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Some aspects of customers impatience with varying rates of Reneging

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Abstract

Objectives: Stochastic modeling of M/M/c/k queuing system in the presence of Position dependent reneging (PDR) and state dependent balking (SDB) along with closed form expression of various performance measures. **Methods:** Markov process and probability generating functions (pgf) are used. **Findings:** Closed form expression of a number of traditional as well as newly designed performance measures are derived along with explicit derivation of the steady state probabilities of the model. **Novelty:** Analysis of finite buffer multi server Markovian queuing model in the presence of balking and reneging.

Keywords: Queue; finite buffer; multiserver; reneging; balking; impatience

1 Introduction

Waiting lines or queues are a common occurrence both in everyday life and in variety of business and industrial situations. Waiting for service is inevitable and is typically a negative consumer experience causing unhappiness, frustration and anxiety. Being in a fast-paced world, customers are hard pressed for time. Waiting for service is an act, which they would prefer not to be involved in. Such constraints on time induce impatience on customers' behavior. In queuing parlance, such impatience may find reflection through the concepts of balking and reneging. These were included in recent works on two server Markovian model [1,2].

If a customer finds the queuing system non-empty on arrival, it might decide not to join the queue. In queuing theory, this is known as balking. A rationale which might influence a person to balk has been proposed [3]. It relates to the perception of the importance of being served which induces an opinion somewhere in between urgency, so that a queue of certain length will not be joined, to indifference where a non-zero queue is also joined. Balking can occur in two ways viz. state independent balking (SIB) and state dependent balking (SDB). When balking probability remains constant across all the states of the system i.e. when it does depend on the state of the system, it is called SIB. On the other hand, if balking probability depends on the state of the system, then it is known as SDB. In SDB, balking probability will usually go up as the queue length goes up [4]. In this paper all the analysis have been carried out considering SDB.

Reneging is a phenomenon wherein customers join a queue but leave the same without completing service. It is not unknown that reneging does not find wide mention in the literature of queuing theory. In most of the theoretical models, it is assumed that customers are patient and are willing to wait as long as it is necessary to complete receiving service i.e. no reneging is assumed. This is ironic because it is seldom possible to find a queuing system in practice where customers are not of reneging type. In our fast paced life, customers are unwilling to wait indefinitely. If they are required to do so, they get impatient and may leave without completely receiving service. To a business manager, this implies loss of immediate revenue as well as reputation. Modern businesses cannot afford it. Understanding the different dimensions of impatience is therefore a priority for them. Even in the not so vast reneging literature, the general approach has been to assume constant reneging rates. Irrespective of the position of the customer in the queue, it is assumed that its reneging rate is constant[5–7]. In practice, one can often find many queuing systems where the customer is aware of its position in the queue. 'For example in the OPD of a hospital, patients usually know of their position in the queue. In such circumstances, it is natural to observe that customers who are positioned at a distance from the server have higher propensity to renege than those who are positioned near the server'[8]. This is unlike the constant reneging rate assumption. There is thus a strong case to model systems with varying rates of reneging impatience. This paper is motivated by these considerations.

It needs mention here that reneging can be of two types - reneging till beginning of service (henceforth referred to as R_BOS) and reneging till end of service (henceforth referred to as R_EOS). R_BOS can be observed in queuing systems where customers can renege only as long as they are in the queue. Once they begin receiving service, they do not renege. A common example is the barbershop. A customer can renege while he is waiting in queue. However once service starts i.e. hair cut begins, the customer cannot leave till hair cut is over. On the other hand, R_EOS can be observed in queuing systems where customers can renege not only while waiting in queue but also while receiving service. "An example is processing or merchandising of perishable goods"[9]. The patience time commences from the moment the customer joins the system. In case the reneging discipline is R_BOS, the customer may renege i.e. leave the system if service does not begin before expiry of his patience time. On the other hand, in case of R_EOS, the customer could renege if service is not complete before the expiry of his patience time. Thus, in case of R_EOS, the customer may depart either from the queue or from the service with partial and incomplete service whereas in case of R_BOS, the customer can renege only from the queue. Both R_BOS and R_EOS have been treated separately in this paper.

Even though queuing models of various types have been assumed in queuing literature, it is not often that reneging and balking have been analyzed[10–12]. Even if these have been dealt with, closed form expressions of performance measures are not always available[13,14]. Recently, a multi-server Markovian queuing system was analyzed under the assumption that customers may balk as well as renege[15]. However, they did not include finite buffer restriction. In another relevant work where a M^X /M/c Bernoulli feedback queueing system with variant multiple working vacations and impatience timers was considered[16] but their model did not assume balking or position dependent reneging and neither did they include finite buffer restriction. Analysis of queuing system under general reneging distribution in the presence of balking is also available in literature[17] but that was carried out for basic M/M/1 queue. Two server Markovian queuing model with discouraged arrivals, reneging and retention of reneged customers was analyzed in[18] but balking was not assumed.

The modelling approach of a finite buffer multi server Markovian queuing system incorporating the additional challenges of position dependent reneging and balking is the hallmark of this paper. Importance of the queuing model stems from the fact that in the classical M/M/1 model," it is assumed that the system can accommodate any number of units. In practice, this may seldom be the case. We have thus to consider the situation such that the system has limited waiting space and can hold a maximum number of k units (including the one being served)"[19]. Even though this model has been analyzed, however, to the best of our knowledge the said model with the added complexity of PDR and SDB has not been dealt with in the literature. Only a restricted version of the multi-server queuing model for just two servers assuming impatient customers is available[1,2]. This work therefore aims to fill this gap in literature.

The subsequent sections of this paper are structured as follows. Section 2 contains the description of the model. The derivation of steady state probabilities are discussed in Section 3. Closed form expression of the various performance measures are described in Section 4. We perform sensitivity analysis in section 5. Concluding remarks are presented in section 6. The appendix contains some derivation.

2 Description of the model

The model we deal with in this paper is the traditional M/M/c/k model with the restriction that customers may balk from a non-empty queue as well as may renege after they join the queue. We shall assume that the inter arrival and service rates are λ and μ respectively. Customers joining the system are assumed to be of Markovian reneging type. We shall assume that on joining the system the customer is aware of its position in the system. Consequently, the reneging rate will be a function of the customer's position in the system. In particular, a customer who is at state 'n' will be assumed to have random patience time following exp (ν_n). In case of R_EOS, it is perhaps natural to expect that the reneging rate of a customer receiving service, would be different from the reneging rate in case the customer is in the queue. This motives our assumption that under R_EOS

$$v_n = \begin{cases} 0 & \text{for n} = 0 \\ v & \text{for n} = 1,2,\dots \text{c} \\ v+\gamma_n & \text{for n} = \text{c}+1,\text{c}+2,\dots,\text{k} \end{cases}$$

Under R_BOS, there is no reneging while at service and hence we assume reneging rate to be

$$v_n = \begin{cases} 0 & \text{forn } = 0,1,\dots \text{ c} \\ v+\gamma_n & \text{forn } = \text{c}+1,\dots \text{k} \end{cases}$$

Where c>1 is the number of servers and ν>0.

Clearly, r_n can have many formulations. In this paper, we shall assume that $r_{n=}$ $r^{\text{n-c}}$, where r>1.The rest of the paper is based.

As regards balking, we assume that each customer has a state dependent balking probability. It will be assumed that if the customer on arrival observes the system to be in state 'n', the probability that he will balk is $\frac{n-c+1}{k-c+1}; n = c, c+1,\dots,k$. Under this set up, the finite buffer restriction can also be seen as the state from which customer balks with probability 1 $\left(= \frac{k-c+1}{k-c+1}\right)$. There is no balking from an empty queue.

3 System state probabilities

The steady state probabilities are derived by the Markov process method. We first analyze the case where customers renege only from the queue. Under R_BOS, let 'p_n' denote the probability that there are 'n' customers in the system. The steady state equations are given below

$$\lambda p_0 = \mu p_1 \tag{3.1}$$

$$\lambda p_{n-1} + (n+1)\mu p_{n+1} = \lambda p_n + n\mu p_n; \text{n} = 1_2 2,\dots \text{c}-1 \tag{3.2}$$

$$\lambda p_{c-1} + (c\mu + v + \gamma) p_{c+1} = \lambda[1-\{1/(k-c+1)\}]p_c + c\mu p_c \tag{3.3}$$

$$\begin{aligned} &\lambda[1-\{(n-c)/(k-c+1)\}]p_{n-1} + \left\{c\mu + (n-c+1)v + \gamma\left(\gamma^{n-c+1}-1\right)/(\gamma-1)\right\}p_{n+1} = \lambda[1-\{(n-c+1)/(k-c+1)\}]p_n \\ &+ \{c\mu + (n-c)v + \gamma(\gamma^{n-c}-1)/(c-1)\}p_n \; ; \; n = c+1,..,k-1 \end{aligned} \tag{3.4}$$

$$\lambda\left\{1-(\text{k}-\text{c})/((\text{k}-\text{c}+1)\right\}\text{p}_{\text{k}-1} = \left\{\text{c}\mu + (\text{k}-\text{c})v + \gamma\left(\gamma^{\text{k}-\text{c}}-1\right)/(\gamma-1)\right\}\text{p}_\text{k} \tag{3.5}$$

Solving recursively, we get (under R_BOS)

$$p_n = \{\lambda^n/(n!\mu^n)\}p_0 \quad ;\text{n} = 1,2,\dots\text{c}$$

$$\text{p}_\text{n} = \left[\lambda^\text{n}\prod_{\text{r}=1}^{\text{n}-\text{c}}\{1-\text{r}/(\text{k}-\text{c}+1)\}/\left\{\text{c}!\mu^\text{c}\prod_{\text{r}=1}^{\text{n}-\text{c}}(\text{c}\mu+\text{r}v+\gamma(\gamma^\text{r}-1)/(\gamma-1)\}\right]\text{p}_0; \quad \text{n} = \text{c}+1,\dots.\text{k}$$

Where p_0 is obtained from the normalizing condition $\sum_{n=0}^{k} p_n = 1$ and is given as

$$\text{p}_0 = \left[1+\sum_{\text{n}=1}^{\text{c}}\lambda^\text{n}/(\text{n}!\mu^\text{n}) + \sum_{\text{n}=\text{c}+1}^{\text{k}}\lambda^\text{n}\prod_{\text{r}=1}^{\text{n}-\text{c}}\{1-\text{r}/(\text{k}-\text{c}+1)\}/\left\{\text{c}!\mu^\text{c}\prod_{\text{r}=1}^{\text{n}-\text{c}}(\text{c}\mu+\text{r}v+\gamma(\gamma^\text{r}-1)/(\gamma-1)\}\right]^{-1}$$

Under R_EOS where customers may renege from queue as well as while being served, let q_n denote the probability that there are n customers in the system. Applying the Markov theory, we obtain the following set of steady state equations.

$$\lambda q_0 = (\mu+v)q_1 \tag{3.6}$$

$$\lambda q_{n-1} + (n+1)(\mu+v)q_{n+1} = \lambda q_n + n(\mu+v)q_n; \quad \text{n} = 1,2,\dots\text{c}-1 \tag{3.7}$$

$$\lambda q_{c-1} + \{c\mu + (c+1)v + \gamma\} q_{c+1} = \lambda\{1 - 1/(k-c+1)\} q_c + (c\mu + cv) q_c \tag{3.8}$$

$$\lambda\{1-(n-c)/(k-c+1)\}_{n-1} + \left\{c\mu + (n+1)v + \gamma\left(\gamma^{n-c+1}-1\right)/(\gamma-1)\right\} q_{n+1} = \lambda\{1-(n-c+1)/(k-c+1)\} q_n$$
$$+ \{c\mu + nv + \gamma(\gamma^{n-c}-1)/(c-1)\} q_n;\ n = c+1,..,k-1 \tag{3.9}$$

$$\lambda\{1-(\mathrm{k}-\mathrm{c})/(\mathrm{k}-\mathrm{c}+1)\}\mathrm{p}_{\mathrm{k}-1} = \left\{\mathrm{c}\mu + \mathrm{k}v + \gamma\left(\gamma^{\mathrm{k}-\mathrm{c}}-1\right)/(\gamma-1)\right\}\mathrm{p}_{\mathrm{k}} \tag{3.10}$$

Solving recursively, we get (under R_EOS)

$$q_n = [\lambda^n / \{n!(\mu+v)^n\}]\, \mathrm{q}_0; \quad \mathrm{n} = 1, 2, \ldots \mathrm{c}$$

$$q_n = \left[\lambda^n \prod_{r=1}^{n-c}\{1 - r/(k-c+1)\} \Big/ \left\{c!(\mu+v)^c \prod_{r=c+1}^{n}\left(c\mu + rv + \gamma\left(\gamma^{r-c}-1\right)/(\gamma-1)\right\}\right]\right] q_0; \quad \mathrm{n} = \mathrm{c}+1, \ldots, \mathrm{k}$$

where q_0 is obtained from the normalizing condition $\sum_{n=0}^{k} q_n = 1$ and is given as

$$q_0 = \left[1 + \sum_{n=1}^{c} \lambda^n / \{n!(\mu+v)^n\} + \sum_{n=c+1}^{k} \lambda^n \prod_{r=1}^{n-c}\{1 - r/(k-c+1)\} \Big/ \left\{c!(\mu+v)^c \prod_{r=c+1}^{k}\left(c\mu + rv + \gamma\left(\gamma^{r-c}-1\right)/(\gamma-1)\right\}\right]\right]^{-1}$$

4 Performance measures

In general, "performance measures are the specific representation of a capacity, process or outcome deemed relevant to the assessment of performance, which are quantifiable and can be documented" [20]. The main objective of any queuing study is to assess some well-defined parameters, which are designed at striking a good balance between customer satisfaction and economic considerations. In queuing theory, measures through which the nature of the quality of service can be studied are known as performance measures. Performance measures are important as issues or problems caused by queuing situations are often related to customer's dissatisfaction with service or may be the root cause of economic losses in a business. Analysis of the relevant performance measures of queuing models allows the cause of queuing issues to be identified and the impact of proposed changes to be assessed. Some of the performance measures of any queuing system that are of general interest for the evaluation of the performance of an existing queuing system and to design a new system in terms of the level of service a customer receives as well as the proper utilization of the service facilities include mean size, server utilization, customer loss and the like[(8)].

An important measure is the mean number of customers in the system, which is traditionally denoted by 'L'. We have presented the derivation of this important performance measure separately for the two reneging disciplines in the appendix. These are denoted by L_{R_BOS} and $\mathrm{L}_{R\_EOS.}$

Let P(s) be the p.g.f of the steady state probability under R_BOS rule. Then we note that

$$\begin{aligned} L_{R\_\mathrm{BOS}} &= \sum_{n=0}^{k} n p_n \\ &= \mathrm{P}'(1) \\ &= \frac{d}{ds} P(s)\Big|_{s=1} \end{aligned}$$

(See the appendix for more derivations)

From (A.11) and (B.5), the mean system sizes under the two reneging rules are

$$L_{R\_BO\,S} = \left[\begin{array}{l} \lambda\left(1 - S_{R\_BO\,S}\right) + \mu\left(1 - M_{R\_BO\,S} - cp_c\right) + \lambda(1-p)\left(\mathrm{S}_{R\_BO\,S} - p_k\right) - c\mu\left(\mathrm{S}_{R\_BO\,S} - p_c\right) + \\ v\left(M_{R\_BO\,S} + c\,\mathrm{S}_{R\_BO\,S}\right) - \frac{p_0}{(\gamma-1)\left(\gamma^{c-1}\right)\mathrm{p}_0(\gamma\lambda)} + \frac{\{p_0(\gamma\lambda) - p_0 K_{R\_BO\,S}(\gamma\lambda)\}}{(\gamma-1)\left(\gamma^{c-1}\right)\mathrm{p}_0(\gamma\lambda)} + \frac{\gamma\left(S_{R\_BO\,S} - p_c\right)}{(\gamma-1)} \end{array}\right] / (\mu+v)$$

$$L_{R\_\mathrm{EOS}} = \left[\begin{array}{l} \lambda\left(S_{R\_\mathrm{EOS}} - 1\right) + (\mu+v)\left(1 - M_{R\_\mathrm{EOS}} - cp_c\right) + \lambda(1-p)\left(\mathrm{q}_k - \mathrm{S}_{R\_\mathrm{EOS}}\right) + c\mu\left(\mathrm{S}_{R\_\mathrm{EOS}} - q_c\right) + \\ v\left(M_{R\_}\ \mathrm{BOS} - \mathrm{c}q_c\right) + \frac{q_0}{(\gamma-1)\left(\gamma^{c-1}\right)\mathrm{q}_0(\gamma\lambda)} + \frac{\{\mathrm{q}_0(\gamma\lambda) - q_0 K_{R\_\mathrm{EOS}}(\gamma\lambda)\}}{(\gamma-1)\left(\gamma^{c-1}\right)\mathrm{q}_0(\gamma\lambda)} - \frac{\gamma\left(S_{R\_EO\,S} - q_c\right)}{(\gamma-1)} \end{array}\right] / (\mu+2v)$$

Where

$$S_{R\_BOS} = \sum_{n=c}^{k} p_n \text{ and } M_{R\_BOS} = \sum_{n=1}^{c-1} n p_n$$
$$S_{R\_EOS} = \sum_{n=c}^{k} q_n \text{ and } M_{R\_EOS} = \sum_{n=1}^{c-1} n q_n$$

K_{R_BOS} and K_{R_EOS} are defined in (A.8) and (B.6) respectively.

The mean queue size formulas for the two cases can now be obtained and are given by

$$L_{q(R\_BOS)} = \sum_{n=c+1}^{k} (n-c) p_n$$

$$= P'(1) - \sum_{n=1}^{c} n p_n - c\left(S_{R\_BOS} - p_c\right)$$

$$= \left[\begin{array}{l} \lambda\left(1 - S_{R\_BOS}\right) + \mu\left(1 - 2M_{R\_BOS} - cp_c\right) + \lambda(1-p)\left(S_{R\_}\text{BOS} - p_k\right) - c\mu\left(2S_{R\_}\text{BOS} - p_c\right) + vcS_{R\_}\text{BOS} - \frac{p_0\gamma^{1-c}}{(\gamma-1)p_0(\gamma\lambda)} \\ \frac{\left\{p_0(\lambda\gamma,\mu,v,k) - p_0 K_{R\_BOS}(\gamma\lambda,\mu,v,k)\right\}\gamma^{1-c}}{p_0(\lambda\gamma,\mu,v,k)(\gamma-1)} + \frac{\gamma}{(\gamma-1)}\left(S_{R\_BOS} - p_c\right) - cvS_{R\_BOS} \end{array} \right] / (\mu+v)$$

And

$$L_{qR\_\text{EOS}} = \left[\begin{array}{l} \lambda\left(S_{R\_EOS} - 1\right) + (\mu+v)\left(1 - 2M_{R\_EOS} - cq_c\right) - (1-p)\left(S_{R\_\text{EOS}} - q_k\right) - c\mu\left(2S_{R\_\text{EOS}} - q_c\right) - vcS_{R\_\text{EOS}} - \frac{q_0\gamma^{1-c}}{(\gamma-1)q_0(\gamma\lambda)} \\ \frac{\left\{q_0(\lambda\gamma,\mu,v,k) - q_0 K_{R\_\text{EOS}}(\gamma\lambda,\mu,v,k)\right\}\gamma^{1-c}}{q_0(\lambda\gamma,\mu,v,k)(\gamma-1)} - \frac{\gamma}{(\gamma-1)}\left(S_{R\_EOS} - q_c\right) - cvS_{R\_EOS} \end{array} \right]$$

$$/ (\mu+2v)$$

Customers arrive into the system at the rate λ. However, all the customers who arrive do not join the system because of balking and due to finite buffer restriction. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$\lambda^e{}_{(R\_}\text{BOS}) = \lambda \sum_{n=0}^{c-1} p_n + \lambda \sum_{n=c}^{k-1} (1-p) p_n$$
$$= \lambda\left\{1 - (1-p)p_k\right\} - \lambda p S_{R\_}\text{BOS}$$

In case of R_EOS, the effective arrival rate is

$$\lambda^e{}_{(R\_}{}^{eO\,S)} = \lambda \sum_{n=0}^{c-1} q_n + \lambda \sum_{n=c}^{k-1} (1-p) q_n$$
$$= \lambda\left\{1 - (1-p)q_k\right\} - \lambda p S_{R\_}\text{EOS}$$

We have assumed that each customer has a random patience time following exp(v_n). Clearly then, the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rate (Avgrr) under two reneging rules are given by

$$\text{Avgrr}_{(R\_\text{BOS})} = \sum_{n=c+1}^{k} \left\{(n-c)v + \gamma\left(\gamma^{n-1} - 1\right)/(\gamma-1)\right\} p_n$$

$$= vP'(1) - v\sum_{n=1}^{c} n p_n - cv\left(S_{R\_BO\,S} - p_c\right) + \frac{\gamma^{1-c}}{(\gamma-1)} \sum_{n=c+1}^{k} \gamma^n p_n$$

$$= v\left(L_{R\_BO\,S} - M_{R\_BO\,S} - cS_{R\_BO\,S}\right) + \frac{\gamma^{1-c} p_0 K_{R\_BO\,S}(\gamma\lambda,\mu,v,k)}{p_0(\gamma\lambda,\mu,v,k)}$$

$$A\,\text{vgrr}_{(R\_\text{EOS})} = \sum_{n=1}^{c} n v q_n + \sum_{n=c+1}^{k} \left\{nv + \gamma\left(\gamma^{n-1} - 1\right)/(\gamma-1)\right\} q_n$$

$$= vL_{R\_BO\,S} + \frac{\gamma^{1-c} q_0 K_{R\_\text{EOS}}(\gamma\lambda,\mu,v,k)}{q_0(\gamma\lambda,\mu,v,k)}$$

Average balking rate under both the reneging rule are given by

$$\begin{aligned} Avgbr_{R\_BO\ S} &= \lambda p \sum_{n=c}^{k-1} p_n \\ &= \lambda p \left(S_{R\_BO\ S} - p_k\right) \\ Avgbr_{R\_\mathrm{EOS}} &= \lambda p \sum_{n=c}^{k-1} q_n \\ &= \lambda p \left(\mathrm{S}_{\mathrm{R}\_\mathrm{EOS}} - q_k\right) \end{aligned}$$

In any real life situation, customers who balk or renege represent business lost. Customers are lost to the system in three ways, due to balking, due to finite buffer restriction and due to reneging. Management of any firm would like to know the proportion of total customers lost in order to have an idea of total business lost.

Hence the mean rate at which customers are lost (under R_BOS) is

$$\begin{aligned} &\lambda - \lambda^e{}_{(R\_BO\ S)} + \operatorname{avg} rr_{(R\_BO\ S)} \\ &= \lambda(1-p)p_k + \lambda p S_{R\_BO\ S} + v\left(L_{R\_BO\ S} - M_{R\_BO\ S} - cS_{R\_BO\ S}\right) + \tfrac{\gamma^{1-c} p_0 K_{R\_BO\ S}(\lambda\gamma,\mu,v,k)}{(\gamma-1)p_0(\lambda\gamma,\mu,v,k)} \end{aligned}$$

and the mean rate at which customers are lost (under R_EOS) is

$$\begin{aligned} &\lambda - \lambda^e\,(R\_EO\ S) + \mathrm{avgrr}_{(R\_\mathrm{EOS})} \\ &= \lambda(1-p)q_k + \lambda p S_{R\_EO\ S} + vL_{R\_EO\ S} + \frac{\gamma^{1-c} q_0 K_{R\_\ \mathrm{EOS}}(\lambda\gamma,\mu,v,k)}{(\gamma-1)q_0(\lambda\gamma,\mu,v,k)} \end{aligned}$$

These rates helps in the determination of proportion of customers lost which is of interest to the system manager as also an important measure of business lost. This proportion (under R_BOS) is given by

$$\begin{aligned} &\left[\lambda - \lambda^e_{(R\_BO\ S)} + \mathrm{avgrr}_{(R\_BO\ S)}\right]/\lambda \\ &= (1-p)p_k + pS_{R\_BO\ S} + v\left(L_{R\_BO\ S} - M_{R\_BO\ S} - cS_{R\_BO\ S}\right)/\lambda + \tfrac{\gamma^{1-c} p_0 K_{R\_BO\ S}(\lambda\gamma,\mu,v,k)}{\lambda(\gamma-1)p_0(\lambda\gamma,\mu,v,k)} \end{aligned}$$

and the proportion (under R_EOS) is given by

$$\begin{aligned} &\left[\lambda - \lambda^e_{(R\_\ EO\ S}\right) + \operatorname{avg} rr_{(R\_EO\ S)}\Big]/\lambda \\ &= (1-p)q_k + pS_{R\_EO\ S} + (v/\lambda)L_{R\_\ \mathrm{EOS}} + \tfrac{\gamma^{1-c} q_0 K_{R\_\ \mathrm{EOS}}(\lambda\gamma,\mu,v,k)}{\lambda(\gamma-1)q_0(\lambda\gamma,\mu,v,k)} \end{aligned}$$

The proportion of customer completing receipt of service can now be easily determined from the above proportion.

The customers who leave the system from the queue do not receive service. Consequently, only those customers who reach the service station constitute the actual load of the server. From the server's point of view, this provides a measure of the amount of work he has to do. Let us call the rate at which customers reach the service station as λ^s .Then under R_BOS

$\lambda^s{}_{(R\_BOS)} = \lambda^e{}_{(R\_BOS)}$(1-proportion of customers lost due to reneging out of those joining the system)

$$\begin{aligned} &= \lambda^e\,(R\_BO\ S)\left\{1 - \textstyle\sum_{n=c+1}^{k}(n-c)p_n/\lambda^e\,(R\_BO\ S)\right\} \\ &= \lambda\left\{1-(1-p)p_k\right\} - (\lambda p - cc)S_{R\_BO\ S} - U_{R\_BO\ S} + vM_{R\_BO\ S} \end{aligned}$$

In case of R_EOS, one needs to recall that customers may renege even while being served and only those customers who renege from the queue will not constitute any work for the server. Thus

$\lambda^s{}_{(R\_EOS)} = \lambda^e{}_{(R\_EOS)}$(1-proportion of customers lost due to reneging from the queue out of those joining the system)

$$\begin{aligned} &= \lambda^e_{(R\_}\ \mathrm{EOS}\Big)\left\{1 - \textstyle\sum_{n=c+1}^{k}(n-c)v_n/\lambda^e_{(R\_}\ \underline{EO\ S)}\right\} \\ &= \lambda\left\{1-(1-p)q_k\right\} - (\lambda p - vc)S_{R\_\ \mathrm{EOS}} - vL_{R\_\ \mathrm{EOS}} + vM_{R\_\ \mathrm{EOS}} \end{aligned}$$

5 Sensitivity analysis

We have assumed that there are essentially five parameters viz: these may undergo change. From managerial point of view, an idle server is a waste. So also for low server utilization. It is therefore interesting to examine and understand how server utilization varies in response to change in system parameters. We place below the effect of change in these system parameters on server utilization. For this purpose, we shall follow the following notational convention in the rest of this section.

$\mathrm{p_n}(\lambda,\mu,v,c,k.)$ and $\mathrm{q_n}(\lambda,\mu,v,c,k.)$ will denote the probability that there are 'n' customers in a system with parameters λ,μ,v,c,k . in steady state under R_BOS and R_EOS respectively.

Let $\lambda_1 > \lambda_0$, then

$$\frac{p_0(\lambda_1,\mu,v,c,k)}{p_0(\lambda_0,\mu,v,c,k)} < 1$$
$$\Leftrightarrow \frac{(\lambda_0-\lambda_1)}{\mu}+\cdots+\frac{\left(\lambda_0^c-\lambda_1^c\right)}{d\mu^n}+\frac{(1-p)\left(\lambda_0^{c+1}-\lambda_1^{c+1}\right)}{c!\mu^c(c\mu+v+\gamma)}+\cdots+\frac{(1-p)^{k-c}\left(\lambda_0^k-\lambda_1^k\right)}{c!\mu^c(c\mu+v+\gamma)\ldots\left\{c\mu+(k-c)v+\gamma\left(\gamma^{k-c}-1\right)/(\gamma-1)\right\}}<0$$

which is true. Hence $p_0\uparrow$ as $\lambda\uparrow$ i.e. the probability that the system is empty goes down as the arrival rate goes up.

Let $\mu_1 > \mu_0$, then

$$\frac{p_0(\lambda\mu_1,v,c,k)}{p_0(\lambda,\mu_0,v,c,k)} > 1$$

$$\Leftrightarrow \lambda\left(\frac{1}{\mu_0}-\frac{1}{\mu_1}\right)+\ldots.+\frac{\lambda^n}{c!}\left(\frac{1}{\mu_0^c}-\frac{1}{\mu_1^c}\right)+\frac{x^{+1}(1-p)}{c!}\cdot\left\{\frac{1}{\mu_0^c(c\mu_0+v+\gamma)}-\frac{1}{\mu_1^c(c\mu_1+v+\gamma)}\right\}$$

$$+\frac{\lambda^k(1-p)^{k-c}}{c!}\left\{\begin{array}{c}\frac{1}{\mu_0^c(c\mu_0+v+\gamma)\ldots\left\{c\mu_0+(k-c)v+\gamma\left(\gamma^{k-c}-1\right)/(\gamma-1)\right\}}\\ \frac{1}{\mu_1^c(c\mu_1+v+\gamma)\ldots\left\{c\mu_1+(k-c)v+\gamma\left(\gamma^{k-c}-1\right)/(\gamma-1)\right\}}\end{array}\right\}>0$$

which is true. Hence $p_0\uparrow$ as $\mu\uparrow$ i.e. the probability that the system is empty goes up as the service rate goes up.

Let $v_1 > v_0$

$$\frac{p_0(\lambda,\mu,v_1,c,k)}{\mathrm{p}_0(\lambda,\mu,v_0,c,k)} > 1$$

$$\Leftrightarrow \frac{\lambda^{c+1}(1-p)}{c!\mu^c}\left\{\frac{1}{(c\mu+v_0+\gamma)}-\frac{1}{(c\mu+v_1+\gamma)}\right\}+\ldots$$

$$+\frac{\lambda^k(1-p)^{k-c}}{c!\mu^c}\left\{\begin{array}{c}\frac{1}{(c\mu+v_0+\gamma)\ldots\left\{c\mu+(k-c)v_0+\gamma\left(\gamma^{k-c}-1\right)/(\gamma-1)\right\}}\\ \frac{1}{(c\mu+v_1+\gamma)\ldots\left\{c\mu+(k-c)v_1+\gamma\left(\gamma^{k-c}-1\right)/(\gamma-1)\right\}}\end{array}\right\}>0$$

which is true. Hence $p_0\uparrow$ as $v\uparrow$ i.e. the probability that the system is empty goes up as the reneging rate goes up.

Let $\mathrm{k}_1 > \mathrm{k}_0$, then

$$\frac{p_0(\lambda,\mu,v,c,k_1)}{p_0(\lambda,\mu,v,c,k_0)} < 1$$
$$\Leftrightarrow \frac{\lambda^{k_0}(1-p)^{k_0-c}}{c!\mu^c(c\mu+v+\gamma)\ldots\left\{c\mu+(k_0-c)v+\frac{\gamma\left(\gamma^{k_0-c}-1\right)}{(\gamma-1)}\right\}}-\frac{\lambda^{k_1}(1-p)^{k_1-c}}{c!\mu^c(c\mu+v+\gamma)\ldots\left\{c\mu+(k_1-c)v+\frac{\gamma\left(\gamma^{k_1-c}-1\right)}{(\gamma-1)}\right\}}<0$$

which is true and hence $p_0 \uparrow$ as $k \uparrow$ i.e. the probability that the system is empty goes down as the capacity of the system goes up.

If $c_1 > c_0$, then

$$\frac{p_0(\lambda,\mu,v,c,k_1)}{p_0(\lambda,\mu,v,c,k_0)} > 1$$

$$\Leftrightarrow \frac{\lambda^{c_0}}{c_0!\mu^{c_0}} - \frac{\lambda^{c_1}}{c_1!\mu^{c_1}} + (1-p)\left\{\frac{\lambda^{c_1+1}}{c_0!\mu^{c_0}(c_0\mu+v+\gamma)} - \frac{\lambda^{c_1+1}}{c_1!\mu^{c_1}(c_1\mu+v+\gamma)}\right\} + \ldots$$

$$+(1-p)\left\{\frac{\lambda^{k-c_0}}{c_0!\mu^{c_0}(c_0\mu+v+\gamma)\ldots\left\{c_0\mu+(k-c_0)v+\frac{\gamma\left(\gamma^{k-c_0}-1\right)}{(\gamma-1)}\right\}} - \frac{\lambda^{k-c_1}}{c_1!\mu^{c_1}(c_1\mu+v+\gamma)\cdot\left\{c_1\mu+(k-c_1)v+\frac{\gamma\left(\gamma^{k-c_1}-1\right)}{(\gamma-1)}\right\}}\right\} > 0$$

which is true. Hence $p_0 \uparrow$ as $c \uparrow$ i.e. the probability that the system is empty goes down with the increased in number of server in the system.

Similar observations can be made under R_EOS i.e.

$$q_0 \downarrow \text{as } \lambda \uparrow$$
$$q_0 \uparrow \text{as } \mu \uparrow$$
$$q_0 \uparrow \text{as } v \uparrow$$
$$q_0 \downarrow \text{as } k \uparrow$$
$$q_0 \downarrow \text{as } c \uparrow$$

Under R_BOS, these results state that an increase in arrival rate would result in lowering of the fraction of time the server is idle. An increase in service rate would mean the server is able to work efficiently so that it can process same amount of work quickly. This translates to higher server idle time. An increase in reneging rate would mean the server has fewer work to do and hence higher fraction of idle time. An increase in system size translates to the lowering of the fraction of time the server is idle. Also an increase in number of server translates to higher server idle time. Similar conclusions can be drawn under R_EOS.

6 Conclusion

The analysis of a multi-server finite buffer Markovian queuing system with position dependent reneging and state dependent balking has been presented. Even though balking and reneging have been discussed by others, explicit expression is not available under both the reneging rule. This paper makes a contribution here. Closed form expressions of number of performance measures have been derived. To study the change in the system corresponding to change in system parameters, sensitivity analysis has also been presented. Extension of our result considering general distribution is a pointer to future research.

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