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Gamma coloring of Mycielskian graphs

R Gnanaprakasam[1*], I Sahul Hamid[2]

1 Lecturer in Mathematics, Tamilnadu Government Polytechnic College (Autonomous), Madurai11, India
2 Assistant professor in Mathematics, The Madura College (Autonomous), Madurai-11, India

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[*] **Corresponding author**.

gnanam.rgp@gmail.com

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Abstract

Background: Given a graph G, the gamma coloring problem seeks for a proper coloring C of G with the property that there exists a dominating set of G in which all the vertices receive different colors under the coloring C. The minimum number of colors required for a gamma coloring of G is called the gamma chromatic number of G and is denoted by $\chi_{\gamma}(G)$. Our aim is to find the gamma chromatic number of Mycielskian graphs. **Methods:** Here, we obtain gamma coloring for Mycielskian graph $\mu(G)$ from a gamma coloring of G by generalizing the give gamma coloring of G. To prove $\chi_{\gamma}(\mu(G)) \leq m$ for a graph G, we gave a gamma coloring to $\mu(G)$ using m colors. To prove $\chi_{\gamma}(\mu(G)) = m$ for a graph G, we first proved that $\chi_{\gamma}(\mu(G)) \geq m$ and then gave a gamma coloring to $\mu(G)$ using m colors. **Finding:** In this paper, we have initiated a study on Gamma coloring for Mycielskian graph $\mu(G)$ of a given graph G. We have proved that, the gamma chromatic number χ_{γ} for $\mu(G)$ is either $\chi_{\gamma}(G)$ or $\chi_{\gamma}(G)+1$ and thus, we classify the class of all connected graphs into two classes namely Class-1 and Class-2 graphs. Graphs G for which $\chi_{\gamma}(\mu(G)) = \chi_{\gamma}(G)$ are of Class-1 and rest of the graphs are of Class-2. Conditions under which a graph G becomes Class-1/ Class-2 have been established. **Novelty:** One can investigate towards finding a structural characterization of graph G with $\chi_{\gamma}(\mu(G)) = \chi_{\gamma}(G)$ or $\chi_{\gamma}(\mu(G)) = \chi_{\gamma}(G)+1$. Gamma coloring is a new variation of graph coloring in which the concepts of coloring and domination are linked using the condition that the coloring admits a dominating set in which every vertex receives different colors and, in this paper, we study about the gamma coloring of Mycielskian graph $\mu(G)$ of a graph G.

Keywords: Coloring; Dominating Set; Colorful Set; Mycielskian Graphs; Gamma Coloring

1 Introduction

All graphs considered in this paper are connected, simple, finite and undirected graphs. A coloring of a graph G is a function from V(G) to a set of colors which assigns different colors to adjacent vertices. The minimum number of colors needed for a proper coloring of G is called the chromatic number of G and is denoted by $\chi(G)$. A proper coloring of G using $\chi(G)$ colors is called a $\chi-$ coloring of G. A subset U of V(G) is said to be a dominating set of G if every vertex of V(G) is either in U or has a neighbor in U. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. A $\gamma-$set is a dominating set with cardinality $\gamma(G)$.

A subset U of V(G) is said to be a total dominating set of G if every vertex of V(G) has an adjacent vertex in U. The minimum cardinality of a total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(G)$.

Let C be a proper coloring of a graph G. A subset U of V(G) is said to be C-colorful if every vertex of U receives different color under the coloring C. A subset U of V(G) is said to be color transversal with respect to the coloring C if U intersects every color class of the coloring C. The study of a graph theoretical parameter in Mycielskian construction of graphs is one of the interesting research fields in graph theory. Some of such studies are Dominator coloring of Mycielskian graphs[(1)], strong coloring of Mycielskian graphs[(2)], connectivity of Mycielskian graphs[(3,4)], packing coloring of Mycielskian graphs[(5)], total chromatic number of Mycielskian graphs[(6)], diameter of Mycielskian of graphs[(7)], total weight choosability of Mycielskian graphs[(8)] and Hamilton-connected Mycielskian graphs[(9,10)]. In this paper we define the notion of gamma coloring and discuss about gamma coloring of Mycielskian graphs.

2 Gamma Coloring of graphs

In this section, we introduce the notion of gamma coloring of a graph along with an example.

Definition 2.1: A proper coloring C of a graph G is said to be a gamma coloring of G if there exists a dominating set which is C-colorful. The gamma chromatic number $\chi_\gamma(G)$is the minimum number of colors needed for a gamma coloring. A gamma coloring that uses $\chi_\gamma(G)$colors is called a minimum gamma coloring (or) a $\chi_\gamma-$coloring of G.

Remark 2.2: Certainly, for any graph G, the trivial coloring (that assigns distinct colors to distinct vertices) serves as a gamma coloring of G to which the whole vertex set V(G) is a colorful dominating set. Therefore, every graph admits a gamma coloring and so the parameter $\chi_\gamma(G)$ is well-defined for all graphs.

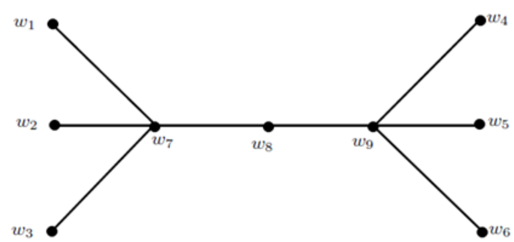

Fig 1. A Graph G with

Example 2.3: Consider the graph G shown in Figure 1. It is clear that, $\chi(G)=2$ and $\gamma(G)=2$. Therefore, $\chi_\gamma(G)\geq 2$. Further, since $C=((w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6,w_8\},(w_7,\ w_9\}\}$ and $S=(w_7,\ w_9\}$ is the only dominating set of G with cardinality 2 which is not C-colorful, it follows that $\chi_\gamma(G)\neq 2$ and thus $\chi_\gamma(G)\geq 3$. Also, $C=((w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6,w_8\},\{w_7\},\{w_9\}\}$ is a coloring of G to which $(w_7,\ w_9\}$ is a colorful dominating set so that C is a gamma coloring of G and hence $\chi_\gamma(G)\leq 3$. Thus, $\chi_\gamma(G)=3$.

Suppose C is a $\chi_\gamma-$coloring of a graph G with a colorful dominating set D. Then D has at most $\chi_\gamma(G)$ vertices so that $\gamma(G)\leq (D|\leq \chi_\gamma(G)$. It is also certain that $\chi_\gamma(G)\geq \chi(G)$ and thus we have the following observation.

Observation 2.4: For any graph G, we have $\chi_\gamma(G)\geq Max(\gamma(G),\ \chi(G)\}$.

3 Gamma Chromatic number of Mycielskian graphs

In this section we discuss about the gamma coloring of Mycielskian graph $\mu(G)$ of a graph G. For the sake of completeness let us recall the definition of Mycielskian graph of a graph G.

Definition 3.1: Let G be a graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$, edge set E and let $V'=\left(v'_1,v'_2,\ldots,v'_n\right\}$. The Mycielskian graph $\mu(G)$ of G, is the graph with vertex set $V(\mu(G))=V\cup V'\cup\{u\}$ and edge set $E(\mu(G))=E\cup(v'_iv_j/v_iv_j\in E\}\cup\left(uv'_i/v'_i\in V'\right\}$. The vertex v'_i is called the twin vertex of $v_i$and u is called the root vertex of $\mu(G)$. The Mycielskian graph of P_4 is given in Figure 2.

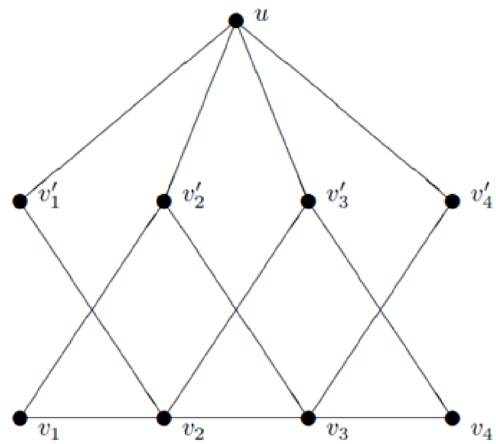

Fig 2. Mycielskian graph of P_4

It has been proved in[11] that, the chromatic number of $\mu\ (G)$ is always $\chi(G)\ +\ 1$and in[12] that, the domination number of $\mu\ (G)$ is $\gamma(G)\ +\ 1$. However, in the case of gamma coloring, we have the following result.

Theorem 3.2: For any graph G, $\chi_\gamma(\mu\,(G)) = \chi_\gamma(G)$ or $\chi_\gamma(\mu\,(G)) = \chi_\gamma(G)+1$.

Proof: Consider a gamma coloring f_1 of G using $\chi_\gamma(G)$ colors with a colorful dominating set D_1. Let us give a gamma coloring g_1 to $\mu(G)$ using $\chi_\gamma(G)$ + 1 colors as follows. Define $g_1\,(v_i) = g_1\left(v'_i\right) = f_1(v_i)$ and assign a new color to the root vertex u. We first prove that g_1 is a proper coloring of $\mu(G)$. Let $e\varepsilon E(\mu\,(G))$. Then $e = v_iv_j$ or $e = v'_iv_j$ or $e = uv'_i$. If $e = v_iv_j$ or $e = v'_iv_j$, then $v_iv_j \in E\,(G)$ which implies that $f_1(v_i) \neq f_1(v_j)$ and hence $g_1(v_i) \neq g_1(v_j)$. If $e = uv'_i$, then $g_1(v_i) \neq g_1(u)$ as $g_1(u)$ is a new color. Thus g_1 is a proper coloring. Certainly, $D_1 \cup (u)$ is a dominating set of $\mu(G)$ as D_1 is a dominating set of G. Since $g_1\,(v_i) = f_1(v_i)$ for all $v_i \in V$ and D_1 is a f_1- colorful set in G, it follows that $D_1 \cup \{u\}$ is a g_1- colorful set in $\mu(G)$. Thus $D_1 \cup \{u\}$ is a colorful dominating set of $\mu(G)$ and therefore $\mu\,(G)$ has a gamma coloring using $\chi_\gamma(G)+1$ colors and hence $\chi_\gamma(\mu\,(G)) \leq \chi_\gamma(G)+1$.

Let us now prove that, $\chi_\gamma(G) \leq \chi_\gamma(\mu\,(G))$. Consider a minimum gamma coloring g_2 of $\mu\,(G)$ using $\chi_\gamma(\mu\,(G))$ colors with a colorful dominating set S. We obtain a gamma coloring f_2 to G using $\chi_\gamma(\mu\,(G))$ colors as follows. Define $f_2\,(v_i) = g_2\left(v'_i\right)\ if\ v'_i \in S$ and $f_2\,(v_i) = g_2\,(v_i)$ otherwise. Let us first show that f_2 is a proper coloring of G. Let $v_iv_j \in E\,(G)$. Then, $v_iv_j,\ v'_iv_j,\ v_iv'_j \in E\,(\mu\,(G))$. If $v'_i,\ v'_j \notin S$, then $g_2\,(v_i) \neq g_2\,(v_j)$ as $v_iv_j \in E\,(\mu\,(G))$ and hence $f_2\,(v_i) \neq f_2\,(v_j)$. If $v'_i \in S$ and $v'_j \notin S$, then $g_2\left(v'_i\right) \neq g_2(v_j)$ as $v'_iv_j \in E\,(\mu\,(G))$ and hence $f_2(v_i) \neq f_2(v_j)$. If $v'_i \notin S$ and $v'_j \in S$, then $g_2(v_i) \neq g_2(v'_j)$ as $v'_iv'_j \in E\,(\mu\,(G))$ and hence $f_2(v_i) \neq f_2(v_j)$. If $v'_i, v'_j \in S$, then as S is colorful, $g_2(v'_i) \neq g_2(v'_j)$ and hence $f_2(v_i) \neq f_2(v_j)$. Thus, whenever we have $v_iv_j \in E\,(G)$, we have $f_2(v_i) \neq f_2(v_j)$ and hence f_2 is a proper coloring of G.

Let $D_2 = (v_i \in V(G)/v_i \in S\ or\ v'_i \in S\}$. Let us claim that D_2 is a colorful dominating set in G. We first verify that D_2 is a dominating set of G. Let $v_j \in V - D_2$. Then, $v_j \notin S$. Since S is a dominating set in $\mu\,(G)$, either there exists a vertex $v_i \in S$ such that $v_iv_j \in E\,(\mu\,(G))$ or there exists a vertex $v'_i \in S$ such that $v'_iv_j \in E\,(\mu\,(G))$. In either case, we have, $v_i \in D_1$ and $v_iv_j \in E\,(G)$. Therefore, D_2 is a dominating set in G. Let $v_i,\ v_j \in D_2$. Then $v'_i,\ v'_j \in S$ or $v'_i \in S,\ v'_j \notin S$ or $v'_i \notin S,\ v'_j \in S$ or $v'_i,\ v'_j \notin S$. If $v'_i,\ v'_j \in S$, then $g_2(v'_i) \neq g_2(v'_j)$ as S is colorful in $\mu\,(G)$ and hence $f_2(v_i) \neq f_2(v_j)$. If $v'_i \in S$ and $v'_j \notin S$, then $v'_i, v_j \in S$ and as S is colorful in $\mu\,(G)$, $g_2(v'_i) \neq g_2(v_j)$ and hence $f_2(v_i) \neq f_2(v_j)$. If $v'_i \notin S$ and $v'_j \in S$, then $v_i, v'_j \in S$ and as S is colorful in $\mu\,(G)$, $g_2(v_i) \neq g_2(v'_j)$ and hence $f_2(v_i) \neq f_2(v_j)$. If $v'_i,\ v'_j \notin S$, then $v_i, v_j \in S$ and as S is colorful in $\mu\,(G)$, $g_2(v_i) \neq g_2(v_j)$ and hence $f_2(v_i) \neq f_2(v_j)$. Thus D_2 is a colorful dominating set and therefore G has a gamma coloring using $\chi_\gamma(\mu\,(G))$ colors and hence $\chi_\gamma(G) \leq \chi_\gamma(\mu\,(G))$ which implies that $\chi_\gamma(G) \leq \chi_\gamma(\mu\,(G)) \leq \chi_\gamma(G)+1$. Thus $\chi_\gamma(\mu\,(G))$ is either $\chi_\gamma(G)$ or $\chi_\gamma(G)+1$. ■

In view of Theorem 3.2, the set of all connected graphs can be classified into two groups namely Class-1 graphs and Class-2 graphs. Class-1 graphs consist of all connected graphs G with $\chi_\gamma(\mu\,(G)) = \chi_\gamma(G)$ whereas Class-2 graphs consist of all connected graphs G with $\chi_\gamma(\mu\,(G)) = \chi_\gamma(G)+1$. The following proposition shows that each of these classes contains infinitely many members.

Proposition 3.3 : Let$k \geq 3$ be an integer. Then

(i) There exists a graph G such that $\chi_\gamma(G) = k$ and $\chi_\gamma(\mu(G)) = k$.

(ii) There exists a graph H such that $\chi_\gamma(H) = k$ and $\chi_\gamma(\mu(H)) = k+1$.

Proof: For the given $k \geq 3$, we construct a required graph G as follows.

Consider a star graph on k vertices with $v_1,\ v_2,\ldots,v_{k-1}$ as pendent vertices and w as the center vertex. Now, attach exactly two pendent vertices at each of $v_1,\ v_2,\ldots,v_{k-1}$ and label them as shown in Figure 3. Let us first show that $\chi_\gamma(G) = k$.

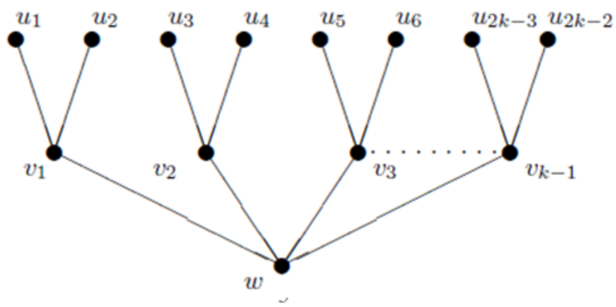

Fig 3. A graph G with

Clearly$\left(\{v_1\},\ \{v_2\},\ldots,\{v_{k-1}\},\ \{w,\ u_1,u_2,\ldots,u_{2k-2}\}\right)\{v_1,\quad v_2,\ldots,v_{k-1}\}$ $\chi_\gamma(G) \leq k.$ $\chi_\gamma(G) \geq k.$ $\gamma(G) = k - 1\{v_1,\ v_2,\ldots,v_{k-1}\}\gamma - set$the colorful dominating set D.$(D| \geq \gamma(G) = k-1$. $(D| \geq k$,

If $(D| = k-1$, then $D = \{v_1,\ v_2,\ldots,v_{k-1}\}$. Since w is a neighbor to all the vertices in D which is colorful, we need a new color for w and hence at least k colors are required for a gamma coloring of G and therefore, $\chi_\gamma(G) \geq k$. Thus, $\chi_\gamma(G) = k$.

Now, we prove that $\chi_\gamma(\mu(G)) = k$. By Theorem 3.2, $\chi_\gamma(\mu(G)) \geq \chi_\gamma(G)$ and hence $\chi_\gamma(\mu(G)) \geq k$. Let us obtain a gamma coloring of $\mu(G)$ with k colors. Consider the coloring $((v_1,\ u\},\ (v_2,\ v'_1, v'_2\},\ (v_3, v'_3\},\ldots,(v_{k-1}, v'_{k-1}\},\ (w,\ u_1,u_2,\ldots,u_{2k-2},\ w',\ u'_1,u'_2,\ldots$ Clearly, it is a gamma coloring of $\mu(G)$ using k colors in which $\{v_1,\ v_2,\ldots,v_{k-1},\ \ w'\}$ is a colorful dominating set and therefore, $\chi_\gamma(\mu(G)) \leq k$. Thus, $\chi_\gamma(\mu(G)) = k = \chi_\gamma(G)$.

Complete graphs on k vertices serve the purpose as proved in Theorem 3 7.

The following theorem provides a necessary and sufficient condition for a graph G in terms of $\mu(G)$ to be of Class-2 graph.

Theorem 3.4 : $\chi_\gamma(\mu(G)) = \chi_\gamma(G) + 1$ if and only if there is a $\chi_\gamma-$coloring of $\mu(G)$ admitting a colorful dominating set containing the root vertex u.

Proof: Suppose there is a $\chi_\gamma-$coloring f of $\mu(G)$ admitting a colorful dominating set D containing u. Let $c = f(u)$. Now, let us give a gamma coloring g to G using $\chi_\gamma(\mu(G)) - 1$ colors as follows. Define $h(v_i) = f(v'_i)$ if $v'_i \in D$ and define $h(v_i) = f(v_i)$ otherwise. Also, define $S = \{\ v_i \in V(G)/v_i \in D\ or\ v'_i \in D\}$. Then by the second part in the proof of Theorem 3.2, h is a gamma coloring of G with S as a colorful dominating set. Note that the color c is not used by any of the vertex in S for if $v'_i \in D$, then $h(v_i) = f\left(v'_i\right) \neq f(u) = c$ and if $v'_i \notin D$ and $v_i \in D$, then $u, v_i \in D$. Since D is f-colorful, we have $h(v_i) = f(v_i) \neq f(u) = c$. Suppose that $h(v_i) = c$ for some i. Then, $v_i, v'_i \notin D$. Now, recolor the vertex v_i such that $h(v_i) = f\left(v'_i\right) \neq c$. The coloring h is still a proper coloring of G because of $N(v'_i) \cap V(G) = N(v_i) \cap V(G)$. Repeat the above process of recoloring until $h(v_i) \neq c$ for all i. Thus, we have a proper coloring h of G using $\chi_\gamma(\mu(G)) - 1$ colors in which S is a colorful dominating set. Hence $\chi_\gamma(G) \leq \chi_\gamma(\mu(G)) - 1$ which implies that $\chi_\gamma(G) + 1 \leq \chi_\gamma(\mu(G))$ and by Theorem 3.2, $\chi_\gamma(\mu(G)) = \chi_\gamma(G) + 1$.

Conversely, let us assume that $\chi_\gamma(\mu(G)) = \chi_\gamma(G) + 1$. Consider a gamma coloring f of G using $\chi_\gamma(G)$ colors with a colorful dominating set D. By similar argument as in the first part of Theorem 3.2, we can give a gamma coloring h to $\mu(G)$ with $\chi_\gamma(G) + 1$ colors in which u is in a colorful dominating set which completes the proof.

Theorem 3.4 is helpful in proving certain families of graph are of Class-2. For example, paths, cycles and complete graphs are of Class-2 as shown below.

Theorem 3.5: Path graphs are of Class-2.

Proof: Let $P_n = (v_1, v_2, \ldots, v_n)$. It is clear that $\gamma(P_n) = \left(\frac{n}{3}\right]$. Also, it has been proved in[5] that for a graph G, $\gamma(\mu(G)) = \gamma(G) + 1$ and therefore $\gamma(\mu(P_n)) = \left(\frac{n}{3}\right] + 1$. Hence by Observation 2.4, $\chi_\gamma(\mu(P_n)) \geq \gamma(\mu(P_n)) = \left(\frac{n}{3}\right] + 1$. So, in view of Theorem 3.4, it is enough to obtain a gamma coloring C of $\mu(P_n)$ using $\left(\frac{n}{3}\right] + 1$ colors with the property that there is a C-

colorful dominating set of $\mu(P_n)$ containing the root vertex u. We do it in the following cases. Let v'_i be the twin vertex of v_i in $\mu(P_n)$ and u be the root vertex of $\mu(P_n)$.

Case 1: $n \equiv 0\ (mod\ 3)$.

In this case, $n = 3k$ for some natural number k and $\left(\frac{n}{3}\right] + 1 = k+1$. Let us give a gamma coloring C to $\mu(P_n)$ using $k+1$ colors as follows.

Let $V_1 = (v_2,\ v'_2,\ v_4,\ v'_4\} \cup \left(v_{3j},\ v'_{3j} : 2 \le j \le k\right\}$,

$V_2 = (v_1,\ v'_1,\ v_3,\ v'_3,\ v_5,\ v'_5\} \cup \left(v_{3j-2},\ v'_{3j-2} : 3 \le j \le k\right\}$,

$V_i = (v_{3i-1},\ v'_{3i-1}\}$ for all $3 \le i \le k$ and $V_{k+1} = (u\}$.

Clearly $C = (V_1,\ V_2, \ldots, V_{k+1}\}$ is a proper coloring of $\mu(P_n)$. The coloring of $\mu(P_6)$ is illustrated in Figure 4. Moreover, as $D = (v_2,\ v_5, \ldots, v_{3k-1}\}$ is a dominating set of P_n, $S = D \cup \{u\}$ is a dominating set of $\mu(P_n)$. Also, $S \cap V_i = (v_{3i-1}\}$ for $i \in \{1, 2, \ldots, k\}$ and $S \cap V_{k+1} = (u\}$. Thus S is C- colorful dominating set of $\mu(P_n)$ which implies that $\chi_\gamma(\mu(P_n)) \le k+1$ and hence $\chi_\gamma(\mu(P_n)) = k+1$. Thus C is a χ_γ- coloring of $\mu(P_n)$ with a colorful dominating set S containing u as desired.

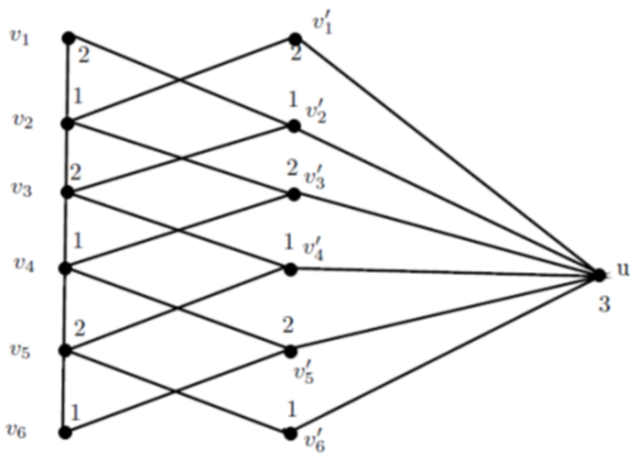

Fig 4. Gamma coloring of with 3 colors

Case 2: $n \equiv 1\ or\ 2\ (mod\ 3)$.

In this case $n = 3k-1$ or $n = 3k-2$ for some natural number k and $\left(\frac{n}{3}\right] + 1 = k+1$. Let us give a gamma coloring C to $\mu(P_n)$ using $k+1$ colors as follows. Let

$$V_1 = \left(\begin{array}{ll} \left(v_2,\ v'_2,\ v_4,\ v'_4\right\} \cup \left(v_{3j-3},\ v'_{3j-3} : 3\ \le j \le k\right\} \cup (v_{3k-1}, v'_{3k-1}\} & if\ n = 3k-1nf\ ce\ by\ o\quad e\ 1roved\ in\ [5 \\ \left(v_2,\ v'_2,\ v_4,\ v'_4\right\} \cup \left(v_{3j-3},\ v'_{3j-3} : 3\ \le j \le k\right\} & if\ n = 3k-2, \end{array}\right.$$

$V_2 = (v_1,\ v'_1,\ v_3,\ v'_3,\ v_5,\ v'_5\} \cup \left(v_{3j-2},\ v'_{3j-2} : 3 \le j \le k-1\right\}$,

$V_i = (v_{3i-1},\ v'_{3i-1}\}$ for all $3 \le i \le k-1$,

$$V_k = \left(v_{3k-2},\ v'_{3k-2}\right\}$$

and $V_{k+1} = (u\}$.

Then $C = (V_1,\ V_2, \ldots, V_{k+1}\}$ is a proper coloring of $\mu(P_n)$. Moreover, as $D = (v_2,\ v_5, \ldots, v_{3k-4},\ v_{3k-2}\}$ is a dominating set of P_n, $S = D \cup \{u\}$ is a dominating set of $\mu(P_n)$. Also, $S \cap V_i = (v_{3i-1}\}$ for $i \in \{1, 2, \ldots, k-1\}$, $S \cap V_k = (v_{3i-2}\}$ and $S \cap V_{k+1} = (u\}$. Thus, S is C- colorful dominating set of $\mu(P_n)$ which implies that $\chi_\gamma(\mu(P_n)) \le k+1$ and hence $\chi_\gamma(\mu(P_n)) = k+1$. Thus C is a χ_γ- coloring of $\mu(P_n)$ with a colorful dominating set S containing u as desired.

By a similar argument we can prove the following theorem for cycle graphs.

Theorem 3.6: Cycle graphs are of Class-2.

Theorem 3.7: Complete graphs are of Class-2.

Proof: Let $v_1, v_2, .., v_n$ be the vertices of K_n. It is clear that $\chi(K_n) = n$. Also, it has been proved in[(12)] that for a graph G, $\chi(\mu(G)) = \chi(G) + 1$ and therefore $\chi(\mu(K_n)) = n+1$. Hence by Observation 2.4, $\chi_\gamma(\mu(K_n)) \ \ge \chi(\mu(K_n)) = n+1$. In view of Theorem 3.4, it is enough to obtain a gamma coloring for $\mu(K_n)$ using $n+1$ colors admitting a colorful dominating set containing the root vertex u. Let v'_i be the twin vertex of v_i in $\mu(K_n)$. Let $V_i = (v_i,\ v'_i\}$ for all $1 \le i \le k$, and $V_{k+1} = (u\}$. As v_i and v'_i are not adjacent in $\mu(K_n)$, it is clear that $C = (V_1,\ V_2, \ldots, V_{k+1}\}$ is a proper coloring of $\mu(K_n)$. Also, $S = \{v_1, v_2, .., v_n,\ u\}$ is a dominating set of $\mu(K_n)$. Clearly, $S \cap V_i = \{v_i\}$ for all $1 \le i \le k$ and $S \cap V_{k+1} = \{u\}$ so that S is a colorful dominating set of $\mu(K_n)$. Hence C is a gamma coloring of $\mu(K_n)$ with $n+1$ colors which implies that $\chi_\gamma(\mu(K_n)) \le n+1$ and thus $\chi_\gamma(\mu(K_n)) = n+1$. Therefore, C is a χ_γ- coloring of $\mu(K_n)$ admitting a colorful dominating set S containing u. Thus, by Theorem 3.4, it follows that, complete graphs are of Class-2.

Theorem 3.4 provides a condition for a graph G to be of Class-2 graph in terms of minimum gamma coloring of $\mu(G)$. In the following theorem, we obtain a sufficient condition for a graph G to be of Class-1 graph in terms of minimum gamma coloring G.

Theorem 3.8 : If a graph G has a $\chi_\gamma(G)-$coloring $\left(V_1,\ V_2,\ \ldots, V_{\chi_\gamma(G)}\right\}$ with the following properties.

(i) There exists a positive integer m such that $N[v]$ is not a color transversal for every vertex$v \in V_m$.

(ii) There exists a colorful total dominating set D such that $D \cap V_p = \phi$ for some $p \ne m$.

Then $\chi_\gamma(\mu(G)) = \chi_\gamma(G)$.

Proof: Let G has a $\chi_\gamma(G)-$coloring $\left(V_1,\ V_2,\ \ldots, V_{\chi_\gamma(G)}\right\}$ with the given properties and let f be the corresponding coloring function. For each $v_i \in V_m$, being $N[v_i]$ is not a color transversal, there exists j such that $N(v_i] \cap V_j = \phi$ and let c_{v_i} be the color used to color the vertices of V_j. Let k be the color used to color the vertices of V_m. Let us give a gamma coloring h to $\mu(G)$ using $\chi_\gamma(G)$ colors as follows. Define h$(v_i) = f(v_i)$ for all $v_i \in V$, $h(v'_i) = f(v_i)$ if $v_i \notin V_m$ and define $h(v'_i) = c_{v_i}$ if $v_i \in V_m$ and $h(u) = k$. Let us first prove that, h is a proper coloring of $\mu(G)$. Let $e \in \mu(G)$. Then $e = v_i v_j$ or $e = v'_i v_j$ or $e = uv'_i$. If $e = v_i v_j$, then $v_i v_j \in E(G)$ which implies that $f(v_i) \ne f(v_j)$ and hence $h(v_i) \ne h(v_j)$. If $e = v'_i v_j$, then $v_i v_j \in E(G)$ which implies that $f(v_i) \ne f(v_j)$. If $v_i \notin V_m$, then $h\left(v'_i\right) = f(v_i) \ne f(v_j) = h(v_j)$. If $v_i \in V_m$, then by the selection of c_{v_i}, the color $c_{v_i}$is not used by adjacent vertices of v_i and therefore $f(v_j) \ne c_{v_i}$. Thus, we have $h\left(v'_i\right) = c_{v_i} \ne f(v_j) = h(v_j)$. Clearly, from the definition of h, $h\left(v'_i\right) \ne k$ for all $v'_i \in V'$. If $e = uv'_i$ then $h(u) = k \ne h\left(v'_i\right)$ and hence h is a proper coloring.

Let $u_d \in V_p$ and $D' = D \cup \{u'_d\}$. Since D is a total dominating set, for every $v_i \in V$ there exists $v_j \in D$ such that v_i is adjacent to v_j and therefore v'_i is adjacent to v_j which implies that D dominates V and V'. Also, u is dominated by u'_d. Hence D' is a dominating set of $\mu(G)$. Since $h(v_i) = f(v_i)$ for all $v_i \in V$and D is $f-$colorful in G, we have D is $h-$colorful in $\mu(G)$. Also, in $\mu(G)$, u'_d receives the color used by the color class V_p and $D \cap V_p = \phi$ which implies that D' is colorful. Thus $\mu(G)$ admits a coloring using $\chi_\gamma(G)$ colors in which a dominating set D' is colorful. Hence $\chi_\gamma(\mu(G)) \le \chi_\gamma(G)$ and by Theorem 3.2, $\chi_\gamma(\mu(G)) = \chi_\gamma(G)$.

Remark 3.9: The converse of Theorem 3.8 is not true. For example, consider the graph G shown in Figure 5 which is the graph constructed as in Proposition 3.3 (i) with k=3. From the construction of the graph, it is clear that$\chi_\gamma(G) = 3 = \chi_\gamma(\mu(G))$. Consider a χ_γ-coloring C=$\{V_1,\ V_2, V_3\}$ of G. Let D be a C-colorful total dominating set. Clearly, $| \le \chi_\gamma(G) = 3$ But $\{v_1,\ v_2,\ w\}$ is the only total dominating set with at most three elements. Hence D= $(v_1,\ v_2,\ w\}$ is the only C - colorful total dominating set. Since D uses all the three colors of C, there does not exist a color class V_p with $D \cap V_p = \phi$

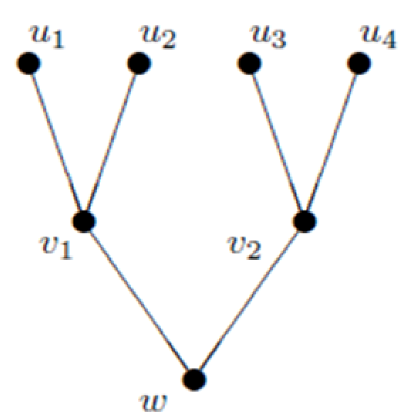

Fig 5. A counter example to the converse of Theorem 3.8

4 Conclusion

We have initiated a study on Gamma coloring for Mycielskian graph $\mu(G)$ of a given graph G. We have proved that the gamma chromatic number χ_γ for $\mu(G)$ is either $\chi_\gamma(G)$ or $\chi_\gamma(G)+1$ and thus the class of all connected graphs is classified into two classes namely Class-1 and Class-2 graphs. Graphs G for which $\chi_\gamma(\mu(G)) = \chi_\gamma(G)$ are of Class-1 and the rest of graphs are of Class-2. Conditions under which a graph G becomes Class-1/ Class-2 have been established by which some families of Class-1 / Class-2 graphs have been characterized. There are still scopes for further research on this topic. For instance, the following are some interesting problems.

(1) Theorem 3.4 provides a necessary and sufficient condition under which a graph falls in Class-2; however, it does not infer about the structure of those graphs. So, it is worthy finding a structural characterization of Class-1/ Class-2 graphs.

(2) Characterize trees which are of Class-1/ Class-2.

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