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RESEARCH ARTICLE

Some special structures of S*and A* semirings

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Abstract

Objectives: The main objective of this research article is to study the semiring structures, we have majorly focused on the constrains under which the structures of S*and A* semirings are additively and/or multiplicatively idempotent. We have also concentrated on the study of structures of totally ordered S* and A* semirings. **Methods:** We have imposed singularity, cancellation property, Integral Multiple Property (IMP) and some other constrains on both semirings. **Findings:** when we imposed totally ordered condition on these two semirings we observed that the additive structure takes place as a maximum addition. **Applications:** The proposed idempotents have wide applications to computer science, dynamical and logical systems, cryptography, graph theory and artificial intelligence.

Keywords: Almost idempotent; idempotent; integral multiple property; multiplicatively subidempotent; periodic; rectangular band; singular semigroup; zeroid

1 Introduction

The first formal definition of semiring was introduced in the year 1934 by Vandiver. However the developments of the theory in semirings have been taking place since 1950. A semiring is basic structure in Mathematics. The ring theory and semigroup theory influenced on the developments of the semiring theory and its ordering. The first mathematical structure we encounter is the natural number set N is a semiring [(1)].

Discrete optimization problems would be linearized as idempotent semirings [(1)]. This semirings are connected with automata theory, optimization theory. This idempotent semiring is richly applicable to Computer Science, we can suitably prototypical the structures of plans and evolution ordering and systems to associate with the dynamical systems as well as logics [(2)]. Some other applications of semiring areas are cryptography, graph theory artificial intelligence.

The research article is drafted as follows: In section 1, we have given basic introduction on semiring theory and we have also presented the definitions used in theorems, Section 2 contains the structures of S^* semiring and totally ordered S^* semiring. In section 3, we have given the structures of A^* semiring and its ordering and the last section is conclusion.

Preliminaries

Definition 1.1:

An algebraic structure (S, +,●) is termed as semiring if the additive and multiplicative reducts are semigroups and u(x + y) = ux + uy and (x + y)u = xu + yu for every u, x, y in S.

Definition 1.2:

An additive semigroup is said to be additively idempotent if u + u = u for all u in S.

A multiplicative semigroup is multiplicatively idempotent or band if $u^2 = u$ for all u in S.

If both (S, +) and (S, ●) are idempotents then S is known as an idempotent semiring[2,3].

Definition 1.3:

A semiring is termed as mono-semiring if u + x = ux for all u, x in S.

Definition 1.4:

A multiplicative semigroup is assumed to be left (right) singular if ux = u (xu = x) for all u, x in S.

An additive semigroup is said to be left (right) singular if u + x = u (u + x = x) for all u, x in S.

Definition 1.5:

An element u is periodic if $u^m = u^n$, where m and n are positive integers.

A multiplicative semigroup is said to be periodic if every one of their elements is periodic.

An element u is periodic if mu = nu, where m and n are positive integers.

An additive semigroup is said to be periodic if every one of their elements is periodic.

Definition 1.6:

An additive semigroup (multiplicative semigroup) is rectangular band if u = u + x + u (u = uxu) for all u, x in S.

Definition 1.7:

A semiring is said to be zerosum if u + u = 0 for all u in S.

A semiring is said to be zero square if $u^2 = 0$ for all u in S

Definition 1.8:

In a semiring S, the semigroup (S, ●) is zeroid if for all u in S such that ux = x or xu = x for some x in S.

In a semiring S, the additive semigroup is zeroid if for all u in S such that u + x = x + u = x for some x in S.

Definition 1.9:

An additive semigroup (multiplicative semigroup) is commutative if u + x = x + u (ux = xu) for all u, x in S.

Definition 1.10:

A component u in a multiplicative semigroup is known as left and right cancellable, if ux = uy (xu = yu) for any x, y in S implies x equals to y.

An element u in an additive semigroup is known as left and right cancellable, if u + x = u + y and x + u = y + u for any x, y in S implies x equals to y.

Definition 1.11:

A semiring is almost idempotent if $u + u^2 = u^2$ for all u, x in S.

Definition 1.12:

A semiring S is said to satisfy the Integral Multiple Property (IMP) if u^2 = na for all a in S where the positive integer n depends on the element u.

Definition 1.13: In a semiring S, an element u is Multiplicatively Subidempotent if $u + u^2 = u$.

Definition 1.14:

In a totally ordered semiring (S, +, ●, $\leq$) (i) (S, +, $\leq$) is p.t.o, if u + x $\geq$ u, x for all u, x in S. (ii) (S, ●, $\leq$) is p.t.o, if ux $\geq$ u, x for all u, x in S.

Definition 1.15:

A totally ordered semigroup (S, +, $\leq$) is assumed to be non-negatively (non-positively) ordered if every element of S is non-negative/non-positive.

Definition 1.16:

An element *u* in a totally ordered semiring is said to be a minimal/maximal if u $\leq$ x (u $\geq$ x) for every u $\in$S.

Note:

1. In this paper a semiring S is said to be a S^* semiring if it satisfies the identity $u^2 + x^2 = ux$ for all u, x in S.[4]
2. In this paper a semiring S is said to be a A^* semiring if it satisfies the identity u + ux + u = u for all u, x in S[5,6].
3. In this research article totally ordered is represented by t.o and positively totally ordered by p.t.o, Integral multiple property by IMP.

2 Some Structures of S* Semiring

Lemma 2.1: Suppose S is a S^* semiring. Then (S, +) is idempotent in the following cases.[7] [8]

(a) S contains multiplicative identity.

(b) (S, •) is either left or right cancellative.

(c) (S, •) is either left singular or right singular semigroup.

Proof: (a) By the definition of S^* semiring we have $u^2 + x^2 = ux$ for all u, x in S
Also S contains multiplicative identity which implies $1^2 + 1^2 = 1$
$\Rightarrow$ u + u = u for all u in S
Thus (S, +) is idempotent
(b) By hypothesis we have $u^2 + u^2 = u^2$ for all u in S$\Rightarrow$ u (u + u) = u.u
On application of (S, •) left cancellation law to above equation it takes the form
u + u = u for all u in S
Therefore (S, +) is idempotent
In a similar manner we can prove in the case of (S, •) right cancellation also.
(c) Given that $u^2 + u^2 = u^2$ for all u in S
Since (S, •) is left singular semigroup u.u = u for all u in S $\Rightarrow$ u + u = u for all u in S
Therefore (S, +) is idempotent
Similarly, we prove that (S, +) is idempotent if (S, •) is right singular semigroup
Example: Here S = {u, x}is a set with two elements u and x which satisfies S^* semiring property, (S, +) and (S, •) idempotent conditions only.

+	u	x
u	u	u
x	u	x

•	u	x
u	u	u
x	u	x

Proposition 2.2: Let S be a S^* semiring. If S is almost idempotent semiring and (S, +) is right cancellative, then S is an idempotent semiring.
Proof: By hypothesis we have $u^2 + u^2 = u^2$ for all u in S $\rightarrow$(1)
Also S is an almost idempotent semiring $u + u^2 = u^2$ for all u in S $\rightarrow$ (2)
From the above two equations we obtain $u^2 + u^2 = u + u^2$
By applying (S, +) right cancellation we get $u^2 = u \rightarrow (3)$
Thus (S, •) is idempotent
Again let us take first equation $u^2 + u^2 = u^2$
Using equation(3) in above we get u + u = u
Therefore (S, +) is idempotent
Hence S is an idempotent semiring.
Remark : In S* semiring we can prove square of an element is periodic without any conditions.
Proof: Since $u^2 + u^2 = u^2 \Rightarrow 2u^2 = u^2$ which implies $3u^2 = 2u^2$ for all u in S
Thus square of an element is periodic with respect to addition
Theorem 2.3: Let S be a S^* semiring.
(a) If S is zerosum semiring, then S is a zerosquare semiring.
(b) If S is IMP, then (S, +) is periodic.
Proof: (a) By S^* semiring we have $u^2 + u^2 = u^2$ for all u in S
By the definition of zerosum semiring $u^2 + u^2 = 0$
then above equation becomes $u^2 = 0$for all u in S
Thus S is a zerosquare semiring
(b) We have u2 + u2 = u2 for all u in S $\rightarrow$ (1)
Since S is IMP, $u^2 = nu$ for some positive integer n

Then first equation takes the form nu + nu = nu which implies 2nu = nu

Therefore (S, +) is periodic

Proposition 2.4: Let S be a S^* semiring.

(i) If (S, •) is idempotent, then S is a mono semiring.

(ii) If S is multiplicative subidempotent and (S, +) is left cancellative, then (S, •) is periodic.

Proof: (i) We have $u^2 + x^2 = ux$ for all u, x in S

Using (S, •) idempotent in above equation then it becomes u + x = ux for all u,x in S

Therefore, S is mono semiring

(ii) By the definition of S^* semiring $u^2 + u^2 = u^2$ for all u in S $\rightarrow$ (1)

Since S is multiplicative subidempotent $u + u^2 = u$ for all u in S

This implies $u^2 + u^3 = u^{2 \rightarrow (2)}$

From (above two equations we get $u^2 + u^2 = u^2 + u^3$

By application of (S, +) left cancellation the above equation reduces to $u^2 = u^3$

Therefore (S, •) is periodic

Theorem 2.5: Let S be a S^* semiring and mono semiring. Then (S, +) and (S, •) are periodic.

Proof: By hypothesis $u^2 + u^2 = u^2$ for all u in $S^{\rightarrow (1)}$

Since S is mono semiring $u + u = u^2$ for all u in S

Then first equation (u + u) + (u + u) = (u + u) $\Rightarrow$ 4u = 2u

Therefore (S, +) is periodic

Also from first equation $u^2 . u^2 = u^2 \Rightarrow u^4 = u^2$

Therefore (S, •) is periodic

Theorem 2.6: If S is a S^* semiring and (S, •) is rectangular band, then S contains multiplicatively idempotents.

Proof: We have $u^2 + x^2 = ux$ for all u, x in S $^{\rightarrow (1)}$

Since (S, •) is rectangular band uxu = u $^{\rightarrow (2)}$

From first equation $(u^2 + x^2)$ u = uxu

Using second equation in above we get $(u^2 + x^2)$ u = u $\Rightarrow$ $(u^2 + x^2)$ ux = ux

This implies $(u^2 + x^2)(u^2 + x^2) = (u^2 + x^2)$

Therefore, S is multiplicatively idempotents

Theorem 2.7: Let S be a totally ordered S^* semiring. If S contains multiplicative identity and (S, +) is p.t.o, then

(a) (S, •) is non-positively ordered.[9]

(b) 1 is the minimum element.

(c) u + x = x + u = max (u, x) for all u, x in S.[10,11]

Proof: (a) Consider $u^2 + x^2 = ux$ for all u, x in S

Since S contains multiplicative identity then $1 . x = 1^2 + x^{2 \Rightarrow x = 1 + x^2}$

Using (S, +) p.t.o we get $x \geq x^2$ for all x in S

Therefore (S, •) is non-positively ordered

(b) Again let us consider $1 + x^2 = 1 . x$ for all 1, x in S

Since (S, +) is p.t.o $\Rightarrow x = 1 + x^{2 \geq 1} \Rightarrow x \geq 1$

Therefore 1 is the minimum element

(c) By lemma 2 1 of (a) we have (S, +) is idempotent

Let us suppose u, x $\in$ S

Let u < x then $u + u \leq u + x \leq x + x \Rightarrow u \leq u + x \leq x \rightarrow$ (1)

Thus from above $u + x \leq x \rightarrow$ (2)

By the definition of (S, +) p.t.o we have u + x $^{\geq x} \rightarrow$ (3)

From equations (2) and (3) u + x = x = max (u, x) for all u, x in S $\rightarrow$ (A)

Again let us take u < x

Then $u + u \leq x + u \leq x + x \Rightarrow u \leq x + u \leq x \rightarrow$ (4)

Using (S, +) is idempotent in above we obtain $x + u \leq x \rightarrow$(5)

Also by (S, +) p.t.o we have x + u $^{\geq x} \rightarrow$ (6)

From equations (5) and (6) we obtain x + u = x = max (u, x) for all u, x in S $\rightarrow$ (B)

Therefore from equations A and B we obtain u + x = x + u = max (u, x)

Similarly, in the case, if x < u we can also prove that

u + x = x + u = max (u, x) for all u, x in S

3 Some Structures of A* Semiring

Proposition 3.1: Let S be a A^* semiring, then (S, +) is periodic in the following cases.

(i) S is IMP.

(ii) (S, •) is left singular semigroup.

Proof: **(i)** By the definition of A^* semiring we have $u + u^2 + u = u$ for all u in S

Since S is IMP then $u^2 = nu$ for some positive integer n

$\Rightarrow u + nu + u = u \Rightarrow (n + 2)\,u = u$

Thus (S, +) is periodic

(ii) Again consider $u + ux + u = u$ for all u, x in S

Since (S, •) is left singular semigroup then $ux = u$ for all u, x in S

Then above equation takes the form $u + u + u = u \Rightarrow 3u = u$

Therefore (S, +) is periodic

Theorem 3.2: Let S be a A^* semiring. If S is zerosum semiring, then

(i) (S, •) is idempotent.

(ii) (S, •) is left singular semigroup.

Proof: **(i)** Given that $u + u^2 + u = u$ for all u in S $\rightarrow$ (1)

Since S is zerosum semiring $u + u = 0 \rightarrow (2)$

$\Rightarrow u + u + u^2 + u = 0 + u^2 + u \Rightarrow u + u + u^2 + u = u^2 + u$

Using first equation in above we get $u + u = u^2 + u$

Using second equation in above we get $0 = u^2 + u \Rightarrow u^2 + u + u = u$

Which implies $u^2 = u$ for all u in S

Therefore (S, •) is idempotent

(ii) By hypothesis we have $u + ux + u = u$ for all u, x in S

This implies $u + u + ux + u = u + u$

Since S is zerosum semiring then above equation becomes as

$0 + ux + u = 0 \Rightarrow ux + u = 0$

By adding 'u' to both sides of above equation and by applying definition of zerosum semiring it leads to ux=u for all u, x in S

Hence (S, •) is left singular semigroup

Proposition 3.3: Let S be a A*semiring. If S is mono semiring, then (S, +) and (S, •) are periodic.

Proof: We have $u + u^2 + u = u$ for all u in S $\rightarrow$ (1)

Using mono semiring in equation (1) then above equation implies $u^4 = u$

Therefore (S, •) is periodic

Again if we apply mono semiring definition in first equation we obtain

$4u = u$ for all u in S

Therefore (S, +) is periodic

Hence (S, +) and (S, •) are periodic

Proposition 3.4: Let S be a A^* semiring, Then

(a) S contains additively idempotents.

(b) Zeroid of S is Zd = {u + ux & ux + u} for all u, x in S is multiplicative ideal.

Proof: **(a)** We have $u + u^2 + u = u$ for all u in S $\rightarrow$ (1)

$\Rightarrow u + u^2 + u + u^2 = u + u^{2 \Rightarrow (u+u^2)+(u+u^2)=(u+u^2)}$

Again from (1) $u + u^2 + u = u \Rightarrow u^2 + u + u^2 + u = u^2 + u$

Which implies $(u^2 + u) + (u^2 + u) = (u^2 + u)$

Therefore, S contains additively idempotents

(b) Given that $u + ux + u = u$ for all u, x in S

$\Rightarrow su = s\,(u + ux + u) \Rightarrow su = su + sux + su \Rightarrow su = su + s\,(ux + u)$

$\Rightarrow su = s\,(ux + u) \in Zd$

Also $(ux + u)s \in Zd$

Therefore, Zeroid denoted by Zd is a multiplicative ideal

Theorem 3.5: Let S be a A^* semiring. Then (S, +) is idempotent in the following cases.

(i) S is almost idempotent.

(ii) S is Multiplicatively subidempotent semiring.

Proof: **(i)** By hypothesis we have u + ux + u = u for all u, x in S $\rightarrow$ (1)
Since S is almost idempotent u + u^2 = u^2 for all uin S $\rightarrow$ (2)
From (1) u + u^2 + u = u for all u in S $\Rightarrow$ u^2 + u = u by (2)
$\Rightarrow$ u + u^2 + u = u + u $\Rightarrow$ u = u + u by (1)
Therefore (S, +) is idempotent
(ii) We have u + u^2 + u = u for all u, x in S $\rightarrow$ (1)
Then (1) becomes u + u = u for all u in S
Therefore (S, +) is idempotent
Lemma 3 .6: Let S be a A^* semiring. If S is multiplicatively idempotent, then (S, +) is idempotent.
Proof: A^* semiring let us consider u + ux + u = u for all u, x in S $\rightarrow$ (1)
Since S is multiplicative idempotent u^2 = u for all u in S
Then first equation implies u + ux + u^2 = u $\Rightarrow$ u + u (x + u) = u
$\Rightarrow$ u + u (x + u) + u = u + u
Which implies u = u + u for all u in S
Therefore (S, +) is idempotent
Example: Here S = {u, y} is a set with two elements u and y which satisfies given conditions of lemma 3.6. i,e A*semiring, multiplicatively idempotent and (S, +) idempotent.

+	u	y
u	u	u
y	y	y

•	u	y
u	u	y
y	u	y

Theorem 3.7: Suppose S is a t.o A^* semiring, S is multiplicatively idempotent and (S, +) is p.t.o, then u + x = x + u = max (u, x) for all u, x in S.
Proof: By above lemma 3.6, we have (S, +) is idempotent
Let us suppose u, x $\in$S
Let u < x then u + u $\leq$ u + x $\leq$ x + x $\Rightarrow$ u $\leq$ u + x $\leq$ x$\rightarrow$ (1)
Thus, from above u + x $\leq$ x$\rightarrow$ (2)
By the definition of (S, +) p.t.o we have u + x $\geq$ x$\rightarrow$ (3)
From equations (2) and (3) u + x = x = max (u, x) for all u, x in S $\rightarrow$ (A)
Again, let us take u < x
Then u + u $\leq$ x + u $\leq$ x + x $\Rightarrow$ u $\leq$ x + u $\leq$ x $\rightarrow$ (4)
Using (S, +) is idempotent in above we obtain x + u $\leq$ x $\rightarrow$(5)
Also, by (S, +) p.t.o we have x + u $\leq$ x $\rightarrow$ (6)
From equations (5) and (6) we obtain x + u = x = max (u, x) for all u, x in S $\rightarrow$ (B)
Therefore from equations (A) and (B) we obtain u + x = x + u = max (u, x)
Similarly, in the case, if x < u we can also prove that
u + x = x + u = max (u, x) for all u, x in S
Theorem 3.8: Let S be a totally ordered A^* semiring. If (S, +) is p.t.o and S contains multiplicative identity, then 1 is the maximum element. [9]
Proof: We have u + ux + u = u for all u, x in S
$\Rightarrow$1 + 1 . x + 1 = 1 for all 1, x in S this implies 1 + x + 1 = 1
By applying (S, +) p.t.o to above equation we get 1 $\geq$ 1, x
Which implies 1 $\geq$ x for all x in S
Therefore, 1 is the maximum element

4 Conclusion

In this study we have discussed and compared several additive and multiplicative structures of S^* and A^* semirings. We have also framed examples for the propositions and theorems. Our future work can be continued by applying some other constrains on S* and A* semirings.

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