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The Rotating Metrics with the Mass Function $\hat{M}(u,r)$ in reference to the Theory of General Relativity

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Abstract

Background/Objectives: With reference to the theory of general relativity, the rotating metrics can be designed with mass function $\hat{M}(u,r)$. **Methods/Statistical analysis:** The methods/analysis adapted is the theoretical and mathematical analysis on the theory of general relativity. **Findings**: The line element can be found out with the help of mass function $\hat{M}(u,r)$ following the find of the covariant complex null tetrad vectors for the rotating metrics. Then, the NP spin — coefficients, the Ricci scalars, and the Weyl scalars for the rotating metrics can be found out. The expanded form of the mass function $\hat{M}(u,r)$ with (a=0) can be shown from Wang and Wu (1999). From the expended form of the mass function $\hat{M}(u,r)$ with $(a \neq 0)$, the NP coefficients can be derived. With the help of the scalar k, the surface gravity of the black hole is derived. **Novelty/Applications:** The findings of covariant complex null tetrad vectors for the rotating metrics, the NP spin — coefficients, the Ricci scalars, and the Weyl scalars are new analysis towards the theory of general relativity. Specific applications are the deep studies on the black hole and its surface gravity. It can be concluded that generally the rotating metric possesses a geodesic $(k^* = \varepsilon = 0)$, actually shear free $(\sigma = 0)$, purely expanding $(\hat{\theta} \neq 0)$ and a non-zero twist $(\omega^{*2} \neq 0)$ null vector l_a (Chandrasekhar, 1983). With the help of a scalar k, on a horizon of a Black hole, the surface gravity of the black hole is derived.

Keywords: The Mass Function; NP Spin – Coefficients; The Ricci Scalars; The Weyl Scalars; The Surface Gravity of the Black Hole

1 Introduction

The rotating metrics with mass function $\hat{M}(u,r)$ can be presented with respect to the theory of general relativity. The line element can be of the form of given below (Ibohal, 2005a)[1,2].

$$ds^2 = \left\{ 1 - 2r\hat{M}(u,r)R^{-2} \right\} du^2 + 2du\,dr + 4ar\hat{M}(u,r)R^{-2}\sin^2\theta\,du\,d\phi$$
$$-2a\sin^2\theta\,dr\,d\phi - R^2 d\theta^2 - \left\{ \left(r^2 + a^2\right)^2 - \Delta^* a^2 \sin^2\theta \right\} R^{-2}\sin^2\theta\,d\phi^2 \tag{1}$$

Here, $R^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta^* = r^2 - 2r\hat{M}(u,r) + a^2$.

And the covariant complex null tetrad vectors for the rotating metrics can be written as follows below [3,4].

$$l_a = \delta_a^1 - a \sin^2 \theta \, \delta_a^4$$

$$n_a = \frac{\Delta}{2R^2} \delta_a^1 + \delta_a^2 - \frac{\Delta}{2R^2} a \sin^2 \theta \, \delta_a^4$$

$$\bar{m}_a = -\frac{1}{\sqrt{2}\bar{R}} \left\{ ia \sin \theta \, \delta_a^1 + R^2 \delta_a^3 - i \left(r^2 + a^2 \right) \sin \theta \, \delta_a^4 \right\}$$

$$m_a = -\frac{1}{\sqrt{2}R} \left\{ -ia \sin \theta \, \delta_a^1 + R^2 \delta_a^3 + i \left(r^2 + a^2 \right) \sin \theta \, \delta_a^4 \right\}$$

(2)

The NP spin — coefficients, the Ricci scalars, Weyl scalars for the given rotating metrics a written as follows [5,6]:

The spin — coefficients:

$$k^* = \sigma = \lambda = \varepsilon = 0$$

$$\rho^* = -\frac{1}{\bar{R}}, \mu^* = -\frac{\Delta}{2\bar{R}R^2}$$

$$\alpha = \frac{(2ai - R\cos\theta)}{2\sqrt{2}\bar{R}R\sin\theta}, \beta = \frac{\cot\theta}{2\sqrt{2}R}$$

$$\pi = \frac{ia\sin\theta}{\sqrt{2}\bar{R}R}, \tau = -\frac{ia\sin\theta}{\sqrt{2}R^2}$$

(3)

$$\gamma = \frac{1}{\sqrt{2}\bar{R}R^2} \left[\left(r - \hat{M} - r\hat{M}_{,r} \right) \bar{R} - \Delta^* \right]$$

$$v = \frac{1}{\sqrt{2}\bar{R}R^2} iar \sin \theta \hat{M}_{,u}$$

The notations k^*, ρ^*, μ^* carry the usual NP spin coefficients k, ρ, μ. The notation ρ, μ can be treated as energy density and null density respectively in energy – momentum tensor for the rest of the paper. In Riemannian geometry, the scalar curvatures or the Ricci scalars are the simplest invariant in Riemannian manifold. It actually assigns a single real number determined by the so called intrinsic geometry of that manifold near that very point. Now, the Ricci scalars are as follows [7,8] :

$$\phi_{00} = \phi_{01} = \phi_{10} = \phi_{02} = \phi_{20} = 0$$

$$\phi_{11} = \frac{1}{4R^2R^2} \left[4r^2\hat{M}_{,r} + R^2 \left(-2\hat{M}, -r\hat{M}_{,mr} \right) \right]$$
$$\phi_{12} = \frac{1}{2\sqrt{2}R^2R^2} \left[ia\sin\theta \left\{ R\hat{M}_{,u} - r\hat{M}_{,ru}\bar{R} \right\} \right]$$

$$\phi_{21} = \frac{-1}{2\sqrt{2}R^2R^2} \left[ia\sin\theta \left\{ \bar{R}\hat{M}, -r\hat{M}_{,ru}R \right\} \right]$$

(4)

$$\phi_{22} = -\frac{1}{2R^2R^2} \left[2r^2\hat{M}_{,u} + a^2 r\hat{M}, \sin^2\theta \right\} \right]$$
$$\Lambda^* = \frac{1}{12R^2} \left(2\hat{M}_{,r} + r\hat{M}_{,mr} \right)$$

The Weyl scalars are

$$\varphi_0 = \varphi_1 = 0$$

$$\varphi_2 = \frac{1}{\overline{RR}R^2}\left\{-R\widehat{M} + \frac{\overline{R}}{6}\widehat{M}_r(4r + 2ia\cos\theta) - \frac{r}{6}\overline{RR}\widehat{M}_{mr}\right\}$$

$$\varphi_3 = -\frac{ia\sin\theta}{2\sqrt{2}\overline{RR}R^2}\left\{(4r + \overline{R})\widehat{M}_{,u} + r\overline{R}\widehat{M}_{,ur}\right\}$$

$$\varphi_4 = \frac{a^2r\sin^2\theta}{2\overline{RR}R^2\overline{R}^2}\left\{R^2\widehat{M}_{,uu} - 2r\widehat{M}_{,u}\right\}$$

(5)

From all these NP spin — coefficients we find that generally the rotating metric possesses a geodesic ($k^* = \varepsilon = 0$), actually shear free ($\sigma = 0$), purely expanding ($\hat{\theta} \neq 0$) and a non-zero twist ($\omega^{*2} \neq 0$) null vector l_a (Chandrasekhar, 1983)[9,10], where

$$\hat{\theta} \equiv -\frac{1}{2}(\rho + \overline{\rho}) = \frac{r}{R^2}, \omega^{*2} \equiv -\frac{1}{4}(\rho - \overline{\rho})^2 = -\frac{a^2\cos^2\theta}{R^2\overline{R}^2}$$

(6)

Again, then, the energy momentum tensor for the rotating metric will look like as follows[11,12]:

$$T_{ab} = \mu l_a l_b + 2\rho l_{(a\eta_b)} + 2pm_{(c\bar{m}_b)} + 2\omega l_{(c\bar{m}_b)} + 2\bar{\omega}l_{(am_b)}$$

(7)

And with the following quantities as follows:

$$\mu = -\frac{1}{KR^2\overline{R}^2}\left[2r^2\widehat{M}_u + a^2r\sin^2\theta\widehat{M}_{uut}\right]$$

$$\rho = \frac{2r^2}{KR^2\overline{R}^2}\widehat{M}_{,r}$$

$$p = -\frac{1}{K}\left[\frac{2a^2\cos^2\theta}{R^2\overline{R}^2}\widehat{M}_{rr} + \frac{r}{R^2}\widehat{M}_m\right]$$

$$\omega = -\frac{ia\sin\theta}{\sqrt{2}KR^2\overline{R}^2}\left[R\widehat{M}_u - r\overline{R}\widehat{M}_{,ur}\right]$$

(8)

All the above quantities have the relations with Ricci scalars as follows[13,14]

$$K\mu = 2\phi_{22}, K\omega = -2\phi_{12}$$

$$K\rho = 2\phi_{11} + 6\Lambda, Kp = 2\phi_{11} - 6\Lambda$$

(9)

The result implies that once we obtain Ricci scalars $\phi_{11}, \phi_{12}, \phi_{22}$ and Λ as in equation 4 for a given particular space – time metric, we can always find μ, ρ and ρ which describes energy momentum tensor.

Wang and Wu (1999) actually have expanded the rotating mass function $\hat{M}(u, r)$ for the non – rotating solution (a=0) in the power of as follows

$$\hat{M}(u, r) = \sum_{n=-\infty}^{\infty} q_n(u)r^n$$

(10)

This equation is advantageous because an expanded form always gives us more insight into the system. That is the reason we developed many series expansion like Taylor Series Expansion, Fourier Series Expansion etc. Where $q_n(u)$ is an arbitrary function of u. Wang and Wu consider the above summation as in integral form when the spectrum index $'n'$ is actually continuous in nature. Using the expression in equation 8, we can generate rotating metrics with ($a \neq 0$) as follows below by

replacing mass function $\hat{M}(u,r)$ of equation 10 with the arbitrary function $q_n(u)$. Thus we can rewrite equation 8 as follows [15,16].

$$\mu = -\frac{1}{KR^2R^2}\sum_{n=-\infty}^{\infty}\left[2q_n(u)_u r^{n+2} + a^2\sin^2\theta q_n(u)_{nu}r^{n+1}\right]$$

$$\rho = \frac{2}{KR^2R^2}\sum_{n=-\infty}^{\infty}(n+2)q_n(u)r^{n+1}$$

$$p = -\frac{1}{KR^2}\sum_{n=-\infty}^{\infty}nq_n(u)r^{n-1}\left[\frac{2a^2\cos^2\theta}{R^2}+(n-1)\right]$$

$$\omega = -\frac{ia\sin\theta}{\sqrt{2}KR^2R^2}\sum_{n=-\infty}^{\infty}\left[(R-n\overline{R})q_n(u)_{,u}r^n\right]$$

(11)

The same way, the other NP quantities can be found with the arbitrary function $q_n(u)$. The derivatives of the mass function $\hat{M}(u,r)$ and NP quantities can be easily related by doing derivatives of the equation (10).

In general relativity, the Newtonian acceleration (surface gravity) is not so clear cut concept. For a black hole, the surface gravity should be treated relativistically, one can not define surface gravity as acceleration of a test body at its surface because there is no surface exists in a black hole. It is because the surface gravity of a body at event horizon turns out to be infinity according to general relativity. After renormalization, the surface gravity of a black hole can be analogous to Newtonian surface gravity, but they are not the same thing. In fact, the surface gravity of a black hole is not well-defined. But, one can define the surface gravity of a black hole whose event horizon is a killing horizon. The surface gravity k of a static killing horizon is the acceleration. If k^a is normalized killing vector, then,

$$k^a\nabla_b k^b = kk^b \tag{12}$$

According to Carter (1968, 1973) and also York (1984), we can introduce a scalar k which can be defined by the relation $n^b\nabla_b n^a = kn^a$, here the null vector n^b is parameterized by the coordinate u such that $d/du = n^a\nabla_a$. And again this scalar k, generally, can be expressed as NP spin coefficient γ as follows:

$$k = n^b\nabla_b n^a l_a = -(\gamma+\bar{\gamma}) \tag{13}$$

Where γ is a spin — coefficient cited in equation 3. On a horizon of a Black hole, the scalar k is called the surface gravity of the black hole.

2 Conclusion

With a start of the rotating metrics with Mass Function $\hat{M}(u,r)$, the line element is designed following the covariant complex null tetrad vectors for the rotating metrics with mass function $\hat{M}(u,r)$. Then, it follows, the derivations of NP Spin — Coefficients, the Ricci Scalars, the Weyl Scalars. From all this NP spin — coefficients we found that generally the rotating metric possesses a geodesic $(k^* = \varepsilon = 0)$, actually shear free $(\sigma = 0)$, purely expanding $(\hat{\theta} \neq 0)$ and a non-zero twist $(\omega^{*2} \neq 0)$ null vector l_a. Once we found the Ricci Scalars, we can always find the energy momentum tensors. Wang and Wu actually have expanded the rotating mass function $\hat{M}(u,r)$ for the non — rotating solution $(a = 0)$ as the power of r. With help of the expanded form of mass function $\hat{M}(u,r)$, we have generated the rotating metrics with $(a \neq 0)$. With the help of a scalar k, on a horizon of a Black hole, the surface gravity of the black hole is derived.

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References

1) Bousso R. The cosmological constant. *General Relativity and Gravitation*. 2008;40:607–637. Available from: https://dx.doi.org/10.1007/s10714-007-0557-5.

2) Cai RG, Ji JY, Soh KS. Action and entropy of black holes in spacetimes with a cosmological constant. *Classical and Quantum Gravity*. 1998;15(9):2783–2793. Available from: https://dx.doi.org/10.1088/0264-9381/15/9/023.
3) Chandrasekhar S. The Mathematical Theory of Black Holes. Oxford; New York. Oxford University Press. 1983.
4) Hawking S, Ellis G. The Large Scale Structure of Space- Time. Cambridge. Cambridge University Press. 1973.
5) Ibohal N. Rotating metrics admitting non-perfect fluids. *General Relativity and Gravitation*. 2005;37(1):19–51. Available from: https://dx.doi.org/10.1007/s10714-005-0002-6.
6) Ibohal N, Ishwarchandra N. Non - vacuum conformally flat spacetimes: Dark energy. *Astrophysics and Space Science*;335(2):581–591. doi:10.1007/s10509-011-0767-x.
7) Mallett RL. Radiating Vaidya metric imbedded in de Sitter space. *Physical Review D*. 1985;31(2):416–417. Available from: https://dx.doi.org/10.1103/physrevd.31.416.
8) Mallett R. Pion interferometry of quark - gluon plasma. *Physical Review D*. 1985;33(5). Available from: https://doi.org/10.1103/PhysRevD.33.1314.
9) Newman E, Penrose R. An Approach to Gravitational Radiation by a Method of Spin Coefficients. *Journal of Mathematical Physics*. 1962;3(3):566–578. Available from: https://dx.doi.org/10.1063/1.1724257.
10) Ibohal N. Non - stationary de Sitter cosmological models. *International Journal of Modern Physics D*. 2009;18:853–863. Available from: https://doi.org/10.1142/S0218271809014807.
11) Ibohal N. Non - stationary rotating black holes: Entropy and Hawking's radiation. *International Jounal of Modern Physics D*. 2005;14(8). Available from: https://doi.org/10.1142/S0218271805007127.
12) Ibohal N. On the variable - charged black holes embedded into de Sitter space: Hawking's radiation. *International Jounal of Modern Physics D*. 2005;14(6):973–994. Available from: https://doi.org/10.1142/S0218271805007188.
13) Hawking SW. Information loss in black holes. *Physical Review D*. 2005;72(8):84013. Available from: https://dx.doi.org/10.1103/physrevd.72.084013.
14) Hawking SW, Hertog T. Why does inflation start at the top of the hill? *Physical Review D*. 2002;66(12):1–11. Available from: https://dx.doi.org/10.1103/physrevd.66.123509.
15) Hawking S, Hartle J, Hertog T. No - boundary measure of the Universe. *Physical Review Letter*. 2008;100:1–4. Available from: https://doi.org/10.1103/PhysRevLett.100.201301.
16) Wang A, Wu Y. LETTER: Generalized Vaidya Solutions. *General Relativity and Gravitation*. 1999;31(1):107–114. Available from: https://dx.doi.org/10.1023/a:1018819521971.