{j}\right\rfloor + \left\lfloor\frac{2m}{j}\right\rfloor + \cdots + \left\lfloor\frac{2m}{j}\right\rfloor \\ 2(\mod j) \to \left\lceil\frac{2m}{j}\right\rceil + \left\lceil\frac{2m}{j}\right\rceil + \left\lfloor\frac{2m}{j}\right\rfloor + \cdots + \left\lfloor\frac{2m}{j}\right\rfloor \\ \quad\quad\quad\quad\quad\quad \cdots\cdot \\ j-1(\mod j) \to \left\lceil\frac{2m}{j}\right\rceil + \left\lfloor\frac{2m}{j}\right\rfloor + \left\lfloor\frac{2m}{j}\right\rfloor + \cdots + \left\lfloor\frac{2m}{j}\right\rfloor \end{cases}$$

Where |_| and |_| are usual floor and ceiling functions are the most generalized partial sums to form circuits of lengths 2m. The following result is the primary result of finding all equivalent circuits. We use this result later in section 3.

Lemma 2.6 [14]. The number of classes E_q of equivalent circuits with length $q, q \geq 4$, are precisely $P(q) - 1$.

Proof: Every time in determining all non-equivalent circuits of length q , we obtain a partition of q in the sense that if q entries are appearing in the circuit of length $q = q_1 + q_2 + \cdots + q_k$ for which q_1 are alike, q_2 are alike ,…,q_k are alike. We shall say that a circuit of length q has the circuit decomposition $\{q_1, q_2, \ldots, q_k\}$ if it involves q entries for which q_1 are alike, q_2 are alike,…, q_k are alike. Thus the number of non-equivalent circuits are equal to $P(q) - 1$ as $(a_1, a_1, a_1, \ldots, a_1)$ is not possible for circuits of length $q \geq 4$.

Lemma 2.7: G-midway of a circuit only exists when repeated elements are taken from the circuit generating set.

3 Circuits of Length Eight and their Generating Techniques

In this section, first, we find all the equivalent circuits of length eight, then all the cyclically equivalent circuits corresponding to each circuit are calculated using Lemma 2.6. We have also given a circuit generating technique using reduced positions and G-midway. Further, all the circuits of length eight are classified according to their orbits.

Lemma 3.1: There are precisely 21 equivalent circuits of length eight.

Proof: It is evident from Lemma 2.6 that the number of classes of equivalent circuits are $P(q) - 1$. As the number of integer partitions of 8 are 22, thus total equivalent circuits are 21 in number.

We denote all 21 equivalent circuits by $T_i, i = 1, 2, \ldots, 21$ which are shown in [Table 1] along with their generating sets and integer partitions.

Table 1. All the equivalent circuits of length eight

T_i for i=	Equivalent circuit	Generating Set	Integer Partition
1	$[l_1,l_1,l_1,l_1,l_1,l_1,l_1,l_2]$	X_2	**7,1**
2	$[l_1,l_1,l_1,l_1,l_1,l_1,l_2,l_2]$	X_2	**6,2**
3	$[l_1,l_1,l_1,l_1,l_1,l_2,l_2,l_2]$	X_2	**5,3**
4	$[l_1,l_1,l_1,l_1,l_2,l_2,l_2,l_2]$	X_2	**4,4**
5	$[l_1,l_1,l_1,l_1,l_1,l_1,l_2,l_3]$	X_3	**6,1,1**
6	$[l_1,l_1,l_1,l_1,l_1,l_2,l_2,l_3]$	X_3	**5,2,1**
7	$[l_1,l_1,l_1,l_1,l_2,l_2,l_2,l_3]$	X_3	**4,3,1**
8	$[l_1,l_1,l_1,l_1,l_2,l_2,l_3,l_3]$	X_3	**4,2,2**
9	$[l_1,l_1,l_1,l_2,l_2,l_2,l_3,l_3]$	X_3	**3,3,2**
10	$[l_1,l_1,l_1,l_1,l_1,l_2,l_3,l_4]$	X_4	**5,1,1,1**
11	$[l_1,l_1,l_1,l_1,l_2,l_2,l_3,l_4]$	X_4	**4,2,1,1**
12	$[l_1,l_1,l_1,l_2,l_2,l_2,l_3,l_4]$	X_4	**3,3,1,1**
13	$[l_1,l_1,l_1,l_2,l_2,l_3,l_3,l_4]$	X_4	**3,2,2,1**
14	$[l_1,l_1,l_2,l_2,l_3,l_3,l_4,l_4]$	X_4	**2,2,2,2**
15	$[l_1,l_1,l_1,l_1,l_2,l_3,l_4,l_5]$	X_5	**4,1,1,1,1**
16	$[l_1,l_1,l_1,l_2,l_2,l_3,l_4,l_5]$	X_5	**3,2,1,1,1**
17	$[l_1,l_1,l_2,l_2,l_3,l_3,l_4,l_5]$	X_5	**2,2,2,1,1**
18	$[l_1,l_1,l_1,l_2,l_3,l_4,l_5,l_6]$	X_6	**3,1,1,1,1,1**
19	$[l_1,l_1,l_2,l_2,l_3,l_4,l_5,l_6]$	X_6	**2,2,1,1,1,1**
20	$[l_1,l_1,l_2,l_3,l_4,l_5,l_6,l_7]$	X_7	**2,1,1,1,1,1,1**
21	$[l_1,l_2,l_3,l_4,l_5,l_6,l_7,l_8]$	X_8	**1,1,1,1,1,1,1,1**

Remark 3.2: It is straightforward concluded from [Table 1] that

(i)T_1, T_2, T_3, T_4 are four circuits of length eight that are generated from X_2 **.**

(ii)T_5, T_6, T_7, T_8, T_9 are five circuits of length eight that are generated from X_3 and $T_{10}, T_{11}, T_{12}, T_{13}, T_{14}$ are 5 circuits of length eight that are generated fromX_4 **.**

(iii) T_{15}, T_{16}, T_{17} are three circuits of length eight that are generated from X_5 **.**

(iv) T_{18}, T_{19} are two circuits of length eight that are generated from X_6 **.**

(v)T_{20} and T_{21} are the circuits of length eight that are generated from X_7 and X_8 respectively.

We now discuss each circuit and equivalent classes $E_{T_i}, i = 1, 2, \ldots, 21$.

Theorem 3.3: There are precisely 40320 circuits in the equivalence class $E_{T_{21}}$ of length eight.

Proof: It is clear that T_{21} has all eight different numbers taken from X_8, and there are $8! = 40320$ possible arrangements of eight different objects. All of these are calculated by applying $\theta \in S_8$ on $\left[l_{\theta(1)}, l_{\theta(2)}, l_{\theta(3)}, l_{\theta(4)}, l_{\theta(5)}, l_{\theta(6)}, l_{\theta(7)}, l_{\theta(8)}\right]$.

Now we establish a circuit generating technique for T_{21} . From Lemma 2.7, it is clear that this circuit has one chain of reduced positions without the G-midway. It is the only circuit of length eight which is generated by the set $X_8 = \{l_1, l_2, l_3, \ldots, l_8\}$. We take α_1 on the reduced position after l_8 inside triangles moving in the anti-clockwise direction, and next reduced positions are given by

$$\alpha_1 \to \overline{\alpha_2} \to \overline{\alpha_3} \to -\alpha_4 \to \alpha_5 \to \overline{\alpha_6} \ldots \alpha_{13} \to \overline{\alpha_{14}} \to \overline{-\alpha_{15}} \to -\alpha_{16} \tag{1}$$

This circuit is ultimately shown in Figure 1 with all reduced positions as dots.

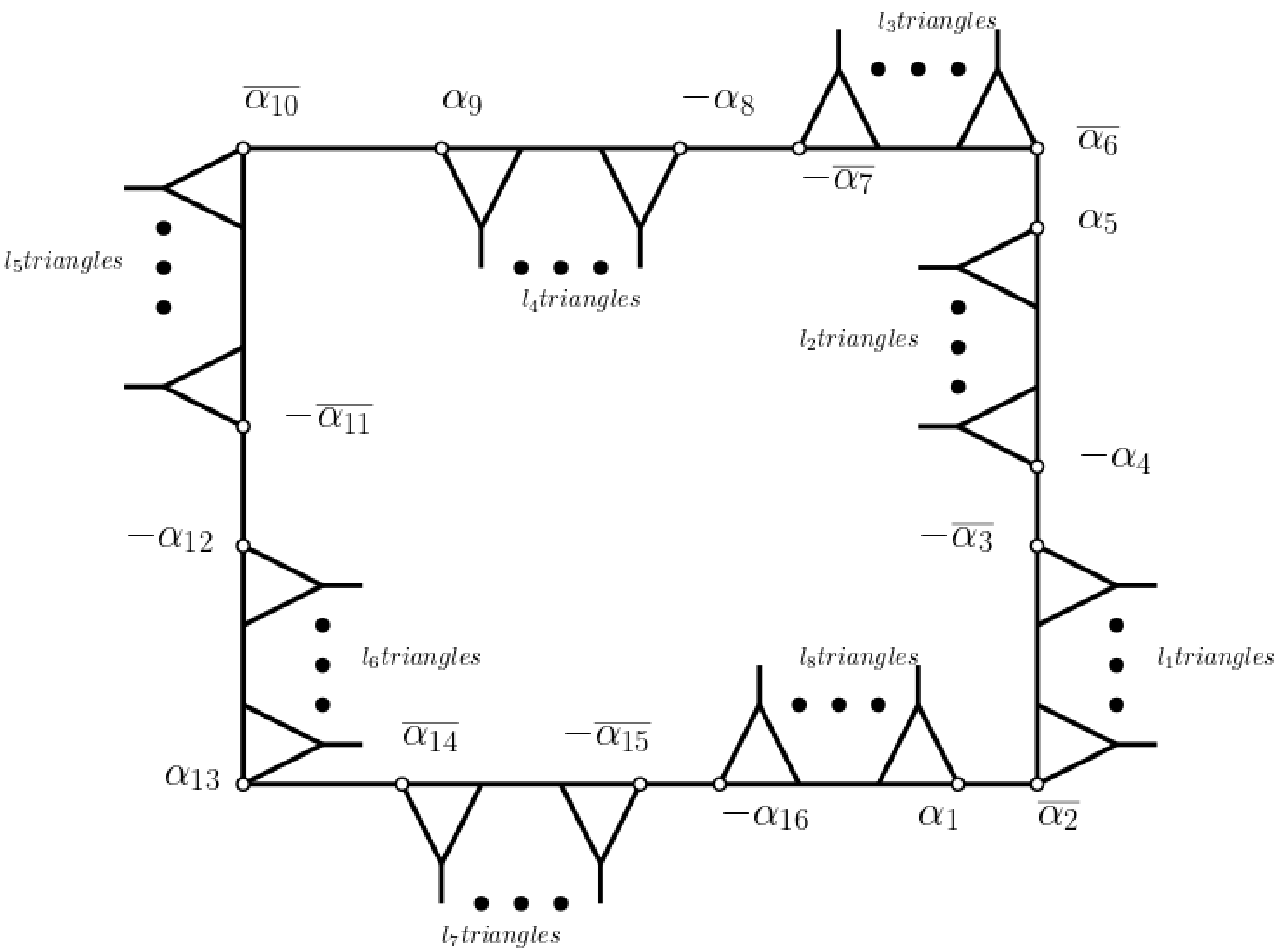

Fig 1. Coset Diagram for the Circuit $T_{21} = [l_1, l_2, \ldots, l_7, l_8]$

This idea is generalized by taking length 2m which is generated by the set $X_{2m} = \{l_1, l_2, l_3, \ldots, l_{2m}\}$. We take α_1 on the reduced position after l_{2m} inside triangles moving in the anti-clockwise direction, and next reduced positions are given by

$$\alpha_1 \to \overline{\alpha_2} \to -\overline{\alpha_3} \to -\alpha_4 \to \alpha_5 \to \overline{\alpha_6} \ldots \alpha_{4m-3} \to \overline{\alpha_{4m-2}} \to \overline{\alpha_{4m-1}} \to -\alpha_{4m} \tag{2}$$

This complete circuit, along with reduced positions, is illustrated in [Figure 2].

Theorem 3.4: For the circuit $T_{21} = [l_1, l_2, \ldots, l_7, l_8]$ of length eight, $(\alpha_i)^G, i = 1, 5, 9, 13$ are the only reduced numbers in the circuit.

Proof: Since there is no G-midway, therefore, all the reduced numbers follow the same pattern in the reduced positions, as shown in [Equation 1], and we have a reduced number after skipping four reduced positions. Hence the result.

Corollary 3.5: For the circuit, $T_{21} = [l_1, l_2, \ldots, l_7, l_8]$ the reduced number α_i , its algebraic conjugate $\overline{\alpha_i}$, negative reduced number $-\alpha_i$, and negative conjugate $-\overline{\alpha_i}$, for each i belong to different circuits.

Proof: [Equation 1] suggests that this circuit contains all distinct 16 subscripts of some reduced numbers. Thus the equivalence classes are all different and belong to all different circuits.

Theorem 3.6: There are precisely 2520 cyclically equivalent circuits in the equivalence class $E_{T_{21}}$.

Proof: $T_{21} = [l_1, l_2, \ldots, l_7, l_8]$ contains eight different numbers, which can be arranged in 16 ways without changing the ordering of numbers, which makes a cyclically equivalent class for $T_{21} = [l_1, l_2, \ldots, l_7, l_8]$. Dividing all 40320 permutations of T_{21} on 16 gives 2520 cyclically equivalent circuits, each containing 16 circuits.

It is not possible here to mention all of these 2520 cyclically equivalent classes; however, few are; $E^c_{[h,l_2,l_3,l_4,l_5,l_6,h_7,l_8]}, E^c_{[l_2,h_7,l_3,l_4,l_5,l_6,l_7,l_8]}, E^c_{[l_2,l_1,l_3,l_5,l_4,l_6,h_7,l_8]}$ and so on.

Now we discuss $T_1 = [l_1, l_1, l_1, l_1, l_1, l_1, l_1, l_2]$ and its cyclically equivalent classes in detail.

Theorem 3.7: There are precisely eight circuits in the equivalent class E_{T_1} .

Proof: Total number of arrangements of 8 elements from which seven are alike are given by $\frac{8!}{7!} = 8$.

Moreover,

$$E_{T_1} = E^c_{T_1} = \{[l_1,l_1,l_1,l_1,l_1,l_1,l_1,l_2], [l_1,l_1,l_1,l_1,l_1,l_1,l_2,l_1], [l_1,l_1,l_1,l_1,l_1,l_2,l_1,l_1], [l_1,l_1,l_1,l_1,l_2,l_1,l_1,l_1]$$
$$[l_1,l_1,l_1,l_2,l_1,l_1,l_1,l_1], [l_1,l_1,l_2,l_1,l_1,l_1,l_1,l_1], [l_1,l_2,l_1,l_1,l_1,l_1,l_1,l_1], [l_2,l_1,l_1,l_1,l_1,l_1,l_1,l_1]\}$$

above is the complete cyclically equivalent class of circuits of length eight.

It is important to note that the circuit $T_1 = [l_1, l_1, l_1, l_1, l_1, l_1, l_1, l_2]$ is generated by the set $X_2 = \{l_1, l_2\}$, which plays a pivotal role in the symmetries of its generating techniques. From the short review above, key findings emerge: G-midway divides the circuit into two segments. We define two different techniques for each segment of a circuit and join these to get a complete circuit.

Reduced positions describe the complete circuit in all aspects. Defining all the reduced positions give complete information about the structure of the orbit. In our case, the first reduced position α_1 is precisely at the ending corner of l_2 inside triangles moving in the anti-clockwise direction, and the G-midway is between the 8th and 9th reduced position. The following pattern shows all the reduced positions of the first half-section before the G-midway.

$$\alpha_1 \to \overline{\alpha_2} \to -\overline{\alpha_3} \to -\alpha_4 \to \alpha_5 \to \overline{\alpha_6} \to -\overline{\alpha_7} \to -\alpha_8 \tag{3}$$

The reduced positions for the second half of the circuit after G-midway follow the following pattern.

$$\alpha_{2i-1} \to \overline{\alpha_{2i}} \to \overline{\alpha_{21-3}} \to -\alpha_{2i-2} \text{ for } i = 4, 2 \tag{4}$$

The following results can be drawn straightforward from the above discussion.

Theorem 3.8. For the circuit $T_1 = [l_1, l_1, l_1, l_1, l_1, l_1, l_1, l_2]$ of length eight, we have $(\alpha_i)^G = (-\overline{\alpha_i})^G$ for $i = 1, 3, 5, 7$

Proof: To prove this theorem, we prove that each α_i and $-\overline{\alpha_i}$ have the same circuit for $i = 1, 3, 5, 7$.

Put $i = 2$ in [Equation 4], clearly shows that $-\overline{\alpha_i}$ lies on the second segment after G-midway. Similarly, if one reduced number is at one side of the G-midway, its negative conjugate lies on the other side. Hence the result.

Remark 3.9: The literature review shows that for the circuit $T_1 = [l_1, l_1, l_1, l_1, l_1, l_1, l_1, l_2]$ containing α_i , there exists another equivalent circuit containing $\overline{\alpha_i}$ for each reduced number of the circuit. Thus it is evident that $(\bar{\alpha}_i)^G = (-\alpha_i)^G$ for $i = 1, 3, 5, 7$ is true for other equivalent circuits. Moreover, that equivalent circuit also has the property $(\alpha_i)^G = (-\bar{\alpha}_i)^G$ for $i = 2, 4, 6, 8$ as vice versa.

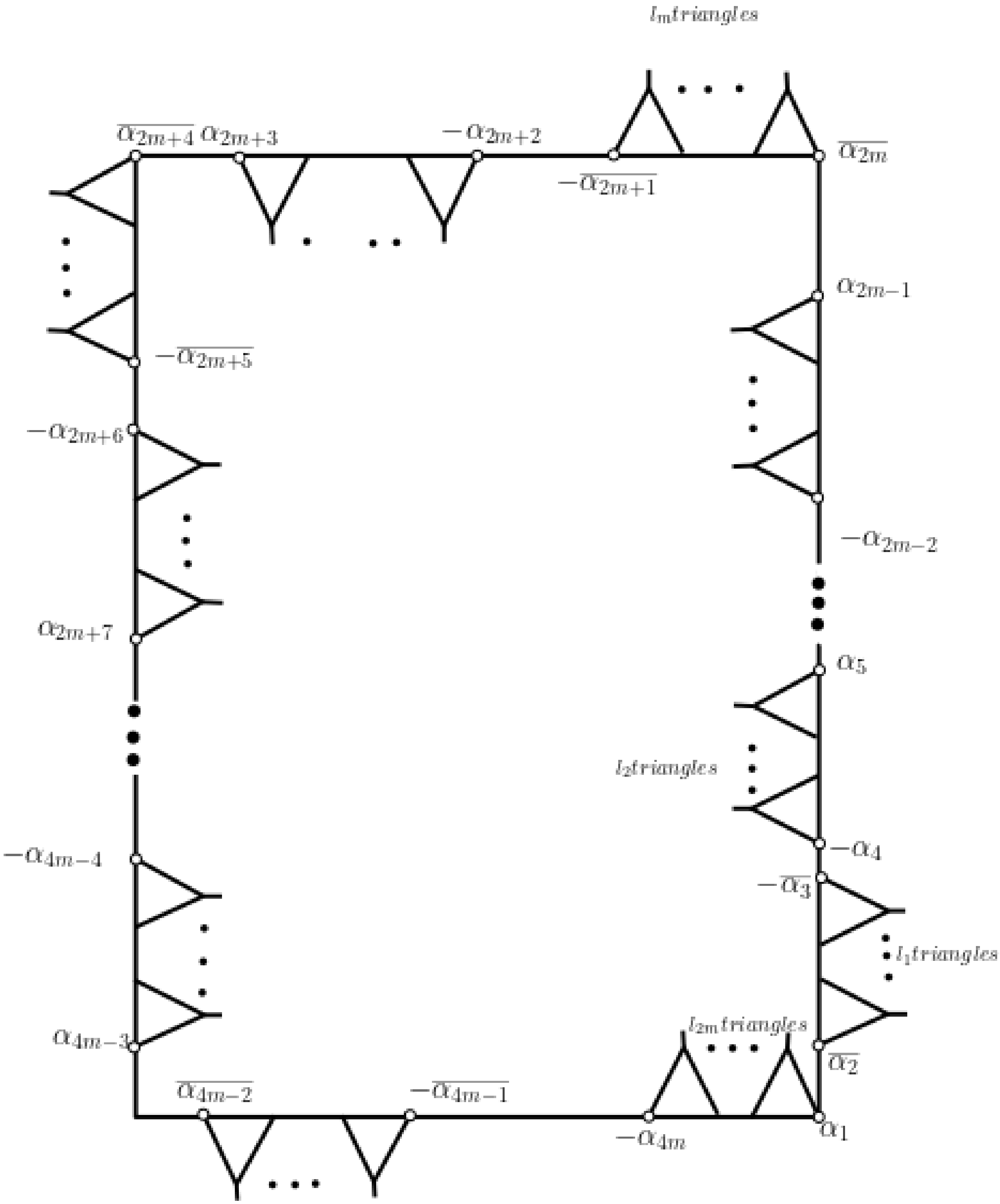

Fig 2. A Generalized Circuit of the type $[l_1, l_2, \ldots, l_{2m-1}, l_{2m}]$

[Figure 2] summarizes the findings and contributions for $T_1 = [l_1, l_1, l_1, l_1, l_1, l_1, l_1, l_2]$ via the coset diagram. Now we give a general result for the cyclically equivalent circuits of length eight.

Theorem 3.9: For any circuit $T_i, i = 1, 2, \ldots, 21$ of length eight with integer partition $8 = \pi_1 + \pi_2 + \ldots + \pi_k$ such that $T_i \in X_{\pi_k}$

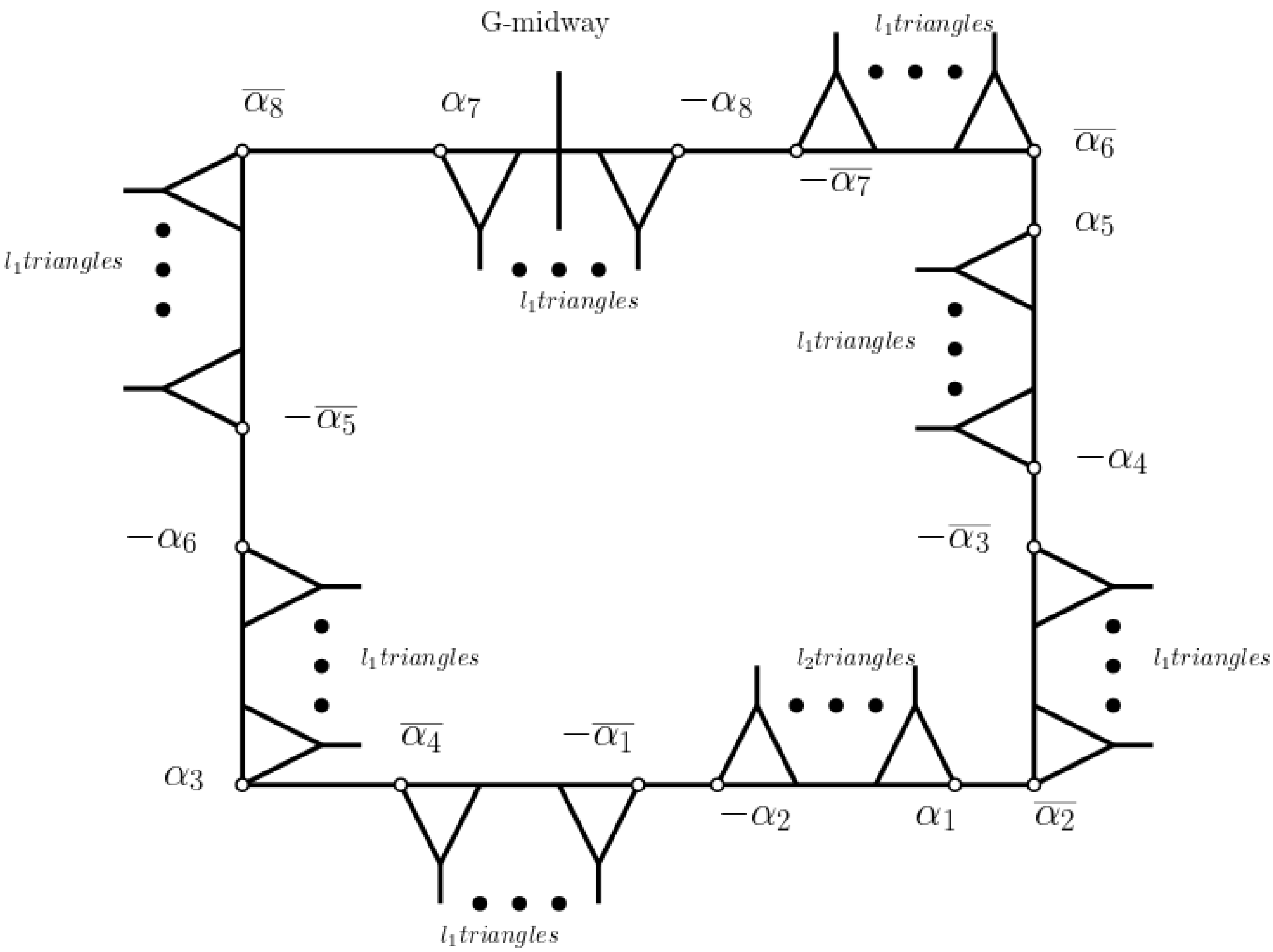

Fig 3. Coset Diagram for the Circuit $T_1 = [l_1, l_1, l_1, l_1, l_1, l_1, l_1, l_2]$

then the number of cyclically equivalent circuits are given by $\left|E^c_{T_i}\right| = \frac{8!}{\pi_1!\pi_2!\pi_3!\ldots\pi_k!}$

Proof: For each $T_i \in X_{\pi_k}$ with integer partition $8 = \pi_1 + \pi_2 + \cdots + \pi_k$ has l_1 elements π_1 times, l_2 elements π_2 times, and some other elements π_k times. Since the number of circuits in each cyclically equivalent class is the total number of arrangements of $T_i \in X_{\pi_k}$ therefore each repeated element with factorial will be in the denominator of $8!$.

Example 3.10: We have 280 cyclically equivalent circuits for
$T_7 = [l_1, l_1, l_1, l_1, l_2, l_2, l_2, l_3] \in X_3 \left|E^c_{T_7}\right| = \frac{8!}{4!3!1!} = 280$.

[Table 2] shows the number of cyclically equivalent circuits for each $T_i, i = 1, 2, \ldots, 21$.

Table 2. Number of Cyclically Equivalent circuits of Length Eight

Equivalence Class E_{T_i}	Number of Cyclically Equivalent Circuits $\left|E^c_{T_i}\right|$
E_{T_1}	$\frac{8!}{7!} = 8$
E_{T_2}	$\frac{8!}{6!2!} = 28$
E_{T_3}	$\frac{8!}{5!3!} = 56$

Continued on next page

Table 2 continued

Equivalence Class E_{T_i}	**Number of Cyclically Equivalent Circuits** $E^c_{T_i}$
E_{T_4}	$\frac{8!}{4!4!} = 70$
E_{T_5}	$\frac{8!}{6!} = 56$
E_{T_6}	$\frac{8!}{5!2!} = 168$
E_{T_7}	$\frac{8!}{4!3!} = 280$
E_{T_8}	$\frac{8!}{4!2!2!} = 420$
E_{T_9}	$\frac{8!}{3!3!2!} = 560$
$E_{T_{10}}$	$\frac{8!}{5!} = 336$
$E_{T_{11}}$	$\frac{8!}{4!2!} = 840$
$E_{T_{12}}$	$\frac{8!}{3!3!} = 1120$
$E_{T_{13}}$	$\frac{8!}{3!2!2!} = 1680$
$E_{T_{14}}$	$\frac{8!}{2!2!2!2!} = 2520$
$E_{T_{15}}$	$\frac{8!}{4!} = 1680$
$E_{T_{16}}$	$\frac{8!}{3!2!} = 3360$
$E_{T_{17}}$	$\frac{8!}{2!2!2!} = 5040$
$E_{T_{18}}$	$\frac{8!}{3!} = 6720$
$E_{T_{19}}$	$\frac{8!}{2!2!} = 10080$
$E_{T_{20}}$	$\frac{8!}{2!} = 20160$
$E_{T_{21}}$	$\frac{8!}{1!} = 40320$

4 Conclusion

We have classified all the circuits of length eight into 21 equivalent circuits $T_i, i = 1, 2, \ldots, 21$. Moreover, by finding all classes of cyclically equivalent circuits corresponding to each, T_i we have completed the classification of G-orbits for PSL(2, Z) with length eight. We have introduced reduced positions and found that for any reduced numbers α_i , starting from α_1 ,and reduced positions have a recurring pattern $\overline{\alpha_2}, \overline{-\alpha_3}, -\alpha_4, \alpha_5$, and so on. It is now easy to draw any circuit of length eight with accurate positions of all the reduced numbers, algebraic conjugates, negatives, and negative conjugates of reduced numbers without calculating all the ambiguous numbers of the circuits of an orbit. We have calculated all the equivalent and cyclically equivalent circuits under five different conditions on $\alpha, -\alpha, \bar{\alpha}$ and $-\bar{\alpha}$.

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