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Connected Restrained Detour Number of a Graph

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Abstract

Objectives: To find the connected restrained detour number for standard graphs and mesh graphs. **Methods:** By determining the connected restrained detour set with minimum cardinality, the connected restrained detour number of a graph is investigated. **Findings:** We study that the connected restrained detour number of the graphs is altered when we add pendent vertices. The minimum and maximum degree vertices of a graph are deleted and the connected restrained detour number of the mesh graph is computed. **Novelty:** Finding the detour path plays a vital role in the network-based systems. Planning the largest route that is connected and restrained is essential in business, industries and radio technologies. We introduce the new concept of connected restrained detour number. We also exhibit the bounds for the connected restrained detour set of a graph.

Keywords: Detour Set; Detour Number; Mesh Graphs; Connected Restrained Detour Set; Connected Restrained Detour Number

1 Introduction

Connected detour number of a graph developed from the notion of detour number was studied by Ali, Ahmed M, and Ali A. Ali [1]. Various parameters of detour number were established by John J, Sunil Kumar VR, Sethu Ramalingam S and Athisayanathan S [2–4]. In 2020, Palani K, Shanthi P, Nagarajan A. established the restrained detour concept [5,6]. Santhakumaran AP, Titus P, Ganesamoorthy K. defined the minimal connected restrained monophonic sets in graphs [7] . In this study, we introduce the connected restrained detour number denoted by $dn^{cr}(G)$. The connected restrained detour number for some standard graphs and mesh graphs is studied. We denote the vertices with minimum degree and maximum degree by δ-vertices and $\triangle$-vertices in a graph M. We discuss the effect of the addition of pendent vertices to the δ-vertices of mesh M. We also investigate the connected restrained detour number of derived graphs obtained from the mesh graphs by deletion of the pendent vertices from the δ-vertices of the mesh M.

Let $G = (V(G), E(G))$ be a connected finite simple graph with order $n \geq 2$.

2 Methodology

Definition 2.1. Any set $R \subseteq V(G)$ is a connected restrained detour set if

(i) R is a detour set of G

(ii) $\langle R \rangle$ is connected and

(iii) no isolated vertices exist in $< V(G) - R >$

Definition 2.2. The connected restrained detour number, $dn^{cr}(G)$ is the minimum cardinality of connected restrained detour set of G

Definition 2.3. A connected restrained detour set with cardinality $dn^{cr}(G)$ is called the minimum connected restrained detour set [dn^{cr}-set] of G.

Example 2. 4. For G_1 shown in Figure 1 , $R' = \{z_2, z_3, z_6, z_7\}$, $R'' = \{z_2, z_3, z_6, z_8\}$ and $R''' = \{z_1, z_2, z_3, z_6\}$ are the dn^{cr} sets of G_1. Hence $dn^{cr}(G_1) = 4$. Thus there can be many dn^{cr} sets for any graph G.

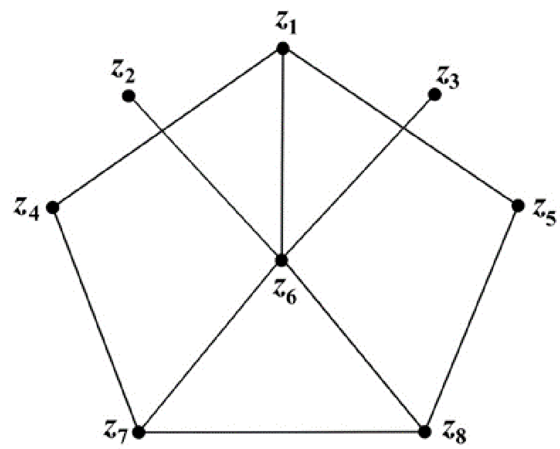

Fig 1. G1

3 Results and Discussion

Theorem 3.1. For a cycle $C_n\,(n \geq 3)$, $dn^{cr}(C_n) = 2$.

Proof. Consider a cycle C_n with $(V(C_n)| = n$. If $R = \{y, z\}$ is a set of vertices that are adjacent in C_n, then all the vertices in $V(C_n)$ lie on some $y - z$ detour. Since y and z are adjacent, $< R >$ is connected and $< V(C_n) - R >$ has no isolated vertices. Hence R is a dn^{cr}-set and $dn^{cr}(C_n) = 2$.

Theorem 3.2. For a path $P_n\,(n \geq 2)$, $dn^{cr}(P_n) = n$.

Proof. Consider a path P_n of order n. Let R be a set of end-vertices. Then all the vertices of P_n lie on some detour joining those two end-vertices of R. Since the vertices of R are not connected, $< R >$ is also not connected but $< V(P_n) - R >$ has no isolated vertices. Thus R is not a dn^{cr}-set. Therefore, R must contain all the internal vertices to generate a dn^{cr}-set. Thus $dn^{cr}(P_n) = n$.

Remark 3.3. By Theorem 3.1, for a cycle C_n, a set of any two adjacent vertices is a detour set and also a dn^{cr}-set. Hence $dn(C_n) = dn^{cr}(C_n) = 2$. Thus

$$2 = dn(G) = dn^{cr}(G) < n, \text{ for } C_n \tag{1}$$

By Theorem 3.2, for the path P_n of order n, $dn^{cr}(P_n) = n$.

$$2 = dn(G) < dn^{cr}(G) = n, \text{ for } P_2 \tag{2}$$

For the given graph G_1 of order 8 as in Figure 1, $R = (z_2, z_3, z_7\}$ is a minimum detour set.

Hence $dn(G) = 3$. Thus

$$2 < dn(G) < dn^{cr}(G) < n, \text{ for } G_1 \tag{3}$$

Observation 3.4. From (1), (2) and (3) of Remark 3.1, we obtain the bound for a dn^{cr}-set.

$2 \le dn(G) \le dn^{cr}(G) \le n$.

The following theorems exhibit the dn^{cr}-set for the cartesian product of any two paths P_a and P_b, which is known as mesh graph $M = P_a \square P_b$ and their derived graphs.

Theorem 3.5 . Let P_a and P_b be the paths of order a and b. Then $dn^{cr}(P_a \square P_b) = 2$.

Proof. Let $M = P_a \square P_b$ be a mesh, where

$V(P_a) = h_1,\ h_2,\ \ldots,\ h_a$ and $V(P_b) = h_1^*,\ h_2^*,\ h_3^*, \ldots,\ h_b^*$;

$V(P_a \square P_b) = \left(\left(h_i,\ h_j^*\right) = h_{ij} :\ 1 \le i \le a,\ 1 \le\ j \le b;\ h_i \in P_a,\ h_j^* \in P_b\right\}$.

Then $(V(P_a \square P_b)| = ab$. Let R be a set of two vertices, which are adjacent in the boundary of the mesh M. We consider four cases.

Case 1. Let a and b be even. Clearly, every vertex of M lies on either the $P^{'} : h_{k1} - h_{(k+1)1}$ or $P'' :\ h_{kb} - h_{(k+1)b}$ detour, similarly on either $P^{'''} : h_{1l} - h_{1(l+1)}$ or $P'''' : h_{rl} - h_{r(l+1)}$ detour for some $k,\ l\,(1 \le k \le a-1;\ 1 \le l \le b-1)$. Suppose S $= \left(h_{k1}, h_{(k+1)1}\right\}$. Then the vertices of M lie on $P' :\ h_{k1} - h_{(k+1)1}$ detour $(1 \le k \le b-1)$. When $k = 1$,

$$P^{'} :\ h_{11}, h_{12}, h_{13}, \ldots, h_{1(b-1)},\ h_{1b}, h_{2b}, h_{2(b-1)},\ \ldots, h_{24},\ h_{23},\ h_{22},\ h_{32}, h_{33}, h_{34},\ \ldots, h_{3(b-1)}, h_{3b}, h_{4b}, ..$$

$\ldots, h_{(a-1)b},\ h_{ab},\ h_{a(b-1)}, .., h_{a3},\ h_{a2}, h_{a1}, h_{(a-1)1}, ..., h_{41},\ h_{31}, h_{21}$.

Similarly, we can derive the detour path of $P^{''}, P^{'''}$ and P'''', where $2 \le k \le a-1; 1 \le l \le b-1$.

Case 2. Let a and b be odd. Obviously, every vertex of M lies on either the $P^{'} : h_{k1} - h_{(k+1)1}$ or $P'' :\ h_{kb} - h_{(k+1)b}$ detour, similarly on either $P^{'''} : h_{1l} - h_{1(l+1)}$ or $P'''' : h_{al} - h_{a(l+1)}$ detour for some $k,\ l\,(1 \le k \le a-1;\ 1 \le l \le b-1)$. Suppose R $= \left(h_{1l} - h_{1(l+1)}\right\}$. Then the vertices of M lie on either $P_1^{'} :\ h_{k1} - h_{(k+1)1}$ detour or $P_2^{'} :\ h_{k1} - h\,(1 \le k \le a-1)$. If $k = 1$,

$$P_1' : h_{11}, h_{12}, h_{13}, \ldots, h_{1(b-1)}, h_{1b}, h_{2b}, h_{2(b-1)}, \ldots, h_{24}, h_{23}, h_{22}, h_{32}, h_{33}, h_{34}, \ldots, h_{3(b-1)}, h_{3b}, \ldots$$
$$\ldots, h_{(a-2)b}, h_{(a-1)b}, h_{ab}, h_{a(b-1)}, \ldots, h_{a4}, h_{a3}, h_{a2}, h_{a1}, h_{(a-1)1}, h_{(a-2)1}, \ldots, h_{41}, h_{31}, h_{21}$$
$$P_2' : h_{11}, h_{12}, h_{13}, \ldots, h_{1(b-1)}, h_{1b}, h_{2b}, h_{2(b-1)}, \ldots, h_{24}, h_{23}, h_{22}, h_{32}, h_{33}, h_{34}, \ldots, h_{3(b-1)}, h_{3b}, ..$$
$$\ldots, h_{(a-1)b}, h_{(a-1)(b-1)}, \ldots, h_{(a-1)4}, h_{(a-1)3}, h_{(a-1)2}, h_{a2}, h_{a1}, h_{(a-1)1}, h_{(a-1)2}, h_{a2}, h_{a1}, h_{(a-1)1}, \ldots$$
$$\ldots, h_{41}, h_{31}, h_{21}$$

Similarly, we find two different detour paths of $P^{''}, P^{'''}$ and P'''', where $2 \le k \le a-1;\ 1 \le l \le b-1$.

Case 3. Let a be even. Clearly, every vertex of M lies on $P^{'}$ or P'' detour, where $P^{'} : h_{k1} - h_{(k+1)1}$ and $P'' :\ h_{kb} - h_{(k+1)b}$. Then R= $\left(h_{k1},\ h_{(k+1)1}\right\}$ or R $= \left(h_{kb},\ h_{(k+1)b}\right\}$. As in Case 1, consider $P^{'} : h_{11} - h_{21}$, then

$$P^{'} :\ h_{11}, h_{12}, h_{13}, \ldots, h_{41},\ h_{31}, h_{21}.$$

Consider $P'' : h_{1b} - h_{2b},\ \ where\ P'' :\ h_{1b}, h_{1(b-1)}, h_{1(b-2)}, \ldots, h_{a(b-1)}, h_{ab},\ h_{(a-1)b}, \ldots, h_{3b}, h_{2b}$.

In the same manner, we can find different detour paths of $P^{'}$ and $P^{''}$, where $2 \le k \le a-1$

Case 4. Let b be even. Let R $= \left(h_{al}, h_{a(l+1)}\right\}$ or R $= \left(h_{al}, h_{a(l+1)}\right\}$. Then the vertices of M lie on $P^{'''} : h_{1l} - h_{1(l+1)}$ or $P'''' :\ h_{al} - h_{a(l+1)}$ detour $(1 \le l \le b-1)$

Now, consider $P^{'''} :\ h_{11} - h_{12.}$

$$P^{'''} :\ h_{11}, h_{21}, h_{31}, \ldots, h_{a1},\ h_{a2}, h_{(a-1)2},\ \ldots,\ h_{32},\ h_{22},\ h_{23},\ h_{33}, \ldots,\ h_{(a-1)(b-1)},\ h_{ab}, h_{(a-1)b}, \ldots, h_{3b},$$

$h_{2b}, h_{1b},\ h_{1(b-1)}, h_{1(b-2)}, ...,\ h_{13}, h_{12}$.

Similarly,

$P'''' :\ h_{a1} - h_{a2} :\ \ h_{a1},\ h_{(a-1)1}, \ldots,\ h_{31}, h_{21}, h_{11}, h_{12}, h_{22}, h_{32},\ \ldots,\ h_{(a-1)2}, h_{(a-1)3},$

$h_{(r-2)3},\ \ldots, h_{33}, h_{23}, h_{13}, h_{14}, \ldots, h_{1(b-1)}, h_{1_b}, h_{2b}, h_{3b}, \ldots, h_{(a-1)b},\ h_{ab},\ h_{a(b-1)}, \ldots, h_{a4}, h_{a3}, h_{a2}$.

In the same manner, we can find different detour paths of $P^{'''}$ and P'''', where $2 \le l \le \alpha - 1$.

All the above four cases show that R is connected and $< V(P_a \square P_b) - R >$ has no isolated vertices and so R is the dn^{cr}-set. Hence $dn^{cr}(G) = 2$.

Remark 3.6 . Let $M = P_a \square P_b$ be the mesh as shown in Figure 2 . If $b = 2$, then by Theorem 3.5 $P_a \square P_2$ is a ladder graph L_a and $dn^{cr}(L_a) = 2$.

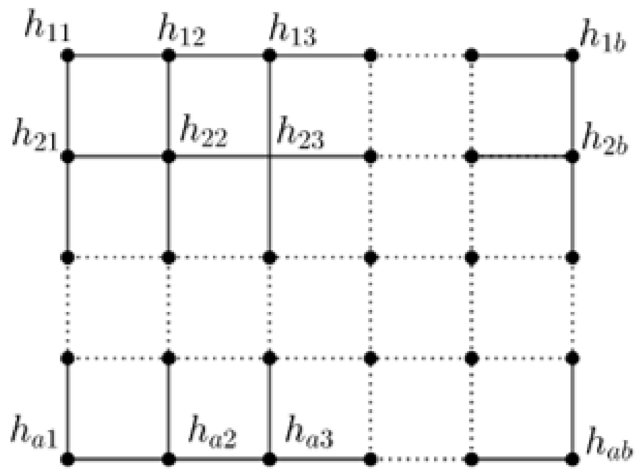

Fig 2. $M = P_a \square P_b$

Theorem 3.7. Let $M = P_a \square P_b$ be a mesh of order $ab\,(2 \le a \le b)$. Let M^{+p} be the graph expanded from M by adding a pendent vertex to the δ-vertices of M. Then $dn^{cr}(M^{+p}\ ) = 2\,(a+b+2)$.

Proof. Let M^{+p} be a graph derived from $M = P_a \square P_b\,(2 \le a \le b)$ by adding a pendent vertex to the δ-vertices of M. Let R^p be a set of all pendent vertices of M^{+p}. Then $R^p\ = (p_i : 1 \le i \le 4\}$. Now obviously, R^p is a dn^{cr}-set but not a connected restrained detour set by Theorem 3.4. Therefore, a dn^{cr}-set R must contain all the pendent vertices and the boundary vertices. Suppose $R = R^p \cup R^b$, where R^b is the set of all boundary vertices. Then $(R| = (R^p\,| + |\left(h_{ik,}\ h_{lj}\right\}| = 4+2(a+b) = 2\,(2+a+b)$. Hence all the vertices of M^{+p} lie on some detour joining the vertices of R. Moreover, R is connected and $< V\left(M^{+p}\right) - R >$ has no isolated vertices. Hence $dn^{cr}(M^{+p}\ ) = 2\,(a+b+2)$.

Theorem 3.8 . Let M^{+kp} be the graph obtained from $M = P_a \square P_b\,(2 \le a \le b)$ by the addition of k-pendent vertices at each δ-vertices of M. Then $dn^{cr}\left(M^{+kp}\ \right) = 2\,(a+b+2k)$.

Proof. Let M^{+kp} be a graph with $\left(V\left(M^{+kp}\ \right)\right| = ab+4k$. Let R^p be a set of all pendent vertices added to M. Then $(R^p\,| = 4k$. Since all the vertices of M^{+kp} lie on some detour joining the pendent vertices in R^p, R^p is the minimum detour set. Since M^p is not connected, we consider the sets R and R^b, the set of all boundary vertices. Then $R^b = \left((h_{ik} : 1 \le k \le b;\ i = 1,\ a\right\} \cup \{h_{lj} :\ 1 \le l \le a;\ j = 1,\ b\big\}$. Suppose $R = R^p \cup R^b$. Therefore, $|R| = |R^p| + \left|\left\{h_{ik}, h_{lj}\right\}\right| = 4k + 2(a+b) = 2(a+b+2k)$ Since R is connected and $< V\left(M^{+kp}\ \right) - R >$ has no isolated vertices, R is a dn^{cr}-set. Therefore $dn^{cr}(M^{+kp}\ ) = 2\,(a+b+2k)$.

Now, we discuss how the connected restrained detour number is altered by deleting the δ-vertices and $\triangle$-vertices of the mesh graph M.

Theorem 3.9. Let $M^{-\delta}$ be the graph derived from $M = P_a \square P_b\,(2 \le a \le b)$ by deleting the δ-vertices of M. Then –

Proof. Let $M = P_a \square P_b\,(2 \le a \le b)$. Let $M^{-\delta}$ be the graph derived by deletion of δ-vertices of M. Then we consider three cases.

Case 1. Let $M = P_a \square P_b\,(a = b)$. Consider R to be a connected restrained detour set of $M^{-\delta}$. We have two subcases.

Subcase 1. Let $a = b = 2$. Since all the vertices of $V(P_2 \square P_2)$ are the δ-vertices, the graph $M^{-\delta}$ is the null graph. Hence $dn^{cr}\left(M^{-\delta}\right) = 0$.

Subcase 2. Let $a = b = 3$. Then all the vertices of $M^{-\delta}$ become the elements of R and so $R = \{h_{12}, h_{21}, h_{22}, h_{23}, h_{32}\}$ is a dn^{cr}-set. Hence $dn^{cr}\left(M^{-\delta}\right) = 5$.

Case 2. When $a \ne 3$, we have two subcases.

Subcase 1. Let $a = 2$ and $b \ge 3$. When $b = 3$, the graph $M^{-\delta}$ derived from $P_a \square P_b$ becomes K_2. Hence $R = V(M^{-\delta})$, $dn^{cr}\left(M^{-\delta}\right) = 2$. When $b \ge 4$, $M^{-\delta}$ is the Ladder graph $L_{a-2} = P_{a-2} \square P_2$. Hence $dn^{cr}\left(\ L_{a-2}\right) = 2$.

Subcase 2. If $a \ge 4$ and $b \ge 3$, then $R = \left(h_{1q},\ h_{(q+1)1}\right\}$ or $\left(h_{qb},\ h_{(q+1)b}\right\}$, where $2 \le q \le b-2$, is a set of any two adjacent vertices. Since all the vertices of $M^{-\delta}$ lie on some detour joining the vertices of R, it is a minimum detour set. Moreover, R is connected and $< V(M^{-\delta}\ ) - R >$ has no isolated vertices. Thus R becomes the dn^{cr}-set of $M^{-\delta}$and so $dn^{cr}\left(M^{-\delta}\right) = 2$.

Case 3. Let $a=3, b\geq 4$. Since $M^{-\delta}$ is a graph obtained from M by deletion of δ-vertices, produces the pendent vertices h_{21} and h_{2b} of $M^{-\delta}$. Then the dn^{cr}-set R consists of the pendent vertices and all the internal vertices of $h_{21}-h_{2b}$ detour. Thus $R=(h_{21}, h_{22},\ h_{23},\ldots,h_{2b}\}$ and so $dn^{cr}\left(M^{-\delta}\right)=b$.

Theorem 3.10. Let $M^{-\triangle}$ be the graph derived from $M=P_a\square\, P_b(2\leq a\leq b)$ by deleting the $\triangle$-vertices of M. Then $-$

Proof. Let $M=P_a\square\, P_b(2\leq a\leq b)$ be a mesh. Let $M^{-\triangle}$ be a graph derived from M by the deletion of $\triangle$-vertices of M. Let R be the dn^{cr}-set of $M^{-\triangle}$. We have two cases.

Case 1: Let $r=2$. Consider the two subcases.

Subcase 1. When $s=2$, the graph $M^{-\triangle}$ derived from $P_2\square\, P_2$ is the null graph. Since the $\triangle$-vertices are also the δ-vertices of M, the result follows from subcase 2 of case1of Theorem 3.6 that $dn^{cr}\left(M^{-\triangle}\right)=0$.

Subcase 2. Let $b>2$. Since the boundary vertices of degree 3 are the $\triangle$-vertices of $M=P_2\square\, P_b$, the graph $M^{-\triangle}$ becomes a disjoint union of two paths of order 2. In this case, we cannot have the detour set R that is connected. Therefore $dn^{cr}\left(M^{-\triangle}\right)=0$.

Case 2: Let $a=b\geq 3$. Let $R^{\triangle}$ be a set of all $\triangle$-vertices of M. Since $\triangle(M)=4, R^{\triangle}=(h_{ij}:\ i\neq 1,\ a;\ j\neq 1,\ b\big\}$ and so $\left(R^{\triangle}\right|=(a-2)\,(b-2)$. Thus $M^{-\triangle}=M-R^{\triangle}$. Hence $|M^{-\triangle}|=(M|-\left(R^{\triangle}\right|=ab-((a-2)\,(b-2))=2\,(a+b-2)$. Thus $M^{-\triangle}$ is isomorphic to the even cycle of order $2\,(a+b-2)$. Thus $M^{-\triangle}=C_{2(a+b-2).}$ Then by Theorem 3.1, R is a set of two adjacent vertices of $M^{-\triangle}$ and so $dn^{cr}(M^{-\triangle})=2$.

4 Conclusion

In this study, we have computed the connected restrained detour number of some standard graphs, the mesh graph and their derived graphs. Derivation of similar results in the context of some other variants of detour number is an open area of research.

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