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RESEARCH ARTICLE

A Note on Full k-Ideals in Ternary Semirings

T Sunitha[1,2], U Nagi Reddy[3]*, G Shobhalatha[4]

1 Research Scholar, Department of Mathematics, Rayalaseema University, Kurnool, Andhra Pradesh
2 Lecturer in Govt. Degree College, Nandikotkur, Kurnool, Andhra Pradesh
3 Assistant Professor, Department of Mathematics, Rayalaseema University, Kurnool, Andhra Pradesh
4 Professor, Department of Mathematics, Sri Krishnadevaraya University, Anantapur, Andhra Pradesh

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* **Corresponding author**.

nagireddyppr@gmail.com

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Abstract

Objectives: k – ideals plays a vital role in ternary semirings. Ternary algebraic systems is a generalization of algebraic structures and it is the most natural way for the further development, deeper understanding of their properties. **Methods**: We have imposed Integral Multiple Property (IMP) and some other different constrains on a ternary semiring. **Findings**: In this study, we have described more results on the full k-ideal in the ternary semirings. Finally, we provide the characterization of full k-ideal in ternary semirings and studied their related properties. **Applications**: The structures of ideals in ternary semirings are widely applicable to computer sciences, dynamical and logical systems, cryptography, graph theory and artificial intelligence.

Keywords: Ternary Semiring; Ideal; k- Ideal; Full k- Ideal; Inverse

1 Introduction

The first formal definition of semiring was introduced in the year 1934 by Vandiver[(1)]. Several researches have characterized the many type of ideals on the algebraic structures such as: In 1958, Iséki considered and proved some theorems on quasi-ideals in semirings. However the developments of the theory in semirings have been taking place since 1950. A semiring is basic structure in Mathematics. The semiring theory and semigroup theory influenced on the developments of the semiring theory and its ordering. Nagi Reddy U, Rajani K, and Shobhalatha G have studied the fuzzy bi-ideals in ternary semigroups[(2)]. Ternary rings are introduced with their structures[(3)]. Some properties of ternary semirings are derived with the quasi ideals and Bi ideals[(4)]. S* semirings and A* semirings, which are studied with the some special structures[(5)]. Certain type of ring congruences on an additive inversive semirings with the help of full k-ideals is studied[(6)]. Sen and Adhikari gave some characterizations of maximal k-ideals of semiring.

Our main purpose of this paper is to introduce the notions of k- ideals and full k - ideals in ternary semirings and study the set of all full k-ideals of an additively inverse ternary semiring in which addition is commutative forms a complete lattice which is also modular.

2 Preliminaries

Definition 2.1: A Ternary semiring is a nonempty set S together with the binary operation addition and ternary operation multiplication denoted by $+, \cdot$ respectively, satisfying the following conditions:

1. $(S,+)$ is a commutative semigroup.
2. $(S,\cdot)$ is ternary semigroup.
3. Distributive laws holds, i.e., $a\cdot b(c+d) = a\cdot b.c + a\cdot b.d \quad a(b+c)d = a\cdot b\cdot d + a.c.d$ and $(a+b)\cdot c\cdot d = a\cdot c\cdot d + b.c\cdot d$

Definition 2.2: An element a of a ternary semiring S is said to be additive idempotent element provided a+a=a .

Note: The set of all additive idempotent elements in a ternary semiring S is denoted by $E^{+}(S)$.

That is $E^{+}(S) = \{a \in S / a+a = a\}$.

Definition 2.3: A ternary semiring S is called E-inverse, if for every $a \in S$, there exists $x \in S$ such that $a+x \in E^{+}(S)$.

Note : Let S be a ternary semi ring, then $E^{+}(S)$ is an ideal of S.

Definition 2.4: A subset I of a ternary semiring S is called a left (resp. a right, lateral) ideal of S if

1. $a+b \in I$ for all $a,b \in I$
2. for any $a \in I,$ and $b,c \in S, bca \in I($ resp. $abc \in I, bac \in I)$

A subset I is called an ideal if I is left, lateral and right ideal.

Note:

1. If A,B are any two ideals of a ternary semiring S, then $A \cap B$ is an ideal.
2. Let A,B be two ideals of a ternary semiring S, then the sum of A,B denoted by $A+B$ is an ideal of S where $A+B = \{x = a+b \mid a \in A, b \in B\}$

Definition 2.5: An ideal I of a ternary semiring S is called full if $E^{+}(S) \subseteq I$

Example: In any ternary ring R, the set $E^{+}(R) = \{0\}$, and so every ideal of R is a full ideal.

Definition 2.6. An ideal I of a ternary semiring S is called k-ideal or subtractive if for any two elements $a \in I$ and $x \in S$ such that $a+x \in I$, then $x \in I$.

Example. In any ternary ring R, every ideal I is k-ideal, since for any $a \in I, x \in R$ such that $a+x \in I$ then $a+x+(-a) \in I$ so $x \in I$

Definition 2.7. A k-ideal I of a ternary semiring S is called full k-ideal if the set of all additive idempotents of S, $E^{+}(S)$ is contained in I.

Example 1: In any ternary ring R every ideal I is a full k-ideal. Since 0 is the only additive idempotent element in R which belongs to any ideal I of R. So I is full k-ideal.

Example 2: In a distributive lattice L with more than two elements, a proper ideal I is k-ideal but not full k-ideal. Let $a \in I, x \in L$ such that $a \vee x \in I,$ then $x \leq a \vee x$. But I is an ideal so $x = x \wedge (a \vee x) \in I$. Hence I is k-ideal. Moreover, the set of all additive idempotents of L is L itself, since $a \vee a = a$ for all $a \in L$. So I is not full k-ideal.

Example 3: In $Z \times Z^{+} = \{(a,b) : a,b$ are integers and $b > 0\}$ we define $(a,b)+(c,d) = (a+c, lcm(b,d))$ and $(a,b)\cdot(c,d)\cdot(e,f) = (a.c.e, \gcd(b,d,f))$, then $Z \times Z^{+}$ is an additive inversive ternary semiring.

Solution: Let $(a,b),(c,d),(e,f) \in Z \times Z^{+}$

Additive commutative:

$$(a,b)+(c,d) = (a+c, lcm(b,d)) = (c+a, lcm(b,d)) = (c,a)+(a,b)$$

Additive associative:

$$\begin{aligned}((a,b)+(c,d))+(e,f) &= ((a+c, lcm(b,d))+(e,f) \\ &= (((a+c)+e, lcm(lcm(b,d),f)) \\ &= ((a+(c+e), lcm(b, \mathrm{lcm}(d,f))) \\ &= (a,b)+((c+e), \mathrm{lcm}(d,f)) \\ &= (a,b)+((c,d)+(e,f)).\end{aligned}$$

Multiplicative associative: Similarly as additive associative Distributive:

$$\begin{aligned}(a,b)\cdot(c,d)((e,f))+(g,h)) &= (a,b)\cdot(c,d)(e+g,\mathrm{lcm}(f,h)) \\ &= (a\cdot c(e+g),\gcd(b,d,lcm(f,h))) \\ &= (a\cdot c.e+a\cdot c.g,\quad lcm(\gcd(b,d,f),\gcd(b,d,h))) \\ &= (a\cdot c.e,\gcd(b,d,f))+(a\cdot c.e,\gcd(b,d,h)) \\ &= ((a,b)\cdot(c,d)(e,f)+(a,b)\cdot(c,d)(g,h))\end{aligned}$$

Similarly, $((e,f))+(g,h))(a,b)\cdot(c,d) = ((e,f)(a,b)\cdot(c,d)+(g,h)(a,b)\cdot(c,d))$

Additive inverse: For any $(a,b)\in Z\times Z^{+}$ there exists a unique $(-a,b)\in Z\times Z^{+}$ such that

$$\begin{aligned}&(a,b)+(-a,b)+(a,b) = (a+-a+a,lcm(b,b,b)) = (a,b), \\ &(-a,b)+(a,b)+(-a,b) = (-a+a+-a,lcm(b,b,b)) = (-a,b)\end{aligned}$$

Moreover, the set $A=\{(a,b)\in Z\times Z^{+}-a=0,b\in Z^{+}\}$ is full k-ideal of $Z\in Z^{+}$.

Since $E^{+}(Z\times Z^{+})=\{0\}\times Z^{+}\subseteq A$ and for any $(0,b)\in A,(c,d)\in Z\times Z^{+}$ such that $(0,b)+(c,d)=(c,lcm(b,d))\in A$, then $c=0$, so $(c,d)\in A$.

Definition 2.8: Let A be an ideal of an additive inversive ternary semiring S. We define the k-closure of A, denoted by A by:

$$\bar{A}=\{a\in S\cdot a+x\in A\quad \text{for some } x\in A\}$$

Definition 2.9: A lattice L is called a modular lattice simply modular, if for $a,b,c\in L, a\leq b\quad a\wedge c=b\wedge c\quad a\vee c=b\vee c$ implies $a=b$.

3 Main Results

Theorem 3.1. Let A and B be two full k-ideal of a ternary semiring S. then $A\cap B$ is full k-ideal.

Proof. Let A and B be two full k-ideal of S, then $A\cap B$ is an k ideal which is full as $E^{+}(S)\subseteq A$ and $E^{+}(S)\subseteq B$

Let $x\in S$ such that $a+x\in A\bigcap B$ for some $a\in A\cap B$. Then $a+x\in A, a\in A$ and $a+x\in B, a\in B$ which implies that $x\in A$ and $x\in B$.

Hence, $x\in A\bigcap B$

Therefore, $A\cap B$ is full k-ideal.

Theorem 3.2. Every k-ideal of ternary semiring S is an inversive sub semiring of S.

Proof. Clearly that every ideal of S is sub semiring of S. Let $a\in I$, then $a\in S$, so there exist $a'\in S$ such that $a=a+a'+a=a+(a'+a)\in I$.

But I is a k-ideal and $a\in I$, so $a'+a\in I$. Again I is a k-ideal and $a\in I$, so $a'\in I$.

Hence I is an inversive sub semiring of S.

Theorem 3.3. Let A be an ideal of ternary semirin S. Then A is a k -ideal of S. Moreover $A\subseteq\bar{A}$.

Proof. Let $a,b\in\bar{A}$, then $a+x, b+y\in A$ for some $x,y\in A$.

Now $(a+b)+(x+y)=(a+x)+(b+y)\in A$.

But $x+y\in A$, so $a+b\in\bar{A}$. Next let $p,r\in S$, then pra + prx $=\mathrm{pr}(a+x)\in A$.

But prx $\in A$, *so*, $pra\in\overline{A}$. *Similarly*, $apr\in A$.

Since $\overline{A}$ is an ideal of S.

To show that $\overline{A}$is k-ideal.

Let $c, c+d\in\bar{A}$, then there exist x and y in A such that $c+x\in A$ and $c+d+y\in A$.

Now $d+(c+x+y)=(c+d+y)+x\in A$ and $c+x+y\in A$.

Hence $d\in\overline{A}$ and so $\overline{A}$ is a k-ideal of S.

Finally, since $a+a\in A$ for all $a\in A$, it follows that $A\subseteq\bar{A}$.

Corollary 3.1 Let A be an ideal of ternary semiring S. Then $\overline{A}=A$ if and only if $\overline{A}$is a k-ideal.

Proof. Suppose $\overline{A}=A$, then by theorem 3.3 $\overline{A}$is k-ideal, and so A is k-ideal.

Conversely, assume that A is a k-ideal. Again by theorem 3.3 $A\subseteq\bar{A}$.

On the other hand, let $a\in\overline{A}$ then $a+x\in A$ for some $x\in A$. But A is a k-ideal and $x\in A$, implies $a\in A$, so $\overline{A}\subseteq A$. Therefore $A=\overline{A}$.

Corollary 3.2: Let A and B be two ideals of a ternary semiring S such that $A\subseteq B$, Then $\bar{A}\subseteq\bar{B}$.

Proof. Let A and B be two ideals of S such that $A \subseteq B$, let $a \in \bar{A}$, then $a+x \in A$ for some $x \in A$, but $A \subseteq B$, so $a+x \in B$ for some $x \in B$.

Hence $a \in \bar{B}$, Therefore $\overline{A} \subseteq \overline{B}$.

Corollary 3.3: Let A be an ideal of ternary semiring S. Then $\overline{A}$ is the smallest k-ideal containing A.

Proof. Let B be a k-ideal of S such that $A \subseteq B$, let $x \in \bar{A}$. Then $x+a_1 = a_2$ for some $a_1, a_2 \in A$.

Since $A \subseteq B$ and B is a k-ideal, then $x \in B$.

This implies that $\bar{A} \subseteq B$.

Therefore $\overline{A}$ is the smallest k-ideal containing A.

Theorem 3.4: Let A and B be two full k-ideals of ternary semiring S. then $\overline{A+B}$ is a full k-ideal of S such that $A \subseteq \overline{A+B}$ and $B \subseteq \overline{A+B}$.

Proof. Let A and B be two full k-ideals of ternary semiring S. Then $A+B$ is an ideal of S.

Then by theorem 3.3 $\overline{A+B}$ is a k-ideal and $A+B \subseteq \overline{A+B}$.

Now $E^+(S) \subseteq A$ and $E^+(S) \subseteq B$. So for any $e \in E^+(S), e = e+e$.

Hence $E^+(S) \subseteq A+B \subseteq \overline{A+B}$. Which implies that $\overline{A+B}$ is a full k-ideal.

Finally let $a \in A$, Then $a = a+a'+a = a+(a'+a) \in A+B$ as $a'+a \in E^+(S) \subseteq B$

Hence $A \subseteq \overline{A+B}$ and similarly $B \subseteq \overline{A+B}$.

Theorem 3.5: The set of all full k-ideals of ternary semi ring S. denoted by I(S), is a complete lattice which is also modular.

Proof. Firstly we note that I(S) is a partially ordered set with respect to usual set inclusion. Let $A, B \in I(S)$. Then by Theorem 3.1 $A \cap B \in I(S)$, and by Theorem 3.4, $\overline{A+B} \in I(S)$.

Define $A \wedge B = A \bigcap B$ and $A \vee B = \overline{A+B}$.

It is clearly that $A \cap B = \inf\{A, B\}$, let $C \in I(S)$ such that $A, B \subseteq C$.

Then $A+B \subseteq C$ and $\overline{A+B} \subseteq \bar{C}$. But $C = \bar{C}$.

Which implies that $\overline{A+B} \subseteq C$.

Hence $\overline{A+B} = \sup\{A, B\}$. Thus we find that I(S) is a lattice.

If S be a ternary semiring, then $E^+(S)$ is an ideal of S.

Thus $E^+(S)$ is an ideal of S, which contained in every ideal in I(S).

Hence $\overline{E^+(S)}$ is the smallest full k-ideal in I(S), and also $S \in I(S)$.

Consequently I(S) is a complete lattice.

Finally to show that I(S) is modular.

Suppose that $A, B, C \in I(S)$ such that $A \wedge B = A \wedge C$ and $A \vee B = A \vee C$ and $B \subseteq C$.

Let $x \in C$. we have $C \subseteq \overline{A+C} = A \vee C$, so $x \in A \vee C = \overline{A+B}$.

Hence there exists $a+b \in A+B$ such that $x+a+b = a_1+b_1$ for some $a_1 \in A, b_1 \in B$.

Then $x+a+a'+b = a_1+b_1+a'$.

But $x \in C, a+a' \in E^+(S) \subseteq C$

Since C is full ideal and $b \in B \subseteq C$, then $a_1+b_1+a' \in C$. But $b_1 \in B \subseteq C$, which is k-ideal.

So $a_1+a' \in C$, also $a_1+a' \in A$ which implies that $a_1+a' \in C \cap A = A \bigcap B$.

Hence $a_1+a' \in B$.

So from (1), We find that $x+a+a'+b = a_1+a'+b \in B$. But $(a+a')+b \in B$, which is a k-ideal.

Which implies that $x \in B$.

Hence B=C.

Therefore I(S) is a modular lattice.

4 Conclusions

We considered the notion of k-ideals and fully k ideals in ternary semirings and studied their properties and relations between them.

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