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RESEARCH ARTICLE

Proper d-Lucky Labeling of Rooted Products and Corona Products of Certain Graphs

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Abstract

Objectives: To examine rooted products graph and corona product of path graph with itself and cycle graph with itself for the existence of d-lucky labeling. **Methods:** In this study, d-lucky number for Rooted product graph of path graph to path graph ($P_n \circ P_n$) and Corona product graph ($P_n \odot P_n$) are computed. Method of construction is used throughout this paper to prove the theorems. **Findings:** Rooted products and corona products of path with itself and cycle graph with itself admit d-lucky labeling and d-lucky numbers for the same are obtained. **Novelty:** d-lucky numbers for some graphs are obtained by some authors but for rooted product of path with itself and corona products of path with itself and cycle with itself are new findings.

Keywords: Proper Lucky labeling; Rooted product; corona product; d-lucky; d-lucky labeling

1 Introduction

Lucky labeling of graph is an extension of graph labeling, a lot of work has been done in this area. Recently proper lucky labeling of Quadrilateral Snake graphs was published by Sateesh Kumar et al[1]. Another variant of lucky labeling was further extended into d-lucky labeling by Indra Rajasingh et al[2]. d-lucky labeling has been in studied for various graphs, this concept is being extended as proper d-lucky labeling by E. Esakkrammal et al[3]. Lower bound of d-lucky number for certain graphs were found by Sandi Klavzar et al[4].

Proper $d-$lucky labeling for Rooted Product of $P_n o\ P_n$ and $C_n o\ C_n$

For a vertex u in a graph G, let $N(u) = \{v \in V(G)/uv \in E(G)$ and $N[u] = N(u) \cup (u\}$. Let $l\ : V(G) \to \{1,2,\ldots,k\}$ be a labeling of vertices of a graph G by positive integers. Define $C(u) = \sum_{v\in N(u)} l(v) + d(u)$, where $d(u)$ denotes the degree of u. Define a labeling l as d-lucky if $C(u) \neq C(v)$, for every pair of adjacent vertices u and v in G. The d-lucky number of a graph G, denoted by $n_{dl}(G)$, is the least positive k such that G has a d-lucky labeling with $\{1,2,\ldots,k\}$ as the set of labels.[2]

Given a graph G of order n and a graph H with root vertex v , the rooted product graph $G\ \ H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and identifying the vertex u_i of G with the vertex v in the i^{th} copy of H for every $1 \leq i \leq n$. We take first rooted product path P_n with itself and then cycle graph C_n with itself and compute the $d-$lucky labeling for them.

Theorem 1.1 The Rooted product of P_n, $P_n \circ P_n$ admits proper $d-$lucky labeling and $P_n \circ P_n$, where $n \geq 2$.
Proof
The vertices of the product graph are labeled as shown below:

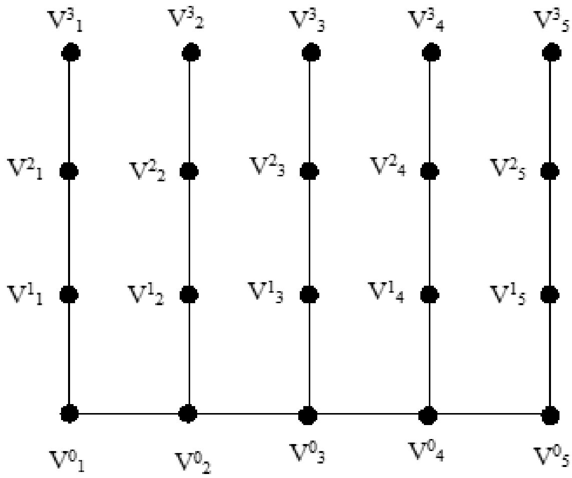

Fig 1. d-Lucky labeling

There are mn vertices of the product graph $P_n \bigcirc P_m$. Here $m = 4$ and $n = 5$. Let $f\left(v_i^j\right)$ = the label assigned to the vertex v_i^j. We define $S\left(v_i^j\right) = \sum_{u_i^j \in N(V_i^j)} f(u_i^j)$, as the sum of neighborhood of vertex v_i^j, where $N(v_i^j)$ denotes the open neighborhood of $v_i^j \in V$.

Label the vertices as given below:
For $j = 1,4,7,10,\ldots,m$.
$i = 1,2,3,\ldots,n$.
$f\left(V_i^{j-1}\right) = (3i-1)\, mod\ 2+1$.
For $j = 2,5,8,11,\ldots,m$.
$i = 1,2,3,\ldots,n$.

$$f\left(V_i^{j-1}\right) = ((i+2)\, mod\ 2+1\} + \{3i\ mod\ 2\}$$

For $j = 3,6,9,\ldots,m$.
$i = 1,2,3,\ldots,n$.
$f\left(V_i^{j-1}\right) = (i+1)\, mod\ 2+2$.

It is seen that the vertices which are labelled as 1 in the row V_i^0 will get neighborhood sum $S\left(v_i^j\right) = 7$ and $C(u) = 10,$ except the corner vertices will get the neighbourhood sum as $S\left(v_i^j\right) = 5$ and $C(u) = 7$. The vertices in the row V_i^0 labeled as 2 will have the neighbourhood sum as $S\left(v_i^j\right) = 3$ and $C(u) = 6$ except the corner vertex which will get the neighbourhood sum as $S\left(v_i^j\right) = 2$ and $C(u) = 4$. In the row V_i^{j-1}, $j = 4,7,10,\ldots,m-1$ the vertices with label 1 will get neighborhood sum as $S\left(v_i^j\right)$= 5 and $C(u) = 7$ and the vertices labeled as 2 will have the neighbourhood sum as $S\left(v_i^j\right) = 4$ and $C(u) = 6$. In the row V_i^{j-1}, $j = m$, the vertices with label 1 will get neighborhood sum as $S\left(v_i^j\right)$=2 and $C(u) = 3$ and the vertices labeled as 2 will have the neighbourhood sum as $S\left(v_i^j\right) = 3$ and $C(u) = 4$. In the row V_i^{j-1}, $j = 2,5,8,\ldots,m-1$ the vertices with label 3 will get neighborhood sum as $S\left(v_i^j\right)$= 3 and $C(u) = 5$ and the vertices labeled as 1 will have the neighbourhood sum as $S\left(v_i^j\right) = 5$

and $C(u) = 7$. In the row $V_i^{j-1},\ j = m$ the vertices with label 3 will get neighborhood sum as $S\left(v_i^j\right) = 1$ and $C(u) = 2$ and the vertices labeled as 1 will get the neighbourhood sum as $S\left(v_i^j\right) = 2$ and $C(u) = 3$. In the row $V_i^{j-1},\ j = 3, 6, 9, \ldots, m-1$ the vertices labelled as 2 will get the neighbourhood sum as $S\left(v_i^j\right) = 4$ and $C(u) = 6$ and the vertices with label 3 will have the neighbourhood sum as $S\left(v_i^j\right) = 3$ and $C(u) = 5$. In the row $V_i^{j-1},\ j = m$ the vertices with label 2 will get neighborhood sum as $S\left(v_i^j\right) = 3$ and $C(u) = 4$ and the vertices labeled as 3 will get the neighbourhood sum as $S\left(v_i^j\right) = 1$ and $C(u) = 2$. We note that $C(u) \neq C(v)$, for every pair of adjacent vertices u and v in $P_n \circ P_n$.

Hence the Rooted product of P_m *and* P_n, $P_m \circ P_n$ admits proper $d-$lucky labeling and $\eta_{dl}(P_m \circ P_n) = 3$, where $n \geq 2$. (For illustration see Figure 2)

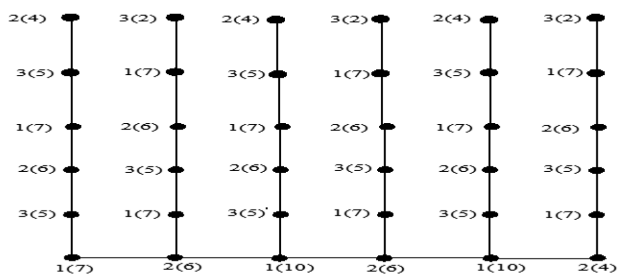

Fig 2. Proper d-Lucky Labeling of P_6 o P_6

Theorem 1.2: The Rooted product of C_n, $C_n \circ C_n$ admits proper d-lucky labeling and

$$\eta_{dl}(C_n \circ C_n) = \begin{cases} 2, & \text{when } n \text{ is even} \\ 3, & \text{when } n \text{ is odd} \end{cases}, \text{ where } \geq 3.$$

Proof

Case 1: When n is even:

Label the vertices of the base cycle in clock wise direction with $1, 2, 1, 2, 1, 2, \ldots, 1, 2$ in a cyclic manner until all the vertices of inner cycles receive a label. Next in the outer cycle whose one vertex is already labeled as 1, label rest of the vertices with 2 and 1 alternately such that no two adjacent vertices have the same labeling. In the outer cycle whose one vertex is labeled as 2, label rest of the vertices with 1 and 2 alternately such the no two adjacent vertices have the same labeling.

It is observed that the neighborhood sum $S(u)$ for the outer cycle which is labeled as 1 are $S(u) = 4$ *and* $C(u) = 6$ and for the vertex with label 2 has $S(u) = 2$ and $C(u) = 4$.

For the inner cycle the neighborhood sum $S(u)$ of the vertices with label as 1 has $S(u) = \left(1\frac{(n-2)}{2} + 2(\frac{n}{2}) + 4\right\}$ and $C(u) = \left(1\frac{(n-2)}{2} + 2(\frac{n}{2}) + 8\right\}$. The vertices which are labeled as 2 will have $S(u) = \left(1(\frac{n}{2}) + 2\frac{(n-2)}{2} + 2\right\}$ and $C(u) = \left(1(\frac{n}{2}) + 2\frac{(n-2)}{2} + 4\right\}$ Thus, it is noticed that no two adjacent vertices have the same $C(u)'s$. Hence the Rooted product of C_n, $\eta_{dl}(C_n \circ C_n) = 2$ admits proper d-lucky labeling and $\eta_{dl}(C_n \circ C_n) = 2$, where $n \geq 3$.

Case 2: when n is odd

Label the vertices of the base cycle in anti-clock wise direction with $1, 2, 3, 1, 2, 3, \ldots., 1, 2, 3$ in a cyclic manner and the last two vertices receive the label as $1, 3$. Next in the outer cycle whose one vertex is labeled as 1, label rest of the vertices in ant-clockwise direction as $3, 1, 3, 1, \ldots .3, 1$ and the last vertex as 2. In the outer cycle whose one vertex is labeled as 2 , label rest of the vertices in anti-clockwise direction as $1, 2, 12, 12, \ldots .1, 2$ alternately and the last vertex as 3. In the outer cycle whose one vertex is labeled as 3, label in anti-clockwise as $1, 3$ alternately and the last vertex as 2.

For $n = 3$, the inner cycle vertices with label 1 have neighbourhood sums as $S(u) = 10$, $C(u) = 14$, with label 2, $S(u) = 8$, $C(u) = 12$, with label 3, $S(u) = 6$ *and* $C(u) = 10$. The neighborhood sum $S(u)$ for the outer cycle are $S(u) = 5$ for the vertex with label as 1 and $C(u) = 7$. The vertex with label 2 of the outer cycle has $S(u) = 4$ and $C(u) = 6$. The vertex with label 3 of the outer cycle has $S(u) = 3$ and $C(u) = 5$.

For $n \geq 4$, it is observed that the neighborhood sum $S(u)$ for the outer cycle are $S(u) = 4$ *or* 5 *or* 6 for the vertex with label as 1 and $C(u) = 6$ *or* 7 *or* 8. The vertex with label 2 of the outer cycle has $S(u) = 2$ *or* 4 and $C(u) = 4$ *or* 6. The vertex with label 3 of the outer cycle has $S(u) = 2$ *or* 3 and $C(u) = 4$ *or* 5.

For the inner cycle the vertices with label 1, which is adjacent to vertices with label as 3 have S$(u) = \left(3\frac{(n-1)}{2} + 1\frac{(n-3)}{2} + 8\right\}$ and $C(u) = \left(3\frac{(n-1)}{2} + 1\frac{(n-3)}{2} + 12\right\}$. For the vertices with label as 1 and which is adjacent to vertices with label as 3 and 2 in the inner cycle have S$(u) = \left(3\frac{(n-1)}{2} + 1\frac{(n-3)}{2} + 7\right\}$ and $C(u) = \left(3\frac{(n-1)}{2} + 1\frac{(n-3)}{2} + 11\right\}$. The vertex with label 2 will get $S(u) = \left(1\frac{(n-1)}{2} + 2\frac{(n-3)}{2} + 7\right\}$ and $C(u) = \left(1\frac{(n-1)}{2} + 2\frac{(n-3)}{2} + 11\right\}$. The vertex with label 3 and adjacent to vertices with label as 1 in the inner cycle will have $S(u) = \left(1\frac{(n-1)}{2} + 2\frac{(n-3)}{2} + 5\right\}$ and $C(u) = \left(1\frac{(n-1)}{2} + 2\frac{(n-3)}{2} + 9\right\}$. The vertex with label 3 and adjacent to vertices with label as 1 and 2 in the inner cycle will have $S(u) = \left(1\frac{(n-1)}{2} + 2\frac{(n-3)}{2} + 6\right\}$ and $C(u) = \left(1\frac{(n-1)}{2} + 2\frac{(n-3)}{2} + 10\right\}$ Thus, it is observed that no two adjacent vertices have the same $C(u)'s$. Hence in this case the Rooted product of C_n, $C_n \circ C_n$ admits proper d-lucky labeling and $\eta_{dl}(C_n \circ C_n) = 3$. (for illustration see Figure 3)

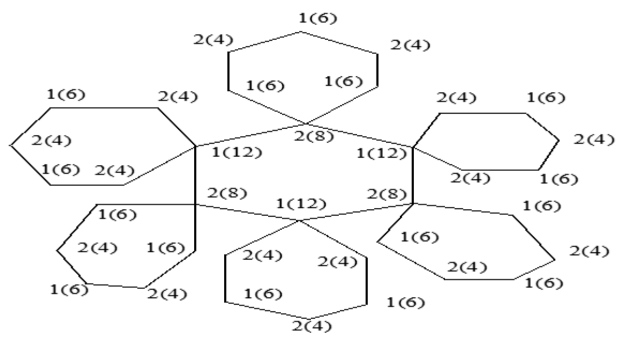

Fig 3. Proper d-lucky labeling of$C_6 \circ C_6$

Hence the result.

2 Proper d- lucky labeling for Corona Product of $P_n \odot P_n$and $C_n \odot C_n$

The corona product of G and H is the graph $G \odot H$ obtained by taking one copy of G, called the center graph, $(V(G)|$ copies of H, called the outer graph, and making the ith vertex of G adjacent to every vertex of the i^{th} copy of H, where $1 \leq i \leq (V(G)|$. In this section corona product of P_n with P_n, C_n with C_n are taken and proper d-lucky number for the same are computed.

Theorem 2.1: The Corona product of P_n, $P_n \odot P_n$ admits proper d-lucky labeling and $\eta_{dl}(P_n \odot P_n) = 4$, where $n \geq 2$.

Proof:

Label the vertices of the base path P_n alternately with $1,2,3,4$. Other vertices of corona are labeled as follows: The vertex which is identified with the base path P_n, if it has the label 1 then label all other vertices alternately as $2,3$. If it has the label as 2 then label all other vertices alternately as $1,4$. If it has label as 3 then label all other vertices as 1, 2 alternately and if it has label as 4, then label all other vertices as $1,2$ alternately. For neighborhood sum two cases arise.

Case 1: when n is even:

$When\ n = 2$, in the base vertices, the neighborhood sum $S(u) = 7$ and $C(u) = 10$, for vertices with label 1, for the vertices labeled as 2, $S(u) = 5$ *and* $C(u) = 8$. The vertices other than base vertices labeled as 1 has $S(u) = 5$, $C(u) = 7$, labeled as 2 has $S(u) = 4$, $C(u) = 6$ and labeled as 3 has $S(u) = 3$ *and* $C(u) = 5$.

$When\ n \geq 3$, in the base vertices, the neighborhood sum $S(u) = \{\frac{n}{2}(5) + 2\}$ and $C(u) = \{\frac{7n+6}{2}\}$ for the end vertices with label 1. All other vertices with label as 1 have $S(u) = \{\frac{n}{2}(5) + 6\}$ and $C(u) = \frac{7n+16}{2}$. $S(u) = \{\frac{n}{2}(5) + 1\}$ and $C(u) = \frac{7n+4}{2}$ for the end vertices with label 2 and all other vertices with label 2 will have $S(u) = \{\frac{n}{2}(5) + 4\}$ and $C(u) = \frac{7n+12}{2}$. The vertex with label as 3 will get $S(u) = \frac{3n+12}{2}$ and $C(u) = \frac{5n+16}{2}$. The end vertex with label as 4 will get $S(u) = \frac{3n+6}{2}$ and $C(u) = \frac{5n+8}{2}$. All other vertices with label as 4 will have $S(u) = \frac{3n+8}{2}$ and $C(u) = \frac{5n+12}{2}$.

For vertices other than the base vertices with label as 2 and adjacent to base vertex 1 will have $S(u) = 4$ *and* 7 and $C(u) = 6$ *and* 10. The vertex labelled as 2 and adjacent to base vertex 3 will have $S(u) = 5,4$ and $C(u) = 8,6$. And adjacent to base vertex with label as 4 will have $S(u) = 6,5$ *and* $C(u) = 9,7$. Thus, it is noticed that no two adjacent $C(u)^{'}s$ are same.

Case 2: When n is odd:

In the base vertices, the neighborhood sum $S(u) = \frac{5n+3}{2}$ and $C(u) = \frac{7n+5}{2}$, for the end vertices with label 1 and adjacent to vertex with label as 2. The vertices adjacent to vertex as label 4 will have $S(u) = \frac{5n+7}{2}$ and $C(u) = \frac{7n+9}{2}$. All other vertices with label as 1 will have $S(u) = \frac{5n+11}{2}$ and $C(u) = \frac{7n+15}{2}$. $S(u) = \frac{3n+3}{2}$ and $C(u) = \frac{5n+5}{2}$ for the end vertices with label as 3 and all other vertices with label as 3 will have $S(u) = \frac{3n+11}{2}$ and $C(u) = \frac{5n+15}{2}$. The Neighbourhood sum for all the vertices with label as 2 will be $S(u) = \frac{5n+5}{2}$ and $C(u) = \frac{7n+9}{2}$. The neighbourhood sum for the vertices with label as 4 will be $S(u) = \frac{3n+5}{2}$ and $C(u) = \frac{5n+9}{2}$. All other vertices will have similar $C(u)'s$ as Case 1.

No two adjacent vertices have the same $S(u)$ and $C(u)$. Hence the result follows. (for illustration see Figure 4).

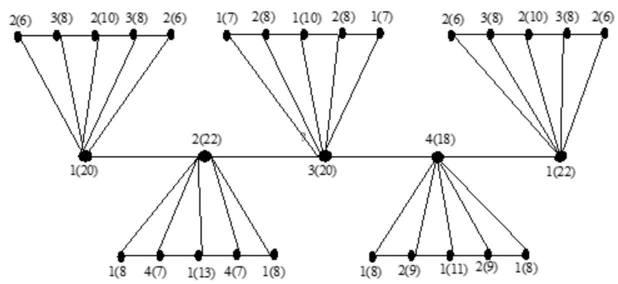

Fig 4. Proper d-Lucky Labeling of $P_5 \odot P_5$

Theorem 2.2: The Corona product of C_n, $C_n \odot C_n$ admits proper d-lucky labeling and $\eta_{dl}(C_n \odot C_n) = \begin{cases} 3, & \text{when } n \text{ is even} \\ 5, & \text{when } n \text{ is odd} \end{cases}$, where $n \geq 3$.

Proof

For this theorem two cases arise.

Case 1: when n is even

Label the vertices of the base cycle as $1,2$ alternately. The vertices of outer cycle are labeled $2,3$ alternately if it is adjacent to 1 of the base cycle and are labelled as $1,3$ alternately if it is adjacent to 2 of the base cycle.

The neighborhood sum $S(u)$ for the vertices of outer cycles will be as follows: $S(u) = 8$ and $C(u) = 11$ for the vertex with label as 1. $S(u) = 5$ and $C(u) = 8$ for the vertex with label as 2. $S(u) = 4$ and $C(u) = 7$ for the vertex with label as 3 and adjacent to the vertex with label as 2 of the base cycle. $S(u) = 5$ and $C(u) = 8$ for the vertex with label as 3 and adjacent to the vertex with label as 1 of the base cycle.

The neighborhood sum $S(u)$ for the base cycle will be as follows: The vertex with label as 1 will receive $S(u) = \frac{5n+8}{2}$ and $C(u) = \frac{7n+12}{2}$. The vertex with label as 2 will have $S(u) = \frac{4n+4}{2}$ and $C(u) = \frac{6n+8}{2}$. It is observed that no two adjacent vertices have the same $S(u)$ and the same $C(u)$ in this case. Hence $\eta_{dl}(C_n \odot C_n)$, when n is even.

Case 2: when n is odd

Label the vertices of the base cycle in clockwise direction alternately with $2,1$ and the last vertex is labeled as 3. Similarly, label the vertices of the outer cycle in anticlockwise direction with $2,3,4$ in cyclic manner and the last vertex is labeled as 5. The vertices with label as 1 in the base cycle, label other vertices alternately as $1,3$ and the last vertex as 4. The vertices labelled as 2 in the base cycle, label other vertices as $1,2$ alternately and the last vertex as 4. The neighborhood sum $S(u)$ and $C(u)$ are calculated similar to case 1 The neighborhood sum $S(u)$ for the base cycle will be as follows: The vertices with label as 1 will receive $S(u) = \left(\frac{5n+8}{2}\right\rceil$ and $C(u) = \left(\frac{7n+12}{2}\right\rceil$. The vertex with label as 2 will have $S(u) = \left(\frac{4n+4}{2}\right\rceil$ and $C(u) = \left(\frac{6n+8}{2}\right\rceil$. It is observed that no two adjacent vertices have the same $S(u)$ and the same $C(u)$ in this case. Hence $\eta_{dl}(C_n \odot C_n) = 5$, when n is odd.

It is noticed that no two adjacent vertices have the same $S(u)$ and $C(u)$ in this case. Hence $\eta_{dl}(C_n \odot C_n) = 5$, when n is odd. (for illustration see the Figure 5)

Hence the result.

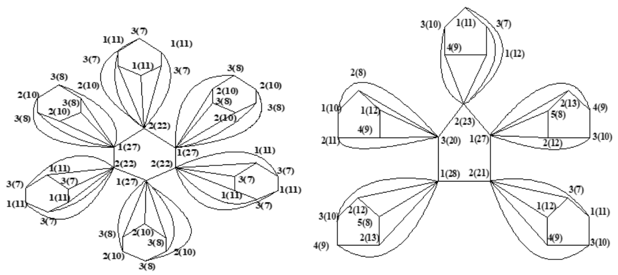

Fig 5. Properd- Lucky Labeling of $C_6 \odot C_6$ and $C_5 \odot C_5$

3 Conclusion

In this Paper proper d-lucky number for rooted graph $P_n \circ P_n$, $P_n \circ P_n$ and corona graph $P_n \odot P_n$, $C_n \odot C_n$ are computed and found as $\eta_{dl}(P_n \circ P_n) = 3$, where $n \geq 2$,

$\eta_{dl}\left(C_n \circ C_n\right) = \begin{cases} 2, \text{ when } n \text{ is even} \\ 3, \text{ when } n \text{ is odd} \end{cases}$, where ≥ 3, and

$\eta_{dl}(P_n \odot P_n) = 4$, where $n \geq 2$, and

$\eta_{dl}\left(C_n \odot C_n\right) = \begin{cases} 3, \text{ when } n \text{ is even} \\ 5, \text{ when } n \text{ is odd} \end{cases}$, where $n \geq$.

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