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A result on Banach Space using Property E.A

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Abstract

Objective/Aim: To establish the existence of common fixed point theorem for four self-mappings in Banach space. **Method:** This method includes using weaker conditions like property E.A and weakly compatible mappings. **Findings:** Unique common fixed point is generated using different conditions. This result is supported by the provision of a suitable example. **Novelty/Improvement:** The concept of property E.A along with weakly compatible mappings is employed and is weaker than the concept of continuity and compatibility.

Keywords: Fixed point; metric space; Normed space; Banach space; EA property; weakly compatible mappings

1 Introduction

Fixed point theory is one of the most interesting topics of modern mathematics and might be taken as the main subject of analysis. For the past many years, fixed point theory has been evolved as the area of research for many researchers. Fixed point theorems contribute major share in the development of the research. To mention a few Banach contraction principle is one such result. Common fixed point theorems for a family of mappings proved in complete metric space[(1)]. The study of discontinuous and non-compatible mappings in fixed point theory is visible from the contribution of[(2)] and[(3)]. A fixed point theorem using the continuity and weakly compatible mappings established on complete metric space[(4)]. Thereafter some more results proved on Banach space[(5)]. Further several theorems[(6–8)] and[(9)] are being generated on Banach space using various conditions. Now the emphasis of this paper is to prove a result on Banach space without continuity condition and also adopting the property E.A to establish a common fixed point theorem. At the end of the paper two corollaries are being generated as consequences of our main result.

2 Preliminaries

Now we give some definitions which are useful in proving our result.

Definition 2.1[(2)]:We define mappings G and H are *self mappings on* Banach space X ,then the pair (G,H) is called weakly commuting on X if

$\|GHx - HGx\| \leq \|Gx - Hx\|$ for all $x \in X$.

Definition 2.2: In a Banach space X, We define mappings G and H are compatible if $\|GH\alpha_k - HG\alpha_k\| = 0$ as $k \to \infty$, when ever $\{\alpha_k\}$ is a sequence in X such that $G\alpha_k = H\alpha_k = \mu$ for some $\mu \in X$.

Definition2.3

A Banach space$(X, \|.\|)$ in which two self-mappings G and H are said to satisfy the property E.A if there is a sequence $\{\alpha_k\}$*in* X with $G\alpha_k = H\alpha_k = \mu$ as $k \to \infty$ as $\mu \in X$.

Now we present some examples on compatible, weakly compatible mappings and E.A properties.

Example 2.4

Let $X = [0, 10]$ then we define norm $\|.\|$ in R by $\|\alpha - \beta\| = |\alpha - \beta|$.

Define

$$G(\alpha) = \begin{cases} 2+\alpha \text{ if } 0 \leq \alpha \leq 2 \\ \frac{3}{2} \text{ if } 2 < \alpha \leq 10 \end{cases} ; \quad J(\alpha) = \begin{cases} \frac{9-2\alpha}{2} & \text{if } 0 \leq \alpha < 2 \\ \frac{3}{2} & 2 \leq \alpha \leq 10 \end{cases}$$

Let α_n be a sequence defined by $\alpha_n = 2 + \frac{1}{n}$ for $n \geq 1$.

Then $G\alpha_n = \frac{3}{2}$ and $J\alpha_n = \frac{3}{2}$ as $n \to \infty$.

Therefore $\lim_{n\to\infty} G\alpha_n = \lim_{n\to\infty} J\alpha_n = \frac{3}{2}$.

Which gives $\lim_{n\to\infty} |GJ\alpha_n - JG\alpha_n| = 0$.

Hence G and J compatible.

Example 2.5

Let $X = [3, 30]$ then we define norm $\|.\|$ in R by $\|\alpha - \beta\| = |\alpha - \beta|$.

Define

$$\mathrm{G}(\alpha) = \begin{cases} 3 & \text{if } \alpha = 3 \\ 43+\alpha & \text{if } 3 < \alpha \leq 6 \\ \alpha - 3 & \text{if } \alpha > 6 \end{cases} ; \quad \mathrm{J}(\alpha) = \begin{cases} 3 & \text{if } \alpha \in \{3\} \cup (6, 30) \\ 9 & 3 < \alpha \leq 6 \end{cases}$$

Let α_n be a sequence defined by $\alpha_n = 6 + \frac{1}{n}$ for $n \geq 1$.

Then $G\alpha_n = \alpha_n - 3 \to 3$ and $J\alpha_n = 3$ as $n \to \infty$.

Therefore $\lim_{n\to\infty} G\alpha_n = \lim_{n\to\infty} J\alpha_n = 3$.

Clearly $G(3) = J(3) = 3$ and $GJ(3) = JG(3)$.

It can be observed that G and J are *weakly compatible maps as* they commute at coincident point but $GJ\alpha_n = 3$ and $JG\alpha_n = J(\alpha_n - 3) \to 9$

which gives $\lim_{n\to\infty} |GJ\alpha_n - JG\alpha_n| = 6 \neq 0$.

But G and J *not compatible*.

Example 2.6

Let $X = [4, 25]$ then we define norm $\|\cdot\|$ in R by $\|\alpha - \beta\| = |\alpha - \beta|$.

Define

$$G(\alpha) = \begin{cases} 4 & \text{if } \alpha = 4 \\ 14+\alpha & \text{if } 4 < \alpha \leq 9 \\ \alpha - 5 & \text{if } \alpha > 9 \end{cases} ; J(\alpha) = \begin{cases} 4 & \text{if } \alpha \in \{4\} \cup (9, 25) \\ 10 & 4 < \alpha \leq 9 \end{cases}$$

Let α_n is a sequence defined by $\alpha_n = 9 + \frac{1}{n}, n \geq 1$,

Then $G\alpha_n = \alpha_n - 5 \to 4$ and $J\alpha_n = 4$.

Now $\lim_{n\to\infty} G\alpha_n = \lim_{n\to\infty} J\alpha_n = 4$

Therefore G and J satisfies E.A property.

3 Main Results

The following Theorem was proved in [4].

Theorem 3.1

Suppose X is a complete metric space, G, H ,I and J are mappings defined on X holding the conditions

$$(C1) G(X) \subseteq H(X) \text{ and } I(X) \subseteq J(X)$$

$(C2)$

$$d(G\alpha, I\beta)^{2p} \leq \left[a\phi_0\left(d(J\alpha, H\beta)^{2p}\right) + (1-a)\max\left\{ \begin{array}{l} \phi_1\left(d(J\alpha, H\beta)^{2p}\right), \phi_2\left(d(J\alpha, G\alpha)^q \cdot d(H\beta, I\beta)^{q'}\right), \phi_3\left(d(J\alpha, I\beta)^r\, d(H\beta, G\alpha)^{r'}\right) \\ \phi_4\left(\frac{1}{2}d(j\alpha, G\alpha)^s\, d(H\beta, G\alpha)^{s'}\right), \phi_5\left(\frac{1}{2}d(J\alpha, I\beta)^l d(H\beta, I\beta)^{l'}\right) \end{array} \right\} \right]$$

for all $\alpha, \beta \in X$, where $\phi_k \in \phi$ $k = 0,1,2,3,4,5, 0 \leq a \leq 1, 0 < p, q, q', r, r's, s', l, l' \leq 1$ such that $2p = q + q' = r + r' = s + s' = l + l'$.

(C3) Either of the mapping G or I is continuous

(C4) the pair of mappings (G, J) and (I, H) are weakly compatible.

Then the above mappings will be having unique common fixed point.

We prove the existence of above Theorem on Banach space under some modified conditions.

For this we need to recall the following lemmas.

Lemma 3.2 [6]

If $\phi_k \in \phi$ and $k \in \{0,1,2,3,4,5\}$ where ϕ is upper semi-continuous and contractive modulus such that $\max\{\phi_k(t)\} \leq \phi(t)$ for all $t > 0$ and $\phi(t) < t$ for $t > 0$.

Lemma 3.3 [4]

Let $\phi_j \in \phi$ and $\{\beta_j\}$ be a sequence of positive real numbers. If $\beta_{j+1} \leq \phi(\beta_j)$ for $j \in N$, then the sequence converges to 0.

Now we prove our theorem on a Banach space.

Theorem 3.4

Suppose in a Banach space $(X, \|.\|)$, there are four mappings G,J,H and I holding the conditions

$$(C1) G(X) \subseteq H(X) \text{ and } I(X) \subseteq J(X)$$

$(C2)$

$$||G\alpha - I\beta||^{2p} \leq \left[\begin{array}{l} a\phi_0\left(||J\alpha - H\beta||^{2p}\right) + (1-a)\max \\ \left\{ \begin{array}{l} \phi_1\left(||J\alpha - H\beta||^{2p}\right), \phi_2\left(||J\alpha - G\alpha||^q\ ||H\beta - I\beta||^{q'}\right), \phi_3\left(||J\alpha - I\beta||^r\ ||H\beta - G\alpha||^{r'}\right) \\ \phi_4\left(\frac{1}{2}||J\alpha - G\alpha||^s\ ||H\beta - G\alpha||^{s'}\right), \phi_5\left(\frac{1}{2}||J\alpha - I\beta||^l\ ||H\beta - I\beta||^{l'}\right) \end{array} \right\} \end{array} \right]$$

for all $\alpha, \beta \in X$, where $\phi_k \in \phi$ $k = 0,1,2,3,4,5, 0 \leq a \leq 1, 0 < p, q, q', r, r's, s', l, l' \leq 1$ such that $2p = q + q' = r + r' = s + s' = l + l'$.

(C3) One of pairs (G, J) or (I, H) satisfies the property E.A

(C4) the pair of mappings (G, J) or $(I, H)$are weakly compatible.

Then the above mappings will be having unique common fixed point.

Proof

Begin with using the condition (C1), there is a point α_1 such that $G\alpha_0 = H\alpha_1$ and for this point α_1 there exists a point α_2 in X such that $I\alpha_1 = J\alpha_2$ and so on. Continuing this process, it is possible to construct a Sequence $\{\beta_j\}$ for $j = 1,2,3\ldots in$ X

Such that $\beta_{2j} = G\alpha_{2j} = H\alpha_{2j+1}, \beta_{2j+1} = I\alpha_{2j+1} = J\alpha_{2j+2}$

We now prove $\{\beta_j\}$ is a cauchy sequence.

Putting $\alpha = \alpha_{2j}$ and $\beta = \alpha_{2j+1}$

$$\left\|\beta_{2j}-\beta_{2j+1}\right\|^{2p} \leq a\phi_0\left(\left\|\beta_{2j-1}-\beta_{2j}\right\|^{2p}\right)+$$

$$(1-a)\max\left\{\begin{array}{l}\phi_1\left(\left\|\beta_{2j-1}-\beta_{2j}\right\|^{2p}\right)\phi_2\left(\left\|\beta_{2j-1}-\beta_{2j}\right\|^{q}\left\|\beta_{2j}-\beta_{2j+1}\right\|^{q'}\right),\phi_3\left(\left\|\beta_{2j-1}-\beta_{2j+1}\right\|^{r}\left\|\beta_{2j}-\beta_{2j}\right\|^{r'}\right)\\ \phi_4\left(\frac{1}{2}\left[\left\|\beta_{2j-1}-\beta_{2j}\right\|^{s}\left\|\beta_{2j}-\beta_{2j}\right\|^{s'}\right]\right),\phi_5\left(\frac{1}{2}\left[\left\|\beta_{2j-1}-\beta_{2j+1}\right\|^{l}\left\|\beta_{2j}-\beta_{2j+1}\right\|^{l'}\right]\right)\end{array}\right\}$$

Denote $\rho_j = \left\|\beta_j-\beta_{j+1}\right\|$

$$(\rho_{2j})^{2p} \leq a\phi_0(\rho_{2j-1})^{2p}+(1-a)\max\left\{\phi_1(\rho_{2j-1})^{2p},\phi_2(\rho_{2j-1})^{q}\,(\rho_{2j})^{q'},\phi_3(0),\phi_4(0),\phi_5\left(\frac{1}{2}\left[(\rho_{2j-1})^{l}+(\rho_{2j})^{l}\right](\rho_{2j})^{l'}\right)\right\}$$

$$\leq a\phi_0(\rho_{2j-1})^{2p}+(1-a)\max\left\{\phi_1(\rho_{2j-1})^{2p},\phi_2(\rho_{2j-1})^{q}(\rho_{2j})^{q'},\phi_3(0),\phi_4(0),\phi_5\left(\frac{1}{2}\left[(\rho_{2j-1})^{l}(\rho_{2j})^{l'}+(\rho_{2j})^{l}(\rho_{2j})^{l'}\right]\right)\right\}$$

If $\rho_{2j} > \rho_{2j-1}$ then we have

$$(\rho_{2j})^{2p} \leq a\phi_0(\rho_{2j})^{2p}+(1-a)\max\left\{\phi_1(\rho_{2j})^{2p},\phi_2(\rho_{2j})^{q+q'},\phi_3(0),\phi_4(0),\phi_5\left(\frac{1}{2}\left[(\rho_{2j})^{l+l}+(\rho_{2j})^{l+l'}\right]\right)\right\}\ (\rho_{2j})^{2p}$$

$$\leq a\phi_0(\rho_{2j})^{2p}+(1-a)\max\left\{\phi_1(\rho_{2j})^{2p},\phi_2(\rho_{2j})^{2p},\phi_3(0),\phi_4(0),\phi_5(\rho_{2j})^{2p}\right\}$$

using Lemma (3.2)

$$(\rho_{2j})^{2p} \leq \phi\left((\rho_{2j})^{2p}\right) < (\rho_{2j})^{2p}$$

which is a contradiction.

Thus we must have $\rho_{2j} \leq \rho_{2j-1}$

Then using this inequality the condition (C2) yields $\rho_{2j} \leq \phi(\rho_{2j-1})$ ——— (1)

Similarly taking $\alpha = \alpha_{2j+2}$ and $\beta = \alpha_{2j+1}$ in (C2) , we get

$$\left\|\beta_{2j+1}-\beta_{2j+2}\right\|^{2p} \leq a\phi_0\left(\left\|\beta_{2j}-\beta_{2j+1}\right\|^{2p}\right)+\ (1-a)\max$$

$$\left\{\begin{array}{l}\phi_1\left(\left\|\beta_{2j}-\beta_{2j+1}\right\|^{2p}\right)\phi_2\left(\left\|\beta_{2j+1}-\beta_{2j+2}\right\|^{q}\left\|\beta_{2j}-\beta_{2j+1}\right\|^{q'}\right),\phi_3\left(\left\|\beta_{2j+1}-\beta_{2j+1}\right\|^{r}\left\|\beta_{2j}-\beta_{2j+1}\right\|^{r'}\right)\\ \phi_4\left(\frac{1}{2}\left[\left\|\beta_{2j+1}-\beta_{2j+2}\right\|^{s}\left\|\beta_{2j}-\beta_{2j+2}\right\|^{s'}\right]\right),\phi_5\left(\frac{1}{2}\left[\left\|\beta_{2j+2}-\beta_{2j+1}\right\|^{l}\left\|\beta_{2j}-\beta_{2j+1}\right\|^{l'}\right]\right)\end{array}\right\}$$

$$(\rho_{2j+1})^{2p} \leq a\phi_0(\rho_{2j})^{2p}+(1-a)\max$$

$$\left\{\phi_1(\rho_{2j})^{2p},\phi_2(\rho_{2j+1})^{q}(\rho_{2j})^{q'},\phi_3(0),\ ,\ \phi_4\left(\frac{1}{2}\left[(\rho_{2j+1})^{s}(\rho_{2j})^{s'}+(\rho_{2j+1})^{s'}\right]\right),\phi_5(0)\right\}$$

$$(\rho_{2j+1})^{2p} \leq a\phi_0(\rho_{2j})^{2p}+(1-a)\max$$

$$\left\{\phi_1(\rho_{2j})^{2p},\phi_2(\delta_{2j+1})^{2}(\rho_{2j})^{i},\phi_3(0),,\phi_4\left(\frac{1}{2}\left[(\rho_{2j+1})^{s}(\rho_{2j})^{j}+(\rho_{2j+1})'(\rho_{2j+1})\right)\right],\phi_5\right.$$

$\rho_{2j+1} > \rho_{2j}$, then we have

$$(\rho_{2j+1})^{2p} \leq a\phi_0(\rho_{2j+1})^{2p}+(1-a)\max\left\{\phi_1(\rho_{2j+1})^{2p},\phi_2(\rho_{2j+1})^{q+q'},\phi_3(0),,\phi_4(\rho_{2j+1}),\phi_5(0)\right\}$$

Since by Lemma (3.2)

$$(\rho_{2j+1})^{2p} \leq \phi(\rho_{2j+1})^{2p} < (\rho_{2j+1})^{2p}$$

which is a contradiction.

Thus we must have $\rho_{2j+1} \le \rho_{2j}$

again from (C2), we obtain $\rho_{2j+1} \le \phi(\rho_{2j})$ ———————-(2)

From (1) and (2), ingeneral

$$\rho_{j+1} \le \phi(\rho_j), \text{ for } j = 0, 1, 2, 3 \ldots$$

since by Lemma (3.3)

we get $\rho_j \to 0$ as $j \to \infty$.

This shows that $\rho_j = \|\beta_j - \beta_{j+1}\| \to 0 \; as \; j \to \infty$

Hence $\{\beta_j\}$ is a cauchy sequence.

By the completeness of X , $\{\beta_j\}$converges to some point in X as $j \to \infty$

Since $\{G\alpha_{2j}\}$, $\{J\alpha_{2j+2}\}$, $\{H\alpha_{2j+1}\}$ and $\{I\alpha_{2j+1}\}$ are sub sequences of $\{\beta_j\}$ and hence they also converges to the same point in X as $j \to \infty$

Suppose the pair (G,J) satisfies the property E.A, then there exists a sequence $\{\alpha_j\}$ in X such that $G\alpha_j = J\alpha_j = \mu$ as $j \to \infty$ for some $\mu \in$ X ——————-(3)

Since $G(X) \subseteq H(X)$ then $\exists$ a sequence $\{\beta_j\}$ in X such that $G\alpha_j = H\beta_j$

Therefore $G\alpha_j = J\alpha_j = H\beta_j = \mu$ as j $\to \infty$ for some $\mu \in X$ ————(4)

Now we prove that $I\beta_j = \mu$ as $j \to \infty$.

Now in (C2) putting $\alpha = \alpha_j, \beta = \beta_j$

$$\left\|G\alpha_j - I\beta_j\right\|^{2p} \le \left[a\phi_0\left(\left\|J\alpha_j - H\beta_j\right\|^{2p}\right) + (1-a)\max\left\{ \begin{array}{l} \phi_1\left(\left\|J\alpha_j - H\beta_j\right\|^{2p}\right), \phi_2\left(\left\|J\alpha_j - G\alpha_j\right\|^{q}\left\|H\beta_j - I\beta_j\right\|^{q'}\right), \\ \phi_3\left(\left\|J\alpha_j - I\beta_j\right\|^{r}\left\|H\beta_j - G\alpha_j\right\|^{r'}\right), \\ \phi_4\left(\frac{1}{2}\left\|J\alpha_j - G\alpha_j\right\|^{s}\left\|H\beta_j - G\alpha_j\right\|^{s'}\right), \phi_5\left(\frac{1}{2}\left\|J\alpha_j - I\beta_j\right\|^{l}\left\|H\beta_j - I\beta_j\right\|^{l'}\right) \end{array} \right\} \right]$$

letting $j \to \infty$ we get

$$\left\|\mu - I\beta_j\right\|^{2p} \le \left[a\phi_0\left(\left\|\mu - \mu\right\|^{2p}\right) + (1-a)\max\left\{ \begin{array}{l} \phi_1\left(\left\|\mu - \mu\right\|^{2p}\right), \phi_2\left(\left\|\mu - \mu\right\|^{q}\left\|\mu - I\beta_j\right\|^{q'}\right), \\ \phi_3\left(\left\|\mu - I\beta_j\right\|^{r}\left\|\mu - \mu\right\|^{r'}\right), \\ \phi_4\left(\frac{1}{2}\left\|\mu - \mu\right\|^{s}\left\|\mu - \mu\right\|^{s'}\right), \phi_5\left(\frac{1}{2}\left\|\mu - I\beta_j\right\|^{l}\left\|\mu - I\beta_j\right\|^{l'}\right) \end{array} \right\} \right]$$

$$\left\|\mu - I\beta_j\right\|^{2p} \le \left[a\phi_0(0) + (1-a)\max\left\{ \begin{array}{l} \phi_1(0), \phi_2(0), \\ \phi_3(0), \\ \phi_4(0), \phi_5\left(\frac{1}{2}\left\|\mu - I\beta_j\right\|^{2p}\right) \end{array} \right\} \right]$$

since by Lemma(3.2)

$\|\mu - I\beta_j\|^{2p} < \phi\left(\|\mu - I\beta_j\|^{2p}\right) < \|\mu - I\beta_j\|^{2p}$

which is a contradiction.

Hence $I\beta_j = \mu$.

Therefore $G\alpha_j = J\alpha_j = H\beta_j = I\beta_j = \mu$ as j $\to \infty$ for some $\mu \in X$ ——(5)

Since the pair (G, J) is weakly compatible mapping and G and J commute at a point of coincidence with $G\alpha_j = J\alpha_j$ for some $\alpha_j \varepsilon X$ and this gives $GJ\alpha_j = JG\alpha_j$ and this turns $G\mu = J\mu$.

Now in the inequality (C2) substitute $\alpha = \mu$ and $\beta = \beta_{2j+1}$

$$\left\|G\mu - I\beta_{2j+1}\right\|^{2p} \le \left[a\phi_0\left(\left\|J\mu - H\beta_{2j+1}\right\|^{2p}\right) + (1-a)\max\left\{ \begin{array}{l} \phi_1\left(\left\|J\mu - H\beta_{2j+1}\right\|^{2p}\right), \phi_2\left(\left\|J\mu - G\mu\right\|^{q}\left\|H\beta_{2j+1} - I\beta_{2j+1}\right\|^{q'}\right), \\ \phi_3\left(\left\|J\mu - I\beta_{2j+1}\right\|^{r}\left\|H\beta_{2j+1} - G\mu\right\|^{r'}\right), \\ \phi_4\left(\frac{1}{2}\left\|J\mu - G\mu\right\|^{s}\left\|H\beta_{2j+1} - G\mu\right\|^{s'}\right), \phi_5\left(\frac{1}{2}\left\|J\mu - I\beta_{2j+1}\right\|^{l}\left\|H\beta_{2j+1} - I\beta_{2j+1}\right\|^{l'}\right) \end{array} \right\} \right]$$

letting $j \to \infty$we get

$$\left\|G\mu - \mu\right\|^{2p} \le \left[a\phi_0\left(\left\|\mu - \mu\right\|^{2p}\right) + (1-a)\max\left\{ \begin{array}{l} \phi_1\left(\left\|\mu - \mu\right\|^{2p}\right), \phi_2\left(\left\|\mu - G\mu\right\|^{q}\left\|\mu - \mu\right\|^{q'}\right), \phi_3\left(\left\|\mu - \mu\right\|^{r}\left\|\mu - G\mu\right\|^{r'}\right), \\ \phi_4\left(\frac{1}{2}\left\|\mu - G\mu\right\|^{s}\left\|\mu - G\mu\right\|^{s'}\right), \phi_5\left(\frac{1}{2}\left\|\mu - \mu\right\|^{l}\left\|\mu - \mu\right\|^{l'}\right) \end{array} \right\} \right]$$

$$\left\|G\mu - \mu\right\|^{2p} \le \left[a\phi_0(0) + (1-a)\max\left\{ \phi_1(0), \phi_2(0), \phi_3(0), \phi_4\left(\frac{1}{2}\left\|\mu - G\mu\right\|^{s+s'}\right), \phi_5(0) \right\} \right]$$

$$\|G\mu-\mu\|^{2p} \le \left[a\phi_0(0)+(1-a)\max\left\{\phi_1(0),\phi_2(0),\phi_3(0),\phi_4\left(\frac{1}{2}\|Gu-\mu\|^{2p}\right),\phi_5(0)\right\}\right]$$

since by Lemma(3.2)

$\|G\mu-\mu\|^{2p} < \phi\left(\|G\mu-\mu\|^{2p}\right) < \|G\mu-\mu\|^{2p}$

which is a contradiction,and hence $G\mu=\mu$.

Therefore $G\mu=J\mu=\mu$ ———-(6)

Again the pair (I, H) is weakly compatible mapping, I and H commute at a point of coincidence with $I\beta_j=H\beta_j$ for some $\beta_j\in X$ and this gives $IH\beta_j=HI\,\beta_j$ and this implies that $I\mu=H\mu$.

Now we prove that $I\mu=\mu$

Again in the inequality (C2) substitute $\alpha=\alpha_{2j},\beta=\mu$

$$\|G\alpha_{2j}-I\mu\|^{2p} \le \left[a\phi_0\left(\|J\alpha_{2j}-H\mu\|^{2p}\right)+(1-a)\max\left\{\begin{array}{l}\phi_1\left(\|J\alpha_{2j}-H\mu\|^{2p}\right),\phi_2\left(\|J\alpha_{2j}-G\alpha_{2j}\|^{q}\|H\mu-I\mu\|^{q'}\right),\phi_3\left(\|J\alpha_{2j}-I\mu\|^{r}\|H\mu-G\alpha_{2j}\|^{r'}\right),\\ \phi_4\left(\frac{1}{2}\|J\alpha_{2j}-G\alpha_{2j}\|^{s}\|H\mu-G\alpha_{2j}\|^{s'}\right),\phi_5\left(\frac{1}{2}\|J\alpha_{2j}-I\mu\|^{l}\|H\mu-I\mu\|^{l'}\right)\end{array}\right\}\right]$$

letting $j\to\infty$

$$\|\mu-I\mu\|^{2p} \le \left[a\phi_0\left(\|\mu-I\mu\|^{2p}\right)+(1-a)\max\left\{\begin{array}{l}\phi_1\left(\|\mu-I\mu\|^{2p}\right),\phi_2\left(\|\mu-\mu\|^{q}\|I\mu-I\mu\|^{q'}\right),\phi_3\left(\|\mu-I\mu\|^{r}\|I\mu-\mu\|^{r'}\right),\\ \phi_4\left(\frac{1}{2}\|\mu-\mu\|^{s}\|I\mu-\mu\|^{s'}\right),\phi_5\left(\frac{1}{2}\|\mu-I\mu\|^{l}\|I\mu-I\mu\|^{l'}\right)\end{array}\right\}\right]$$

$$\|\mu-I\mu\|^{2p} \le \left[a\phi_0\left(\|\mu-I\mu\|^{2p}\right)+(1-a)\max\left\{\phi_1\left(\|\mu-I\mu\|^{2p}\right),\phi_2(0),\phi_3\left(\|\mu-I\mu\|^{2p}\right),\phi_4(0),\phi_5(0)\right\}\right]$$

since by Lemma(3.2)

$\|\mu-I\mu\|^{2p} < \phi\left(\|\mu-I\mu\|^{2p}\right) < \|\mu-I\mu\|^{2p}$

which is a contradiction.

Therefore $I\mu=\mu$

Which implies that $I\mu=H\mu=\mu$ ————(7)

Therefore From (6) and (7) we get $G\mu=J\mu=I\mu=H\mu=\mu$.

Hence μ is a common fixed point fixed point for four self maps G,H,I and J.

For uniqueness:

Suppose μ and μ^* ($\mu\neq\mu^*$) are common fixed points of G,J,H and I, then

substitute $\alpha=\mu$ and $\beta=\mu^*$ in the inequality (C2)

$$\|G\mu-I\mu^*\|^{2p} \le \left[a\phi_0\left(\|J\mu-H\mu^*\|^{2p}\right)+(1-a)\max\left\{ \begin{array}{l} \phi_1\left(\|J\mu-H\mu^*\|^{2p}\right), \phi_2\left(\|J\mu-G\mu\|^{q}\|H\mu^*-I\mu^*\|^{q'}\right), \phi_3\left(\|J\mu-I\mu^*\|^{r}\|H\mu^*-G\mu\|^{r'}\right), \\ \phi_4\left(\frac{1}{2}\|J\mu-G\mu\|^{s}\|H\mu^*-G\mu\|^{s'}\right), \phi_5\frac{1}{2}\left(\|J\mu-I\mu^*\|^{l}\|H\mu^*-I\mu^*\|^{l'}\right) \end{array} \right\} \right]$$

$$\|\mu-\mu^*\|^{2p} \le \left[a\phi_0\left(\|\mu-\mu^*\|^{2p}\right)+(1-a)\max\left\{\phi_1\left(\|\mu-\mu^*\|^{2p}\right), \phi_2(0), \phi_3\left(\|\mu-\mu^*\|^{2p}\right), \phi_4(0), \phi_5(0)\right\}\right]$$

$\|\mu-\mu^*\|^{2p} < \phi\left(\|\mu-\mu^*\|^{2p}\right) < \|\mu-\mu^*\|^{2p}$
which is a contradiction.
Therefore, $\mu=\mu^*$ this proves the uniqueness.
As a particular case on letting $p=q=q'=r=r'=s=s'=l=l'=1$, we get a corollary from Theorem(3.4)

3.5 Corollary

Suppose in a Banach space $(X,\|.\|)$ there are four mappings G, H, I and J holding the conditions

(C1) $G(X) \subseteq H(X)$ and I(X) $\subseteq$ J(X)

$(C2)$

$$\|G\alpha-I\beta\|^{2} \le \left[a\phi_0\left(\|J\alpha-H\beta\|^{2}\right)+(1-a)\max\left\{ \begin{array}{l} \phi_1\left(\|J\alpha-H\beta\|^{2}\right), \phi_2\left(\|J\alpha-G\alpha\|\|H\beta-I\beta\|\right), \phi_3\left(\|J\alpha-I\beta\|\|H\beta-G\alpha\|\right), \\ \phi_4\left(\frac{1}{2}\|J\alpha-G\alpha\|\|H\beta-G\alpha\|\right), \phi_5\left(\frac{1}{2}\|J\alpha-I\beta\|\|H\beta-I\beta\|\right) \end{array} \right\} \right]$$

for all $\alpha,\beta \in X$, where $\phi_k \in \phi$, k $=0,1,2,3,4,5, 0 \le a \le 1$.
(C3) One of pairs (G, J) or (I, H) follows property E.A
(C4) the pair of mappings (G, J) and (I, H) are weakly compatible.
Then the above mappings will be having unique common fixed point.
Similarly, taking $p=q=q'=r=r'=s=s'=l=l'=\frac{1}{2}$,we get another corollary from Theorem (3.4).

3.6 Corollary

Suppose in a Banach space $(X,\|.\|)$there are four mappings G, H, I and J holding the conditions

(C1) $G(X) \subseteq H(X)$ and I(X) $\subseteq$ J(X)

$(C2)$

$$\|G\alpha - I\beta\| \leq \left[a\phi_0(\|J\alpha - H\beta\|) + (1-a)\max\left\{ \begin{array}{l} \phi_1(\|J\alpha - H\beta\|), \phi_2\left(\|J\alpha - G\alpha\|^{\frac{1}{2}}\|H\beta - I\beta\|^{\frac{1}{2}}\right), \phi_3\left(\|J\alpha - I\beta\|^{\frac{1}{2}}\|H\beta - G\alpha\|^{\frac{1}{2}}\right), \\ \phi_4\left(\frac{1}{2}\|J\alpha - G\alpha\|^{\frac{1}{2}}\|H\beta - G\alpha\|^{\frac{1}{2}}\right), \phi_5\left(\frac{1}{2}\|J\alpha - I\beta\|^{\frac{1}{2}}\|H\beta - I\beta\|^{\frac{1}{2}}\right) \end{array} \right\} \right]$$

for all $\alpha, \beta \in X$, where $\phi_k \in \phi$, $k = 0,1,2,3,4,5, 0 \leq a \leq 1$.
(C3) One of pairs (G, J) or (I, H) follows property E.A
(C4) the pair of mappings (G, J) and (I, H) are weakly compatible.
Then the above mappings will be having unique common fixed point.
Now we provide an example to validate our discussion.

4 Example

Let $X = \left[0, \frac{1}{3}\right]$ then we define norm in R by $\|\alpha - \beta\| = |\alpha - \beta|$.

Define $G(\alpha) = I(\alpha) = \begin{cases} \frac{\alpha+5}{7} & \text{if } 0 \leq \alpha < \frac{1}{4} \\ \frac{1}{2} - \alpha & \text{if } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$ and $J(\alpha) = H(\alpha) = \begin{cases} \frac{5\alpha+4}{7} & \text{if } 0 \leq \alpha < \frac{1}{4} \\ 5\alpha - 1 & \text{if } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$

Then $G(X) = I(X) = \left[\frac{5}{7}, \frac{3}{4}\right) \cup \left[\frac{1}{6}, \frac{1}{4}\right]$ while $J(X) = H(X) = \left[\frac{4}{7}, \frac{3}{4}\right) \cup \left[\frac{1}{4}, \frac{2}{3}\right]$ Hence the condition (C1) satisfied.

Take a sequence $\{\alpha_k\}$ as $\alpha_k = \frac{1}{4} + \frac{1}{k}$ for $k \geq 0$.

Now $G\alpha_k = G\left(\frac{1}{4} + \frac{1}{k}\right) == \frac{1}{2} - \left(\frac{1}{4} + \frac{1}{k}\right) = \frac{1}{4} - \frac{1}{k} = \frac{1}{4}$ as $k \to \infty$ and $J\alpha_k = J\left(\frac{1}{4} + \frac{1}{k}\right) = 5\left(\frac{1}{4} + \frac{1}{k}\right) - 1 = \frac{1}{4} + \frac{5}{k} = \frac{1}{4}$ as $k \to \infty$.

This gives $G\alpha_k = J\alpha_k = \frac{1}{4} = J\left(\frac{1}{4}\right)$ where $\frac{1}{4} \in X$, as $k \to \infty$.

This gives $I\alpha_k = H\alpha_k = \frac{1}{4} = H\left(\frac{1}{4}\right)$ where $\frac{1}{4} \in X$, as $k \to \infty$.

Therefore, pairs (G,J) and (I,H) are satisfies E.A property.

Also $G\left(\frac{1}{4}\right) = \frac{1}{4}$ and $J\left(\frac{1}{4}\right) = \frac{1}{4}$ which implies $G\left(\frac{1}{4}\right) = J\left(\frac{1}{4}\right)$.

Now $H\left(\frac{1}{4}\right) = \frac{1}{4}$ and $I\left(\frac{1}{4}\right) = \frac{1}{4}$ which implies $H\left(\frac{1}{4}\right) = J\left(\frac{1}{4}\right)$.

Also $GJ\left(\frac{1}{4}\right) = G\left(\frac{1}{4}\right) == \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

$IH\left(\frac{1}{4}\right) = I\left(\frac{1}{2} - \frac{1}{4}\right) = I\left(\frac{1}{4}\right) = 5\left(\frac{1}{4}\right) - 1 = \frac{1}{4}$ and $GJ\left(\frac{1}{4}\right) = JG\left(\frac{1}{4}\right)$ and $IH\left(\frac{1}{4}\right) = HI\left(\frac{1}{4}\right)$ which gives (G, J), (I, H) are weakly compatible mapping.

Now $GJ\alpha_k = GJ\left(\frac{1}{4} + \frac{1}{k}\right) = G\left(5\left(\frac{1}{4} + \frac{1}{k}\right) - 1\right) = G\left(\frac{1}{4} + \frac{5}{k}\right) = \frac{1}{2} - \left(\frac{1}{4} + \frac{5}{k}\right) = \frac{1}{4} - \frac{5}{k} = \frac{1}{4}$ as $k \to \infty$.

Also $JG\alpha_k = JG\left(\frac{1}{4} + \frac{1}{k}\right) = J\left(\frac{1}{2} - \left(\frac{1}{4} + \frac{1}{k}\right)\right) = J\left(\frac{1}{4} - \frac{1}{k}\right) = \frac{5\left(\frac{1}{4} - \frac{1}{k}\right)+4}{7} = \frac{\frac{5}{4} - \frac{5}{k} + 4}{7} = \frac{21}{28} + \frac{5}{7k} = \frac{3}{4}$ as $k \to \infty$.

So that $\lim_{k\to\infty} \|GJ\alpha_k, JG\alpha_k\| = \left\|\frac{1}{4} - \frac{3}{4}\right\| = \left|\frac{1}{4} - \frac{3}{4}\right| \neq 0$. Similarly $\lim_{k\to\infty} \|HI\ \alpha_k, I\ H\alpha_k\| == \left\|\frac{1}{4} - \frac{3}{4}\right\| = \left\|\frac{1}{4} - \frac{3}{4}\right\| \neq 0$ showing that the compatibility condition is not fulfilled.

We now establish that the mappings G, H,I and J satisfy the condition(C2).

Case (i):

If $\alpha, \beta \in (0,1)$, we define $\|G\alpha - I\beta\| = |G\alpha - I\beta|$.

Putting $\alpha = \frac{1}{5}$ and $\beta = \frac{1}{4}$, then the inequality (C2) gives

$$\left\|G\left(\frac{1}{5}\right) - I\left(\frac{1}{4}\right)\right\|^{2p} \leq \left[a\phi_0\left(\left\|J\left(\frac{1}{5}\right) - H\left(\frac{1}{4}\right)\right\|^{2p}\right) + (1-a)\max\left\{ \begin{array}{l} \phi_1\left(\left\|J\left(\frac{1}{5}\right) - H\left(\frac{1}{4}\right)\right\|^{2p}\right), \phi_2\left(\left\|J\left(\frac{1}{5}\right) - G\left(\frac{1}{5}\right)\right\|^{q}\left\|H\left(\frac{1}{4}\right) - I\left(\frac{1}{4}\right)\right\|^{q'}\right), \phi_3\left(\left\|J\left(\frac{1}{5}\right) - I\left(\frac{1}{4}\right)\right\|^{r}\left\|H\left(\frac{1}{4}\right) - G\left(\frac{1}{5}\right)\right\|^{r'}\right), \\ \phi_4\left(\frac{1}{2}\left\|J\left(\frac{1}{5}\right) - G\left(\frac{1}{5}\right)\right\|^{s}\left\|H\left(\frac{1}{4}\right) - G\left(\frac{1}{5}\right)\right\|^{s'}\right), \phi_5\left(\frac{1}{2}\left\|J\left(\frac{1}{5}\right) - I\left(\frac{1}{4}\right)\right\|^{l}\left\|H\left(\frac{1}{4}\right) - I\left(\frac{1}{4}\right)\right\|^{l'}\right) \end{array} \right\} \right]$$

for $a = \frac{1}{2}$ and $p = p' = q = q' = r = r' = s = s' = l = l' = \frac{1}{2}$

$$\|0.24\| \le \left[a\,\phi_0\left(\frac{13}{28}\right) + (1-a)\max \left\{ \begin{array}{l} \phi_1\left(\frac{13}{28}\right), \phi_2(0), \phi_3(0.68 \times 0.70), \\ \phi_4(0.17 \times 0.7), \phi_5(0) \end{array} \right\} \right]$$

$$|0.24| < |0.468|.$$

Hence the condition (C2) is satisfied.
Similarly we can prove the condition (C2) in other cases.
It is evident from the above mappings that $\frac{1}{4}$ is the unique common fixed point.

5 Conclusion

This study has focussed on a Banach space to establish a common fixed point theorem without continuity condition and also employing the property E.A. Further at the end of this article, two corollaries are generated from the main result. Moreover, the result is also substantiated by the provision of a suitable example.

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