INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

OPEN ACCESS

Received: 23-05-2021

Accepted: 11-02-2022

Published: 18-11-2022

Citation: Shilpa N (2022) On New Subclasses of Analytic Functions Involving (p,q)-Derivatives. Indian Journal of Science and Technology 15(43): 2336-2341. https://doi.org/10.17485/IJST/v15i43.905

[*] **Corresponding author**.

drshilpamaths@gmail.com

Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

On New Subclasses of Analytic Functions Involving (p,q)-Derivatives

N Shilpa[1*]

1 Assistant Professor, PG Department of Mathematics, JSS College of Arts Commerce and Science, Mysuru, 570025, Karnataka, India

Abstract

Objective: The objectives of the present study are to introduce some new subclasses of analytic functions involving (p,q)-derivatives by using subordination. We derive Fekete-Szegö inequalities for the functions belonging to the new subclasses. **Method:** Using the concept of (p,q)-derivative of a function and the subordination principle between analytic functions we introduce and study new subclasses. **Findings:** The Fekete-Szegö problem may be considered as one of the most important results about univalent functions. It was introduced by Fekete-Szegö in 1933. Coefficient estimates for the second and third coefficients of functions belonging to class of analytic functions with specific geometric properties were obtained. We obtain the Fekete-Szegö inequalities for functions belonging to the new subclasses. Moreover, some special cases of the established results are discussed. **Novelty:** The results of the paper are new and significantly contribute to the existing literature on the topic.

Keywords: Analytic functions; Subordination; q-calculus; Fekete-Szegö inequalities; (p; q)-derivative operator

1 Introduction

Let A specify the category of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $U = \{z : z \in C \ \ and \ \ |z| < 1\}$.

The q-calculus is a generalization of the ordinary calculus without using the limit notation. The theory of q-derivative operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control, q-difference and q-integral equations, as well as Geometric function theory of complex analysis. The first application and usage of the q-calculus was introduced by Jackson [1,2]. After that many researchers have carried out remarkable studies, which play a significant role in the development of Geometric function theory. One may refer the papers [3–11] on this subject.

Recently there is an extension of q-calculus, denoted by (p,q)-calculus. The applications of (p,q)-calculus play important role in many diverse areas of the Mathematical, Physical and Engineering sciences. Quite a number of mathematicians studied the concepts of (p,q)-derivative. For details on (p,q)-calculus one can refer [12–15]. For the convenience, we provide some basic definitions and concept details of (p,q)-calculus which are used in this paper. The (p,q)-derivative of the function $f(z)$ is defined as [16]

$$D_{(p,q)}f(z) = \frac{f(pz)-f(qz)}{(p-q)z}, \quad (z \neq 0, \quad 0 < q < p \leq 1) \tag{2}$$

From equation (2) it is clear that if $f(z)$ and $g(z)$ are two functions, then

$$D_{(p,q)}(f(z)+g(z)) = D_{(p,q)}f(z) + D_{(p,q)}g(z).$$

$$D_{(p,q)}(cf(z)) = cD_{(p,q)}f(z).$$

Note that $D_{(p,q)}f(z) \to f'(z)$ as $p=1$ and $q \to 1^-$, where $f'(z)$ is the ordinary derivative of the function $f(z)$. Further by (2) the (p,q)-derivative of the function $h(z) = z^n$, is as follows

$$D_{(p,q)}h(z) = [n]_{(p,q)}z^{n-1} \tag{3}$$

where $[n]_{(p,q)}$ denotes the (p,q)-number and is given as:

$$[n]_{(p,q)} = \frac{p^n-q^n}{p-q}, \ (0 < q < p \leq 1). \tag{4}$$

Note that $[n]_{(p,q)} \to n$ as $p=1$ and $q \to 1^-$, therefore in view of equation (3), $D_{(p,q)}h(z) = h'(z)$ as $p=1$ and $q \to 1^-$, where $h'(z)$ denotes the ordinary derivative of the function $h(z)$ with respect to z.

The (p, q)-derivative of the function $f(z)$, given by equation (1) is defined as

$$D_{(p,q)}f(z) = 1 + \sum_{n=2}^{\infty}[n]_{(p,q)}a_n z^{n-1} \ (0 < q < p \leq 1) \tag{5}$$

where $[n]_{(p,q)}$ is given by (4).

For the analytic functions $f(z)$ and $g(z)$ in U, we say that the function $g(z)$ is subordinate to $f(z)$ in U [17], and write $g(z) \prec f(z)$ if there exists a Schwarz function $\omega(z)$, which is analytic in U, with $\omega(0)=0$ and $|\omega(z)|<1$ such that

$$g(z) = f(\omega(z)), \quad (z \in U). \tag{6}$$

Let P denote the class of all functions $\varphi(z)$ which are analytic and univalent in U and for which $\varphi(z)$ is convex with $\varphi(0) = 1$ and $R\{\varphi(z)\} > 0$ for all $z \in U$.

Now using the concept of (p,q)-derivative of a function $f(z)\,\varepsilon A$ and the subordination principle between analytic functions we introduce new subclasses of A as follows.

Definition 1.1: A function $f(z) \in A$ is said to be in the class $R_{(p,q)}(\varphi)$ if it satisfies the following subordination condition

$$D_{(p,q)}(f(z)) \prec \varphi(z) \tag{7}$$

where $\varphi(z) \in P$ and $0 < q < p \leq 1$.

Definition 1.2: A function $f(z) \in A$ is said to be in the class $N_{(p,q)}(\varphi)$ if it satisfies the following subordination condition

$$(1-\alpha)\frac{f(z)}{z} + \alpha \ D_{(p,q)}(f(z)) \prec \varphi(z) \tag{8}$$

where $\varphi(z) \in P$ and $0 \leq \alpha \leq 1$, $0 < q < p \leq 1$.

Note that for p=1, we have $R_{(p,q)}(\varphi) = R_{(q)}(\varphi)$ [18] and $N_{(p,q)}(\varphi) = N_{(q)}(\varphi)$ [18] respectively.

2 Main results

The Fekete-Szegö problem [19] is to obtain the coefficient estimates for the second and third coefficients of functions belonging to class of analytic functions with a specific geometric properties. Now we find the Fekete-Szegö inequalities for functions belonging to the classes $R_{(p,q)}(\varphi)$ and $N_{(p,q)}(\varphi)$.

The following lemma is necessary to prove our main results.

Lemma 2.1. [20] Let $p(z) = 1+\sum_{n=1}^{\infty} c_n z^n, \quad (z \in U)$ be a function with positive real part in U and μ is a complex number, then

$$\left(c_2 - \mu\, c_1^2 \right| \leq 2max\{1;\ (2\mu-1|\}. \tag{9}$$

The result is sharp for the functions given by $p(z) = \frac{1+z}{1-z}$ and $p(z) = \frac{1+z^2}{1-z^2}$.

Theorem 2.1: Let $\varphi(z) = 1+B_1 z+B_2 z^2+\ \ldots \varepsilon\ P$, with $B_1 \neq 0$. If $f(z)$ given by (1) belongs to the class $R_{(p,q)}(\varphi)$ then

$$\left(a_3 - \mu\, a_2^2 \right| \leq \frac{B_1}{[3]_{p,q}} max\left\{1, \left(\frac{B_2}{B_1} - \frac{[3]_{p,q}\mu B_1}{[2]_{p,q}^2}\right|\right\} \tag{10}$$

where μ is a complex number, and 0 < q < p ≤1. The result is sharp.

Proof: If $f(z)\varepsilon\, R_{(p,q)}(\varphi)$, then in view of Definition (1.1) there is a Schwarz function $\omega(z)$ in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ in U such that

$$D_{(p,q)}(f(z)) = \varphi(\omega(z)). \tag{11}$$

We define the function

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1+p_1 z+p_2 z^2+\ \ldots \tag{12}$$

Since $\omega(z)$ is a Schwarz function, we have $R\{p(z)\} > 0$ and $p(0) = 1$. Let

$$g(z) = D_{(p,q)} f(z) = 1+d_1 z+d_2 z^2+\ \ldots \tag{13}$$

Using equations (11), (12) and (13) we obtain

$$g(z) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right)$$

Since

$$\frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(p_1 z+\left(p_2 - \frac{p_1^2}{2}\right)z^2+\left(p_3+\frac{p_1^3}{4}-p_1 p_2\right)z^3+\ \ldots\right)$$

which gives

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1+\frac{1}{2}B_1 p_1 z+\left(\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2 p_1^2\right)z^2+\ldots \tag{14}$$

Using equations (13) and (14) we obtain

$$d_1 = \frac{1}{2}B_1 p_1$$

$$d_2 = \frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2 p_1^2.$$

A simple computation gives

$$D_{(p,q)}(f(z)) = 1+(2]_{p,q} a_2 z+(3]_{p,q} a_3 z^2+\ \ldots$$

Inequality (13), yields

$$d_1 = [2]_{p,q} a_2$$

$$d_2 = [3]_{p,q} a_3$$

now comparing the coefficients of z and z^2 and simplifying we get

$$a_2 = \frac{B_1 p_1}{2[2]_{p,q}}$$

and

$$a_3 = \frac{B_1}{2[3]_{p,q}} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4[3]_{p,q}}$$

hence

$$a_3 - \mu\, a_2^2 = \frac{B_1}{2[3]_{p,q}} \left(p_2 - \gamma p_1^2 \right)$$

where

$$\gamma = \frac{1}{2} \left(1 - \frac{B_2}{B_1} - \frac{[3]_{p,q} \mu B_1}{[2]_{p,q}^2} \right)$$

Hence, by applying Lemma 2.1, the result follows.

Note that taking p=1 in Theorem 2.1 we get the following result derived in [18].

Corollary 2.1: Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots \varepsilon\, P$, with $B_1 \neq 0$. If $f(z)$ given by (1) belongs to the class $R_{(q)}(\varphi)$ and μ is a complex number, then

$$\left(a_3 - \mu\, a_2^2 \right| \leq \frac{B_1}{[3]_q} max \left(1, \left(\frac{B_2}{B_1} - \frac{[3]_q \mu B_1}{[2]_q^2} \right| \right\}$$

The result is sharp.

Similarly, we can obtain upper bound for the Fekete-Szegö inequalities for functions belonging to the class $N_{(p,q)}(\varphi)$ as follows.

Theorem 2.2: Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots \varepsilon\, P$, with $B_1 \neq 0$. If $f(z)$ given by (1) belongs to the class $N_{(p,q)}(\varphi)$ then

$$\left(a_3 - \mu\, a_2^2 \right| \leq \frac{B_1}{[(1-\alpha) + (3]_{p,q} \alpha]} max \left\{ 1, \left(\frac{B_2}{B_1} - \frac{\mu B_1 [(1-\alpha) + (3]_{p,q} \alpha]}{[(1-\alpha) + (2]_{p,q} \alpha]^2} \right| \right\}$$

where μ is a complex number, and 0 < q < p ≤1. The result is sharp.

Proof: If $f(z) \varepsilon\, N_{(p,q)}(\varphi)$, then in view of Definition (1.1) there is a Schwarz function $\omega(z)$ in U with $\omega(0) = 0$ and $(\omega(z)| < 1$ in U such that

$$(1-\alpha) \frac{f(z)}{z} + \alpha D_{(p,q)}(f(z)) = \varphi(\omega(z)) \tag{15}$$

We define the function

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + \ldots \tag{16}$$

Since ω is a Schwarz function, we have $R\{p(z)\} > 0$ and $p(0) = 1$. Let

$$g(z) = (1-\alpha)\frac{f(z)}{z} + \alpha D_{(p,q)}(f(z)) = 1 + d_1 z + d_2 z^2 + \ldots \tag{17}$$

using equations (15), (16) and (17) we obtain

$$g(z) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right)$$

Since

$$\frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2\right)z^3 + \ldots\right)$$

which gives

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right)z^2 + \ldots \tag{18}$$

using equations (17) and (18) we obtain

$$d_1 = \frac{1}{2}B_1 p_1$$

$$d_2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2$$

A computation gives

$$(1-\alpha)\frac{f(\mathrm{z})}{\mathrm{z}} + \alpha D_{(p,q)}(f(z)) = 1 + [(1-\alpha) + [2]_{p,q}\alpha]\,\mathrm{a}_2\mathrm{z} + [(1-\alpha) + [3]_{p,q}\alpha]\,a_3\mathrm{z}^2 + \ldots$$

Inequality (17), yields

$$d_1 = [(1-\alpha) + [2]_{p,q}\alpha]\,\mathrm{a}_2$$

$$d_2 = [(1-\alpha) + [3]_{p,q}\alpha]\,\mathrm{a}_3$$

or equivalently we get

$$a_2 = \frac{B_1 p_1}{2\left[(1-\alpha) + [2]_{p,q}\alpha\right]}$$

and

$$\mathrm{a}_3 - \mu a_2^2 = \frac{B_1}{2[(1-\alpha) + [3]_{p,q}\alpha]}\left(\mathrm{p}_2 - \gamma p_1^2\right)$$

hence

$$a_3 - \mu\, a_2^2 = \frac{B_1}{2\left((1-\alpha) + (3]_{p,q}\alpha\right]}\left(p_2 - \gamma p_1^2\right)$$

Where

$$\gamma = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{\mu B_1[(1-\alpha) + [3]_{p,q}\alpha]}{[(1-\alpha) + [2]_{p,q}\alpha]^2}\right)$$

Hence, by applying Lemma 2.1, the result follows.

Note that taking p=1 in Theorem 2.2 we get the following result derived in [18].

Corollary 2.2: Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ \ldots\ \in\ P$, with $B_1 \neq 0$. If $f(z)$ is given by (1) belongs to the class $N_{(q)}(\varphi)$ and μ is a complex number, then

$$\left| a_3 - \mu\, a_2^2 \right| \leq \frac{B_1}{[(1-\alpha)+[3]_q\alpha]} max\left\{1, \left|\frac{B_2}{B_1} - \frac{\mu B_1 [(1-\alpha)+[3]_q\alpha]}{[(1-\alpha)+[2]_q\alpha]^2}\right|\right\}.$$

The result is sharp.

3 Conclusion

The q-difference calculus or quantum calculus was initiated at the beginning of 19$^{\text{th}}$ century, that was initially developed by Jackson. The q-calculus is one of the tool which is used to introduce and investigate many number of subclasses of analytic functions. The quantum calculus has many applications in the fields of special functions and many other areas. Further there is an extension of the q-calculus to postquantum calculus denoted by the (p,q)-calculus. In this paper we introduce and study new subclasses of analytic functions defined by using (p,q)-derivative operator. We derive the Fekete-Szegö inequalities for functions belonging to these classes. Moreover, some special cases of the established results are discussed**.**

References

1) Jackson F. On q-definite integrals. *Quarterly Journal of Pure and Applied Mathematics*. 1910;41(15):193–203.
2) Jackson F. q-Difference Equations. *American Journal of Mathematics*. 1910;32(4):305–314. Available from: https://doi.org/10.2307/2370183.
3) Frasin BA, Darus M. On the Fekete-Szegö problem. *International Journal of Mathematics and Mathematical Sciences*. 2000;24(9):577–581. Available from: https://doi.org/10.1155/S0161171200005111.
4) Frasin BA, Darus M. Subclass of analytic functions defined by $ q $-derivative operator associated with Pascal distribution series. *AIMS Mathematics*. 2021;6(5):5008–5019. Available from: https://doi:10.3934/math.2021295.
5) Frasin BA, Murugusundaramoorthy G. A subordination results for a class of analytic functions defined by Q differential operator. *Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica*. 2020;19(1):53–64. Available from: https://doi:10.2478/aupcsm-2020-0005.
6) Aldweby MH, Darus. On Fekete-Szegö problems for certain subclasses defined by q-derivative. *Journal of Function Spaces*. 2017;p. 1–5. Available from: https://doi.org/10.1155/2017/7156738.
7) Alessa N, Venkateswarlu B, Loganathan K, Karthik TS, Reddy T, Sujatha P, et al. Certain class of analytic functions connected with Analogue of the Bessel function. *Journal of Mathematics*. 2021;9. Available from: https://doi.org/10.1155/2021/5587886.
8) Srivastava R, Zayed H. Subclasses of analytic functions of complex order defined by q-derivative operator. *Studia Universitatis Babes-Bolyai Mathematica*. 2019;64(1):71–80. Available from: https://doi:10.24193/submath.2019.1.07.
9) Mahmood S, Sokół J. New Subclass of Analytic Functions in Conical Domain Associated with Ruscheweyh q-Differential Operator. *Results in Mathematics*. 2017;71(3-4):1345–1357. Available from: https://doi:10.1007/s00025-016-0592-1.
10) Srivastava HM. Operators of Basic (or q-) Calculus and Fractional q-Calculus and Their Applications in Geometric Function Theory of Complex Analysis. *Iranian Journal of Science and Technology, Transactions A: Science*. 2020;44(1):327–344. Available from: https://doi.org/10.1007/s40995-019-00815-0.
11) Wang B, Srivastava R, Liu JL. A Certain Subclass of Multivalent Analytic Functions Defined by the q-Difference Operator Related to the Janowski Functions. *Mathematics*. 2021;9(14):1706–1706. Available from: https://doi.org/10.3390/math9141706.
12) Altınkaya S, Yalçın S. CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH QUASI-SUBORDINATION INVOLVING (p, q) -DERIVATIVE OPERATOR. *Kragujevac Journal of Mathematics*. 2020;44(4):639–649. Available from: https://doi./10.46793/KgJMat2004.639A.
13) Araci S, Duran U, Acikgoz M, Srivastava HM. A certain (p , q) - derivative operator and associated divided differences. *Journal of Inequalities and Applications*. 2016;2016(1):301–301. Available from: https://doi:10.1186/s13660-016-1240-8.
14) Srivastava HM, Raza N, Abujarad ESA, Srivastava GM, Abujarad MH. Fekete-Szegö inequality for classes of (p, q)-Starlike and (p, q)-convex functions. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A Matemáticas*. 2019;113(4):3563–3584. Available from: https://doi.org/10.1007/s13398-019-00713-5.
15) Al-Hawary T, Yousef F, Frasin BA. Subclasses of Analytic Functions of Complex Order Involving Jackson's (p, q)-derivative. *SSRN Electronic Journal*. 2018. Available from: https://doi.org/10.2139/ssrn.3289803.
16) Acar T, Aral A, Mohiuddine SA. On Kantorovich modification of (p , q) -Baskakov operators. *Journal of Inequalities and Applications*. 2016;98(1):1–14. Available from: https://doi.org/10.1186/s13660-016-1045-9.
17) Sanford S, Miller, Petru T, Mocanu. Differential subordinations: theory and applications. CRC Press. 2000. Available from: https://doi.org/10.1201/9781482289817.
18) Janteng A, Hern ALP, Omar R. Fekete-Szegö functional of classes of analytic functions involving the q-derivative operator. *Applied Mathematical Sciences*. 2020;14(10):481–488. Available from: https://doi.org/10.12988/ams.2020.914222.
19) Fekete M, Szegö G. Eine Bemerkung Über Ungerade Schlichte Funktionen. *Journal of the London Mathematical Society*. 1933;1(2):85–89. Available from: https://doi.org/10.1112/jlms/s1-8.2.85.
20) Ma W, Minda D. A unified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis. 1992;p. 157–169.