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REVIEW ARTICLE

Classes of T*-Semiring

M Amala[1]*, **N Sulochana**[2], **G Rajeswari**[3]

1 Assistant Professor, Department of Applied Mathematics, Sri Padmavati Mahila Visvavidyalayam, Tirupati, Andhra Pradesh, India
2 Lecturer, Government Degree college, V.Madugula, Visakhapatnam, Andhra Pradesh, India
3 Yogi Vemana University, Kadapa, Andhra Pradesh, India

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* **Corresponding author**.

amalamaduri@gmail.com

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Abstract

Objectives: The main objective of this research article is to study the semiring structures, we have majorly focused on the constrains under which the structures of T*- semiring are additively and/or multiplicatively idempotent. We have also concentrated on the study of structures of totally ordered of T*- semiring. **Methods:** We have imposed singularity, cancellation property, Integral Multiple Property (IMP) and some other constrains on T*- semiring. **Findings:** when we imposed totally ordered condition on T*- semiring we observed that the additive structure takes place as a maximum addition. **Applications:** The proposed idempotents have wide applications to computer science, dynamical and logical systems, cryptography, graph theory and artificial intelligence.

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1 Introduction

The word idempotent signifies the study of semirings in which the addition operation is idempotent u + u = u. The best-known example for idempotent semiring is the max-plus semiring. Interest has been shown in such structures arose in the late 1950s through the observation that certain problems of discrete optimization could be linearized over suitable idempotent semirings [(1)]. The first mathematical structure we encounteris the natural number set N is a semiring [(1)].

Recently the subject has established connections with discrete event systems automata theory, non-expansive mappings, optimization theory. Idempotent semiring is a fundamental structure that has many applications in Computer Science. Idempotent semiring is a ring with additive idempotent. Recently modal operators of idempotent semirings are introduced to model the properties of programs and transition systems more suitably and to link algebraic and relational formalisms with dynamic and temporal logics [(2)].

Some other applications of semiring areas are cryptography, optimization theory, graph theory, dynamical systems, and automata theory. The paper is organized as follows: Section 1 contains an introduction. In section 2 some definitions are given, Section 3 presents the structure of T* - semiring. In section 4 we study the structure of totally ordered T*- semiring and the last section is the conclusion.

2 Preliminaries

Definition 2.1:

An algebraic structure (S, +, •) is termed as semiring if the additive and multiplicative reducts are semigroups and u(x + y) = ux + uy and (x + y)u = xu + yu for every u, x, y in S.

Definition 2.2:

An additive semigroup is said to be additively idempotent if u + u = u for all u in S.

A multiplicative semigroup is multiplicatively idempotent or band if u^2 = u for all u in S.

If both (S, +) and (S, •) are idempotents then S is known as an idempotent semiring[2,3].

Definition 2.3:

A semiring is termed as mono-semiring if u + x = ux for all u, x in S.

Definition 2.4:

A multiplicative semigroup is assumed to be left (right) singular if ux = u (xu = x) for all u, x in S.

An additive semigroup is said to be left (right) singular if u + x = u (u + x = x) for all u, x in S.

Definition 2.5:

An element u is periodic if $u^m = u^n$, where m and n are positive integers.

A multiplicative semigroup is said to be periodic if every one of their elements is periodic.

An element u is periodic if mu = nu, where m and n are positive integers.

An additive semigroup is said to be periodic if every one of their elements is periodic.

Definition 2.6:

An additive semigroup (multiplicative semigroup) is rectangular band if u = u + x + u (u = uxu) for all u, x in S.

Definition 2.7:

A semiring is said to be zerosum if u + u = 0 for all u in S.

A semiring is said to be zero square if u^2 = 0 for all u in S

Definition 2.8:

In a semiring S, the semigroup (S, •) is zeroid if for all u in S such that ux = x or xu = x for some x in S.

In a semiring S, the additive semigroup is zeroid if for all u in S such that u + x = x + u = x for some x in S.

Definition 2.9:

An additive semigroup (multiplicative semigroup) is commutative if u + x = x + u (ux = xu) for all u, x in S.

Definition 2.10:

A component u in a multiplicative semigroup is known as left and right cancellable, if ux = uy (xu = yu) for any x, y in S implies x equals to y.

An element u in an additive semigroup is known as left and right cancellable, if u + x = u + y and x + u = y + u for any x, y in S implies x equals to y.

Definition 2.11:

A semiring is almost idempotent if u + u^2 = u^2 for all u, x in S.

Definition 2.12:

A semiring S is said to satisfy the Integral Multiple Property (IMP) if u^2 = na for all a in S where the positive integer n depends on the element u.

Definition 2.13: In a semiring S, an element u is Multiplicatively Subidempotent if u + u^2 = u.

Definition 2.14:

In a totally ordered semiring (S, +, •, $\leq$) (i) (S, +, $\leq$) is p.t.o, if u + x $\geq$ u, x for all u, x in S. (ii) (S, •, $\leq$) is p.t.o, if ux $\geq$ u, x for all u, x in S.

Definition 2.15:

A totally ordered semigroup (S, +, $\leq$) is assumed to be non-negatively (non-positively) ordered if every element of S is non-negative/non-positive.

Definition 2.16:

An element *u* in a totally ordered semiring is said to be a minimal/maximal if u $\leq$ x (u $\geq$ x) for every u $\in$S.

Note:

1. In this paper a semiring S is said to be a T^* semiring if it satisfies the identity u^2 + ux = u for all u, x in S[4].

2. In this research article the totally ordered is represented by t.o and positively totally ordered by p.t.o, Integral multiple property by IMP.

3 Classes of T*-Semiring

Lemma 3.1: Let S be a T – semiring. Then S is an idempotent semiring in the following cases.
(i) S contains the left or right multiplicative identity.
(ii) (S, +) is right cancellative.
(iii)(S, +) is left singular.
Proof: (i) Given that S is a T*–semiring then $u^2 + u^2 = u$ for all u in S →(1)
Also S contains multiplicative identity i.e1 + 1 = 1 then u + u = u for all u in S
Therefore (S, +) is idempotent
From above $u + u = u \Rightarrow u^2 + u^2 = u^2 \rightarrow$ (2)
From equation (1) and (2)we get $u^2 = u$ for all u in S
Thus (S, •) is idempotent
Hence S is an idempotent semiring
(ii) By hypothesis $u^2 + ux = u$ for all u, x in S → (1)
$\Rightarrow u^2 + u^2 = u$ for all u in S → (2)
From (1) $u^2 + u(u + u) = u \Rightarrow u^2 + u^2 + u^2 = u \Rightarrow u + u^2 = u$ → (3)
From (2) and (3) $u^2 + u^2 = u + u^2$
Using (S, +) right cancellative in above then $u^2 = u$ for all u in S → (4)
Thus (S, •) is idempotent
By substituting equation (4) in equation (2) we obtain u + u = u for all u in S
Thus (S, +) is idempotent
Hence S is an idempotent semiring
By hypothesis u2 + u2 = u for all u in S → (1
Since (S, +) is left singular then u + x = u for all u, x in S
Then equation (1) becomes $u^2 = u$ →(2)
Therefore (S, •) is idempotent
Using equation (2) in (1) we get u + u = u for all u in S
Thus (S, +) is idempotent, Hence S is an idempotent semiring
Example 3.2: we have framed an example which is an idempotent T*–semiring satisfying lemma 3.1 (i).

+	u	x	y
u	u	x	y
x	u	x	y
y	u	x	y

•	u	x	y
u	u	x	y
x	u	x	y
y	u	x	y

Theorem 3.3: Let S be a T*-semiring and S be a multiplicatively subidempotent semiring. Then $u + u^n = u$ for all u in S.
Proof: Consider $u^2 + u^2 = u$ for all u in S → (1)
Since S is multiplicative subidempotent $u + u^2 = u$ for all u in S → (2)
$\Rightarrow u^2 + u^3 = u^2 \Rightarrow u^2 + u^2 + u^3 = u^2 + u^2$using equation (1) $\Rightarrow u + u^3 = u$
Proceeding in a similar manner we obtain $u + u^n = u$ for all u in S
Theorem 3.4 : If S is a T*– semiring and (S, •) is left or right singular semigroup, then (S, +) is idempotent.
Proof: Given $u^2 + u.u = u$ for all u in S
By hypothesis(S, •) is left singular then u.x = u for all u, x in S
then above equation becomes u + u = u for all u in S
Therefore (S, +) is idempotent
Corollary 3.5 : In a T*–Semiring if (S, •) is idempotent, then (S, +) is idempotent.

Proof: Proof follows from theorem 3.4.

Proposition 3.6 : If S is a T*– semiring and (S, +) is idempotent, then $u^n + u = u$ for all u in S.

Proof: Given $u^2 + ux = u$ for all u, x in S $\rightarrow$ (1)

$\Rightarrow u^2 + u^2 + ux = u^2 + u$ since (S, +) is idempotent implies $u^2 + ux = u^2 + u$

$\Rightarrow u = u^2 + u$ by (1)

Multiplying u on both sides$u^3 + u^2 = u^2 \Rightarrow u^3 + u^2 + ux = u^2 + ux \Rightarrow u^3 + u = u$ by (1)

Continuing like this we get $u^n + u = u$ for all u in S.

Proposition 3.7 : Let S be a T*-semiring. If S is an almost idempotent semiring. Then (S, •) is periodic and (S, +) is idempotent.

Proof: We have $u^2 + u^2 = u$ for all u in S $\rightarrow$ (1)

Since S is almost idempotent semiring $u + u^2 = u^2$ for all u in S $\rightarrow$ (2)

From (1) $u (u + u^2) = u$ using equation (2) we obtain $u (u^2) = u \Rightarrow u^3 = u$

Therefore (S, •) is periodic

Also from equation (1) $u^2 + u = u \rightarrow$ (3)

$\Rightarrow u + u^2 + u = u + u$ using equation (2) implies $u^2 + u = u + u$

$\Rightarrow u = u + u$ using equation (3)

Therefore (S, +) is idempotent

Theorem 3.8: Let S be a T*-semiring and (S, •) be zeroed. Then $u^n=u^{n+1} + x$ or for all $n \geq 1$, where x is the element arising from the zeroid of u.

Proof: Consider $u^2 + ux = u$ for all u, x in S $\rightarrow$ (1)

Let $u \in S$. Then there exists x in S such that ux = x or xu = x for some y in S

Suppose ux = x $\rightarrow$ (2)

$u^2 + ux = u^2 + x \Rightarrow u = u^2 + x \Rightarrow u^2 = u^3 + ux \Rightarrow u^2 = u^3 + x$

$u^n=u^{n+1} + x$ for all $n \geq 1$

4 Some Special Classes of T*–Semiring

In this section, we study the structure of T*–semiring by considering different conditions and we will see the interrelations between different semirings.

Theorem 4.1: Let S be a T*–semiring. If S is mono semiring, then (S, •) and (S, +) are periodic.

Proof: By hypothesis $u^2 + u^2 = u$ for all u in S

Since S is mono semiring u + x = ux for all u, x in S

Then the above equation implies $u^2.u^2 = u\ u^4 = u$ for all u in S

Thus (S, •) is periodic

Since S is mono semiring $u^4 = u$ becomes as4u = u for all u in S for all u in S

Therefore (S, +) is periodic

Theorem 4. 2: If S is a T*–semiring and (S, •) is left cancellative, then $u + u^2 = x + x^2$ for all u, x in S.

Proof: Consider $u^2 + u^2 = u$ for all u in S $\Rightarrow u^2x+ u^2x = ux \rightarrow$ (1)

Also $x^2 + x^2 = x$ for all x in S $\Rightarrow ux^2 + ux^2 = ux \rightarrow$ (2)

From (1) and (2) $u^2x+ u^3x = ux^2 + ux3 \Rightarrow ux(u + u^2) = ux(x + x^2)$

Using (S, •) left cancellative in above then $u + u^2 = x + x^2$ for all u, x in S

Theorem 4.3 : Let S be a T*–semiring and S be zerosum semiring. Then $u^2 + u = u + u^2$ for all u in S.

Proof: We have $u^2 + ux = u$ for all u, x in S $\rightarrow$ (1)

$\Rightarrow u^2 + u^2 + ux = u^2 + u$

Since S is zerosum semiringu + u = u for all u in S

then above equation takes the form $0+ ux = u^2 + u \Rightarrow u^2 + u = ux \rightarrow$ (2)

Also $u^2 + ux = u \Rightarrow u + u^2 + ux = u + u$

Using S is zerosum semiring then $u + u^2 + ux = 0 \Rightarrow u + u^2 + ux + ux = 0 + ux$

$\Rightarrow u + u^2 + 0 = 0 + ux \Rightarrow u + u^2 =ux \rightarrow$ (3)

Equating equations (2) and (3) we obtain $u^2 + u = u + u^2$ for all u in S.

Proposition 4.4 : Let S be a T*- semiring and S satisfies Integral Multiple Property. Then (S, +) is periodic.

Proof: By hypothesis $u^2 + u^2 = u$ for all u in S

Since S satisfies IMP then above equation becomes nu + (nu) = u
$\Rightarrow$2nu = u where n is a positive integer
Therefore (S, +) is periodic.

5 Structures of Totally Ordered T*–Semiring

In this section, we study structures of totally ordered T – semiring.

Theorem 5 .1: If S is an ordered T*–semiring and (S, +) is p.t.o, then (S, $\bullet$) is non-positively ordered.
Proof: We have $u^2 + ux = u$ for all u, x in S
Since (S, +) is p.t.o $u + x \geq u$, x in S
Then $u^2 + ux \geq u^2 \Rightarrow u \geq u^2$ for all u on S
Therefore (S, $\bullet$) is non-positively ordered
Proposition 5 .2: Let S be an ordered T*–semiring. If S has multiplicative identity 1 and (S, +) is p.t.o., then
(i) (S, $\bullet$) is n.t.o.
(ii) 1 is the maximum element.
Proof: **(i)** We have $1^2 + 1 \cdot x = 1$ for all 1, x in S
Since (S, +) is p.t.o $u + x \geq u$, x in S
$\Rightarrow 1 + x = 1 \Rightarrow u\,(1 + x) = u \cdot 1 \Rightarrow u + ux = u$
Again using (S, +) is p.t.othen $u = u + ux \geq ux \rightarrow (1)$
Again by T*–semiring we have $1^2 + 1 \cdot u \cdot 1 = 1$ for all 1, u in S
$\Rightarrow 1 + u = 1 \Rightarrow (1 + u)\,x = 1 \cdot x \Rightarrow x + ux = x$
Since (S, +) is p.t.o$x = x + ux \geq ux \rightarrow (2)$
From (1) and (2) (S, $\bullet$) is n.t.o
(ii) We have $1 + 1 \cdot x = 1$ for all 1, x in S
Since (S, +) is p.t.o $u + x \geq u$, x in S
Then $1 = 1 + x \geq x$
Therefore 1 is the maximum element.
Theorem 5.3: If S is an ordered T*–semiring and (S, $\bullet$) is non-negatively ordered, then (S, +) is non-positively ordered.
Proof: Consider $u^2 + u^2 = u$ for all u in S
Since (S, $\bullet$) is non-negatively ordered $u^2 \geq u$ for all u in S
Therefore $u = u^2 + u^2 \geq u + u \cdot u \geq u + u$
$\Rightarrow u \geq u + u$ for all u in S
Therefore (S, +) is non-positively ordered
Theorem 5.4: Let S be a ordered T*–semiring. If S contains multiplicative identity. Then u + x = x + u = max (u, x) for all u, x in S.
Proof: Using lemma 3.1 (1), we have (S, +) is idempotent
Suppose $u, x \in S$
If $u < x$, then $u + u \leq u + x \leq x + x \Rightarrow u \leq u + x \leq x \rightarrow (1)$ since (S, +) is idempotent
Thus, from above $u + x \leq x \rightarrow (2)$
Since (S, +) is p.t.o $u + x \geq x \rightarrow (3)$
From (2) and (3) u + x = x = max (u, x) for all u, x in S
If $u < x$
Also $u + u \leq x + u \leq x + x \Rightarrow u \leq x + u \leq x \rightarrow (4)$
Using (S, +) idempotent in the above equation we obtain $x + u \leq x \rightarrow (5)$
Since (S, +) is p.t.o $x + u \geq x \rightarrow (6)$
From (5) and (6) x + u = x = max (u, x) for all u, x in S
Therefore u + x = x + u = max (u, x) for all u, x in S
Similarly, in the case, $x < u$ we can also prove that
u + x = x + u = max (u, x) for all u, x in S if $x < u$.

6 Conclusion

We have studied different constraints of T*–semiring. We got meaningful outcomes when applied multiplicative identity, n.t.o, and so on. We have also constructed an example with three examples in first lemma. We are interested to extend the work in diverse areas of study. Our future work can be continued by applying some other constrains on T*–semiring.

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