INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

OPEN ACCESS
Received: 05-03-2022
Accepted: 11-07-2022
Published: 03-09-2022

Citation: Dixit U (2022) On Bounds of Non-Deficient Numbers. Indian Journal of Science and Technology 15(35): 1683-1690. https://doi.org/10.17485/IJST/v15i35.946

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Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

On Bounds of Non-Deficient Numbers

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Abstract

Objectives: To improve the upper bounds of a quasi perfect number and give an important result on its divisibility with primes. **Methods:** A positive integer n is quasi perfect if σ (n) >2n + 1, where $\sigma(n)$ denotes the sum of the positive divisors of n. However, the existence of a quasi perfect number, which is a Non-Deficient number, is still an open problem. We use R(n), the sum of the reciprocals of distinct primes dividing the quasi perfect number, to derive lemmas and improve the bounds obtained by earlier authors. **Findings:** We improve the upper bounds for R(n), when n is quasi perfect with gcd (15, n) = 3 or gcd (15, n) = 5. As a consequence, we establish that a quasi perfect number, if exists, is divisible by both 3 and 5 or by none of them. **Novelty:** The unique method of using R(n) also resulted in finding an important result that 3, 5 and 7 cannot divide any quasi perfect number.

Mathematics Subject Classification: 11A05, 11A25

Keywords: non-deficient number; quasi perfect number; sum of the divisor; sum of the reciprocal; bounds of perfect number; number of divisors.

1 Introduction

let $\sigma(n)$ denote the sum of the positive divisors of n. It is well-known that a positive integer n is said to be abundant, perfect or deficient according as $\sigma(n)$ > 2n, $\sigma(n)$ = 2n or $\sigma(n)$ < 2n. One can see that the set of abundant numbers as well as the set of deficient numbers are both infinite. In fact, the numbers of the form $2^k.3$ with k > 1 are all abundant, while every prime is deficient. But it is not known whether the set of perfect numbers is infinite or not.

Cattaneo [1] has called a positive integer n quasi perfect if σ(n) = 2n + 1. It is not known whether such numbers exist at all. Abbott, Kishore and Cohen [2–5] have made significant contributions to the study of quasi perfect numbers. In 1978, Kishore [4] used the inequality $2-10^{-12} < \sigma(N)/N < 2+10^{-12}$ and proved that there are no odd perfect numbers, no quasi perfect numbers and no odd almost perfect numbers with five distinct prime factors, proving $\omega(n) \geq 6$, where w(n) denotes number of divisors of n.

Cohen [6] proved, if any quasi perfect number n exists, then

(1.1.1)$\omega(n)$ 7 $\geqslant$and $n > 10^{35}$

Subsequently, Peter Hagis and Cohen [7] have used computations to improve some of the results on quasi perfect numbers. In fact, they established that

(1.1.2) for any quasi perfect number n, $\omega(n) \geqslant 15$ if $(15, n) = 1$, $\omega(n) \geqslant 9$ if $3 \nmid n$ and in other case $\omega(n) > 7$ and also n > 10^{35}., where $\omega(n)$ denotes the number of divisors of n.

Cattaneo [1] has proved the following theorem:

1.1 **Theorem:** If n is a quasi perfect number then

(1.1.3)$n = p_1^{2e_1} p_2^{2e_2} \cdots p_r^{2e_r}$

where p_i's are odd primes

Moreover,

(1.1.4) i) $p_i \equiv 1 \pmod 8 \Rightarrow e_i \equiv 0$ or $1 \pmod 4$

ii) $p_i \equiv 3 \pmod 8 \Rightarrow e_i \equiv 0 \pmod 2$

iii) $p_i \equiv 5 \pmod 8 \Rightarrow e_i \equiv 0$ or $-1 \pmod 4$

iv) $p_i \equiv 7 \pmod 8 \Rightarrow e_i \geq 1$

In a different direction, Cohen [8] has considered R(n), the sum of the reciprocals of the distinct primes dividing the quasi perfect number n (if exists) and obtained bounds for R(n). They are

(1.1.5)

i) $0.667450 < R(n) < 0.693148\ if\ (15, n) = 1$

ii) $0.603831 < R(n) < 0.625140\ if\ (15, n) = 3$

iii) $0.647387 < R(n) < 0.670017\ if\ (15, n) = 5$

iv) $0.596063 < R(n) < 0.602009\ if\ (15, n) = 15$

New methods were introduced by Tang and Feng [9] and they established that there are no odd deficient-perfect numbers with three distinct prime divisors. The alternate proof was given in [10] using special components, which is

(1.1.6) If n is a quasi perfect number and is of the form (1.1.3) then for at least one factor $p_i^{2e_i}$, we have either

$$\left(\begin{array}{l} p_i \equiv 1 \,(mod\ 8)\ and\ e_i \equiv 1 (mod\ 4) \\ \qquad\qquad\text{or} \\ p_i \equiv 5 \,(mod\ 8)\ and\ e_i \equiv 3 (mod\ 4) \end{array} \right.$$

Calling ${}_i^{2e_i}$ as a special component if it satisfies (1.1.6)

and proved every quasi perfect number has an odd number of special components.

In 2019, Tomohiro [11] has given some lower bounds concerning quasi perfect numbers of the form N = m^2 where m is square free and Prasad [12] obtained a lower bound for the product of the distinct primes dividing n in terms of ω(n).

In this paper, the upper bounds given in (1.1.5) are improved and subsequently an important result regarding divisibility with primes of n is obtained.

2 Methodology

In this section, we will prove some lemmas used in the sequel. We shall give detailed proof of the required lemmas in [13] which were used for bounds of odd perfect numbers.

2.1 Lemma:

For $o < p \leq \frac{1}{3}$ we have $1 + p + p^2 + p^3 + p^4 > e^{p + t_1 p^2}$ where $t_1 = 0.373$.

Proof: For simplicity of notation we write t for t_1. Consider the function

$$f(p) = e^{\left(p + tp^2\right) - \left(1 + p + p^2 + p^3 + p^4\right)}$$

$= \left(1 + \frac{(p+tp^2)}{1!} + \frac{(p+tp^2)^2}{2!} + \frac{(p+tp^2)^3}{3!} + \dots \right\} - \left(1 + p + p^2 + p^3 + p^4\right)$

$= p^2\left(t - \frac{1}{2}\right) + p^3(t-1) + p^4\left(\frac{t^2}{2} - 1\right) + \frac{1}{3!}\left(p + tp^2\right)^3 + \dots$

Now the inequality of the lemma holds

$$\Leftrightarrow f(p) < 0$$

$$\Leftrightarrow \frac{1}{3!}\left(p + tp^2\right)^3 + \frac{1}{4!}\left(p + tp^2\right)^4 + \dots < p^2\left(\frac{1}{2} - t\right) + p^3(1-t) + p^4\left(1 - \frac{t^2}{2}\right)$$

$$\Leftrightarrow \frac{1}{3!}p(1+tp)^3+\frac{1}{4!}p^2(1+tp)^4+\dots<\left(\frac{1}{2}-t\right)+p\,(1-t)+p^2\left(1-\frac{t^2}{2}\right)$$

$$\Leftrightarrow \frac{1}{6}p(1+tp)^3\left(1+\frac{1}{4}p\,(1+tp)+\dots\right]<\left(\frac{1}{2}-t\right) \qquad (1)$$

But for $0<p\leq\frac{1}{3}$, we have

$$g\,(p)=\frac{1}{6}p(1+tp)^3\left(1+\frac{1}{4}p\,(1+tp)+\frac{1}{20}p^2(1+tp)^2+\dots\right]$$

$< \frac{1}{6}p(1+tp)^3\left(1+\left(p+tp^2\right)+\left(p+tp^2\right)^2+\dots\right]$
$= \frac{1}{6}p(1+tp)^3\left(1-\left(p+tp^2\right)\right]^{-1}$,
Whenever t is chosen such that $\left(p+tp^2\right|<1$. Therefore for $0<p\leq\frac{1}{3}$, we have

$$g\,(p)<\frac{1}{6}p(1+tp)^3\left(\frac{1}{1-(p+tp^2)}\right]$$

$$\leq\frac{1}{18}\left(1+\frac{t}{3}\right)^3\left(\frac{1}{1-\left(\frac{1}{3}+\frac{t}{9}\right)}\right]$$

$$\frac{1}{18}\bullet\frac{(3+t)^3}{3^3}\bullet\frac{9}{6-t} \qquad (2)$$

Now from (1) and (2), the inequality of the lemma holds if

$$\frac{1}{18}\bullet\frac{(3+t)^3}{3^3}\cdot\frac{9}{6-t}<\frac{1}{2}-t,$$

and this holds for $t=0.373$. Hence the lemma.

In similar way, we can prove the following Lemma 2.2, Lemma 2.3 and Lemma 2.4.

2.2 Lemma:

For $0<p\leq\frac{1}{7}$ we have $1+p+p^2>e^{p+t_2p^2}$ where $t_2=0.406$.

2.3 Lemma:

For $0<p\leq\frac{1}{5}$ we have $1+p+p^2+p^3+p^4+p^5+p^6>e^{(p+t_3p^2)}$ Where $t_3=0.496$

2.4 Lemma:

For $0<p\leq\frac{1}{5}$ we have $1+p+p^2+p^3+p^4+p^5+p^6+p^7+p^8>e^{(p+t_4p^2)}$ Where $t_4=0.4998$.

2.5 Lemma:

Suppose p is a quasi perfect number, $R_k\,(p)$ is the sum of reciprocals of the k^{th} powers of the distinct primes dividing p and $t=min(t_1,t_2,t_3,t_4)=0.373$ where t_1,t_2,t_3,t_4 are as given in Lemma 2.1 to 2.4. Then $R_1\,(p)+cR_2\,(p)<log(2.000000001)$.

Proof: Given that p is quasiperfect, so that in view of (1.1.3) and (1.1.6) we can write p as

(2.5.1)

$$p=a_1^{2e_1}a_2^{2e_2}\ldots a_r^{2e_r}\cdot b_1^{2f_1}b_2^{2f_2}\ldots b_s^{2f_s}$$

Where each of the a_i^{2e} is a special component and $b_j^{2f_j}$ is any component.

Then by (1.1.6), r is odd and also for each i, we have either

(2.5.2)

$$\left(\begin{array}{c} a_i\equiv 1\,(mod\ 8)\ \text{ and }\ e_i\equiv 1(mod\ 4)\\ \text{or}\\ a_i\equiv 5\,(mod\ 8)\ \text{ and }\ e_i\equiv 3(mod\ 4)\end{array}\right.$$

Further by (1.1.4), for each j,

(2.5.3)

$$(2.5.3)\quad \begin{cases} b_j\equiv 1(\bmod 8)\Rightarrow f_j\equiv 0(\bmod 4)\\ b_j\equiv 3(\bmod 8)\Rightarrow f_j\equiv 0(\bmod 2)\\ b_j\equiv 5(\bmod 8)\Rightarrow f_j\equiv 0(\bmod 4)\\ b_j\equiv 7(\bmod 8)\Rightarrow f_j\geq 1\end{cases}$$

Therefore by (2.5.2) and (2.5.3) we have

(2.5.4)

$$2+\frac{1}{p}=\frac{\sigma(p)}{p}=\prod_{i=1}^{r}\left(1+\frac{1}{a_i}+\cdots+\frac{1}{a_i^{2e_i}}\right)\prod_{j=1}^{s}\left(1+\frac{1}{b_j}+\cdots+\frac{1}{b_j^{2f_j}}\right)$$

$$>\prod\nolimits_{a_i\equiv 1(mod\ 8)}\left(1+\frac{1}{a_i}+\frac{1}{a_i^2}\right)\prod\nolimits_{a_i\equiv 5(mod\ 8)}\left(1+\frac{1}{a_i}+\frac{1}{a_i^2}+\frac{1}{a_i^3}+\frac{1}{a_i^4}+\frac{1}{a_i^5}+\frac{1}{a_i^6}\right)$$

$$\prod\nolimits_{b_j\equiv 1(mod\ 8)}\left(1+\frac{1}{b_j}+\frac{1}{b_j^2}+\frac{1}{b_j^3}+\frac{1}{b_j^4}+\frac{1}{b_j^5}+\frac{1}{b_j^6}+\frac{1}{b_j^7}+\frac{1}{b_j^8}\right)$$

$$\prod\nolimits_{b_j\equiv 3(mod\ 8)}\left(1+\frac{1}{b_j}+\frac{1}{b_j^2}+\frac{1}{b_j^3}+\frac{1}{b_j^4}\right)\cdot\prod\nolimits_{b_j\equiv 7(mod\ 8)}\left(1+\frac{1}{b_j}+\frac{1}{b_j^2}\right)$$

$$\prod\nolimits_{b_j\equiv 5(mod\ 8)}\left(1+\frac{1}{b_j}+\frac{1}{b_j^2}+\frac{1}{b_j^3}+\frac{1}{b_j^4}+\frac{1}{b_j^5}+\frac{1}{b_j^6}+\frac{1}{b_j^7}+\frac{1}{b_j^8}\right)$$

Now using Lemmas 2.1 to 2.4 on the right of (2.5.4) we get

$$2+\frac{1}{p}>\prod\nolimits_{a_{i\equiv 1(mod\ 8)}}e^{\left(\frac{1}{a_i}+\frac{t_1}{a_i^2}\right)}\cdot\prod\nolimits_{a_{i\equiv 5(mod\ 8)}}e^{\left(\frac{1}{a_i}+\frac{t_3}{a_i^2}\right)}\cdot\prod\nolimits_{b_{j\equiv 1(mod\ 8)}}e^{\left(\frac{1}{b_j}+\frac{t_4}{b_j^2}\right)}\cdot\prod\nolimits_{b_{j\equiv 3(mod\ 8)}}e^{\left(\frac{1}{b_j}+\frac{t_2}{b_j^2}\right)}$$

$$\cdot\prod\nolimits_{b_{j\equiv 5(mod\ 8)}}e^{\left(\frac{1}{b_j}+\frac{t_4}{b_j^2}\right)}\cdot\prod\nolimits_{b_{j\equiv 7(mod\ 8)}}e^{\left(\frac{1}{b_j}+\frac{t_1}{b_j^2}\right)}$$

Which on taking logarithm gives

$$log\left(2+\frac{1}{n}\right)>\sum\nolimits_{a_i\equiv 1(mod\ 8)}\left(\frac{1}{a_i}+\frac{t_1}{a_i^2}\right)+\sum\nolimits_{a_i\equiv 5(mod\ 8)}\left(\frac{1}{a_i}+\frac{t_3}{a_i^2}\right)+\sum\nolimits_{b_j\equiv 1(mod\ 8)}\left(\frac{1}{q_j}+\frac{t_4}{q_j^2}\right)$$

$$+\sum_{b_j\equiv 3(mod\ 8)}\left(\frac{1}{q_j}+\frac{t_2}{q_j^2}\right)+\sum_{b_j\equiv 5(mod\ 8)}\left(\frac{1}{q_j}+\frac{t_4}{q_j^2}\right)+\sum_{b_j\equiv 7(mod\ 8)}\left(\frac{1}{q_j}+\frac{t_1}{q_j^2}\right)$$

$= R_1(p)+t_1\sum_{a_i\equiv 1(mod\ 8)}\frac{1}{a_i^2}+t_3\sum_{a_i\equiv 5(mod\ 8)}\frac{1}{a_i^2}+t_4\sum_{b_j\equiv 1(mod\ 8)}\frac{1}{b_j^2}$

$+t_2\sum_{b_{j\equiv 3(mod\ 8)}}\frac{1}{q_j^2}+t_4\sum_{b_{j\equiv 5(mod\ 8)}}\frac{1}{q_j^2}+t_4\sum_{b_{j\equiv 7(mod\ 8)}}\frac{1}{q_j^2}$

$> R_1(p)+tR_2(p)$

Since $t=min(t_1,t_2,t_3,t_4)$, proving the lemma because $p>10^{35}$.

2.6 Lemma:

Suppose p is quasi perfect and $\alpha=2+\frac{1}{p}$. Then for any divisor of q of p with $q<p$, we have $R_1(p)<R_1(q)+tR_2(q)+log\left(\frac{\alpha q}{\beta\sigma(q)}\right)-tR_2(p)$, where β = 1 if$(p,15)=1$ or $(p,15)=(q,15)=3$

or $(p,15)=(q,15)=5$ or $(p,15)=(q,15)=15$

$1+\frac{1}{3}+\frac{1}{3^2}if\,(p,15)=3$ and $(q,15)=1$ or $(p,15)=15$ and $(q,15)=5$

$1+\frac{1}{5}+\frac{1}{5^2}if\,(p,15)=5$ and $(q,15)=1$ or $(p,15)=15$ and $(q,15)=3$

$\left(1+\frac{1}{3}+\frac{1}{3^2}\right)\left(1+\frac{1}{5}+\frac{1}{5^2}\right)if\,(p,15)=15$ and $(q,15)=1$

Proof: Suppose p is quasiperfect, it is of the form $p=\prod_{i=1}^{r}a_i^{2e_i}$, where a_i's are odd primes, then

(2.6.1)

$$\alpha=2+\frac{1}{p}=\frac{\sigma(p)}{p}=\prod_{i=1}^{r}\left(1+\frac{1}{a_i}+\frac{1}{a_i^2}+\cdots+\frac{1}{a_i^{2e_i}}\right)$$

Suppose $q=\prod_{i=1}^{r}a_i^{b_i}$ is a divisor of p with $q<p$. Then $0\le b_i\le 2e_i$ for each i and $b_i<2e_i$ for at least one i. Also

(2.6.2)

$$\frac{\sigma(q)}{q}=\prod_{i=1}^{r}\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}+\cdots+\frac{1}{p_i^{b_i}}\right)$$

Now from (2.6.1), (2.6.2) and Lemma 2.1, we get

$$\ge\prod_{\substack{i=1\\ a_i|q}}^{r}\left(1+\frac{1}{a_i}+\cdots+\frac{1}{a_i^{b_i}}\right)\prod_{\substack{i=1\\ a_i|a}}^{r}\left(1+\frac{1}{a_i}+\cdots+\frac{1}{a_i^2}\right)$$

$=\frac{\sigma(q)}{q}\exp\left\{\sum_{a|p}\left(\frac{1}{a}+t\cdot\frac{1}{a^2}\right)\right\}\cdot\beta$

Which on taking logarithm gives

$\log\alpha>\log\left(\frac{\sigma(q)}{q}\right)+\sum_{\substack{a|p\\ a\nmid q}}\left(\frac{1}{a}+t\cdot\frac{1}{a^2}\right)+\log\beta$

$=\log\left(\frac{\sigma(q)}{q}\right)+\sum_{a\nmid q}\frac{1}{a}+t\cdot\sum_{a|p}\frac{1}{a^2}+\log\beta$

$$log\left(\frac{\sigma(q)}{q}\right)+R_1(p)-R_1(q)+t(R_2(p)-R_2(q))+log\beta$$

Therefore,

$$R_1(p)<R_1(q)+log\left(\frac{\alpha}{\beta}\right)-log\left(\frac{\sigma(q)}{q}\right)-t(R_2(p)-R_2(q))$$

$=\ R_1(q)+tR_2(q)+log\left(\frac{\alpha(q)}{\beta\sigma(q)}\right)-tR_2(p)$

Proving the lemma.

3 Results and Discussion

3.1 Theorem:

Suppose $3 \mid n$ and $5 \nmid n$. Then

i) $R_1(n) < 0.2727439241$ if $7 \mid n$.

and

ii) $R_1(n) < 0.2839779561$ if $7 \nmid n$.

Proof: (i) Suppose $3 \mid n$ and $7 \mid n$.

It was proved in [2] that if $n \equiv 0(mod\ 3)$ and $p \equiv 1(mod\ 3)$ is a divisor of n then $p^k \mid n$ with $k \geq 4$. Therefore $7^4 \mid n$. Also since $w(n) \geq 7$ we get that $m = 7^4$ is a divisor of n with $m < n$, so that by Lemma 2.6 we have

$R_1(n) < R_1(m) + cR_2(m) + log\left(\frac{\alpha m}{\beta\sigma(m)}\right) - cR_2(n)$

$= \frac{1}{7} + 0.373\left(\frac{1}{7^2}\right) + log\left(\frac{\alpha \cdot 7^4 \cdot 6}{\beta \cdot (7^5-1)}\right] - (0.373)\left(\frac{1}{3^2} + \frac{1}{7^2}\right)$

$= \frac{1}{7} + 0.373\left(\frac{1}{7^2}\right) + log\alpha + log\left(\frac{7^4 \cdot 6}{(1.444444444) \cdot (7^5-1)}\right] - (0.373)\left(\frac{1}{3^2} + \frac{1}{7^2}\right)$

$= \frac{1}{7} + 0.373\left(\frac{1}{7^2}\right) + log2 + log\left(1 + \frac{1}{2n}\right) + log\left(\frac{14406}{24275.33333}\right] - (0.373)\left(\frac{1}{3^2} + \frac{1}{7^2}\right)$

$= 0.272743924 + log\left(1 + \frac{1}{2n}\right)$

$= 0.272743924 + 0.0000000001$, since $n > 10^{35}$. Thus $R_1(n) < 0.2727439241$.

Suppose $3 \mid n$ and $7 \nmid n$

It follows from (1.1.4) (ii) that $3^4 \mid n$. Also let m = 1 is a divisor of n with $m < n$, so that by Lemma 2.6 we have $R1(n) < log\left(\frac{\alpha}{\beta}\right) - cR_2(n)$

$= \log\alpha - \log\beta - (0.373)\left(\frac{1}{3^2}\right)$

$= \log 2 + log\left(1 + \frac{1}{2n}\right) - \log(1.444444444) - (0.373)\left(\frac{1}{3^2}\right)$

$= 0.283977956 + log\left(1 + \frac{1}{2n}\right)$

$= 0.283977956 + 0.0000000001$, since $n > 10^{35}$. Thus $R_1(n) < 0.2839779561$.

3.2 Theorem:

Suppose 3 n and $5 \mid n$. Then

i) $R_1(n) < 0.4547419741$ if $7 \mid n$.

and

ii) $R_1(n) < 0.4631158001$ if $7 \nmid n$

Proof: (i) Suppose $5 \mid n$ and $7 \mid n$.

It follows from (1.1.4) (iii) and (1.1.4) (iv) that $5^6 \mid n$ and $7^2 \mid n$. Also since $w(n) \geq 7$ we get that $m = 7^2$ is a divisor of n with $m < n$, so that by Lemma 2.6 we have

$R_1(n) < R_1(m) + cR_2(m) + log\left(\frac{\alpha m}{\beta\sigma(m)}\right) - cR_2(n)$

$= \frac{1}{7} + 0.373\left(\frac{1}{7^2}\right) + log\left(\frac{\alpha \cdot 7^2 \cdot 6}{\beta \cdot (7^3-1)}\right] - (0.373)\left(\frac{1}{5^2} + \frac{1}{7^2}\right)$

$= \frac{1}{7} + 0.373\left(\frac{1}{7^2}\right) + log\alpha + log\left(\frac{7^2 \cdot 6}{(1.24) \cdot (342)}\right] - (0.373)\left(\frac{1}{5^2} + \frac{1}{7^2}\right)$

$= \frac{1}{7} + 0.373\left(\frac{1}{7^2}\right) + log2 + log\left(1 + \frac{1}{2n}\right) + log\left(\frac{294}{424.08}\right] - (0.373)\left(\frac{1}{5^2} + \frac{1}{7^2}\right)$

$= 0.454741974 + log\left(1 + \frac{1}{2n}\right)$

$= 0.454741974 + 0.0000000001$, since $n > 10^{35}$. Thus $R_1(n) < 0.4547419741$.

Suppose $5 \mid n$ and $7 \nmid n$

It follows from (1.1.4) (iii) that $5^6 \mid n$. Also let m = 1 is a divisor of n with m < n, so that by Lemma 2.6 we have

$R_1(n) < log\left(\frac{\alpha}{\beta}\right) - cR_2(n)$

$= \log\alpha - \log\beta - (0.373)\left(\frac{1}{5^2}\right)$

$= \log 2 + log\left(1 + \frac{1}{2n}\right) - \log(1.24) - (0.373)\left(\frac{1}{5^2}\right)$

= 0.4631158 + $log\left(1+\frac{1}{2n}\right)$
= 0.4631158 + 0.0000000001, since n > 10^{35}. Thus $R_1(n)$ < 0.4631158001.

3.3 Theorem:

If n is quasi perfect then (15, n) = 1 or 15. More explicitly, every quasi perfect number is divisible by both 3 and 5 or by none of them.

Proof: If n is quasi perfect and (15, n) = 3.

Then by Lemma 3.1, $R_1(n)$ < 0.2839779561 while $R_1(n)$ > 0.603831 by (1.1.5) (ii), which gives a contradiction. Hence (15, n) $\neq$3.

Again, if (15, n) = 5.

Then by Lemma 3.2, $R_1(n)$ < 0.4631158001 while (1.1.5) (iii) gives $R_1(n)$ > 0.647387. These two contradict each other. Hence (15, n) $\neq$5.

Thus (15, n) = 1 or 15, proving the theorem.

3.4 Theorem:

If n is quasi perfect then 3.5.7 cannot divide n.

Proof: Suppose 3.5.7 divide n. Then it follows from (1.1.4) (ii), (1.1.4) (iii) and (1.1.4) (iv) that $3^4 \mid n, 5^6 \mid n, 7^2 \mid n$. But it was proved in[2] that if n $\equiv$0 (mod 3) and p $\equiv$1 (mod 3) is a divisor of n then pk | n with k $\geq$ 4. Therefore $7^4 \mid n$. Also since w(n) $\geq$7 we get that m = $3^4.5^6.7^4$ is a divisor of n with m < n, so that by Lemma 2.6 we have

$R_1(n) < R_1(m) + cR_2(m) + log\left(\frac{\alpha m}{\beta\sigma(m)}\right) - cR_2(n)$

$= \frac{1}{3}+\frac{1}{5}+\frac{1}{7}+0.373\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\right)+log\left(\frac{\alpha\cdot 3^4\cdot 5^6\cdot 7^4\cdot 2\cdot 4\cdot 6}{\beta\cdot(3^5-1)\cdot(5^7-1)\cdot(7^5-1)}\right] - (0.373)\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\right)$

$= \frac{1}{3}+\frac{1}{5}+\frac{1}{7}+log\alpha+log\left(\frac{3^4.5^6.7^4.48}{1\cdot(3^5-1)\cdot(5^7-1)(7^5-1)}\right]$

$= \frac{1}{3}+\frac{1}{5}+\frac{1}{7}+log2+log\left(1+\frac{1}{2n}\right)+log\left(\frac{3^4.5^6.7^4.48}{(3^5-1)\cdot(5^7-1)(7^5-1)}\right]$

= 0.590774338 + $log\left(1+\frac{1}{2n}\right)$

= 0.590774338 + 0.0000000001, since n > 10^{35}. Thus $R_1(n)$ < 0.5907743381.

But (1.1.5) (iv) gives $R_1(n)$ > 0.596063, which gives a contradiction.

Hence 3.5.7 cannot divide n.

4 Conclusion

The study is focused on improving upper bounds and in the process concluding about the restrictions on divisors of quasi perfect numbers which is a non-deficient number. The inequality we proved in lemma 2.6 is used to improve bounds in Theorem 3.1 and Theorem 3.2 which helps in the important conclusion that either 3 and 5 together divide a quasi perfect number or none of them divide. We also come to the conclusion that 3.5.7 together cannot divide a quasi perfect number.

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