INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Proper d-Lucky Number for Certain Rooted Product Graphs

Chiranjilal Kujur[1]*

1 Assistant Professor, St. Joseph's College, Darjeeling, 734426, West Bengal, India

OPEN ACCESS

Received: 07-11-2022

Accepted: 22-12-2022

Published: 27-01-2023

Citation: Kujur C (2023) Proper d-Lucky Number for Certain Rooted Product Graphs. Indian Journal of Science and Technology 16(4): 249-253. https://doi.org/10.17485/IJST/v16i4.2146

* **Corresponding author**.

chiranji@yahoo.com

Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objectives: In this study, rooted product of path with cycle, path with complete graph cycle with complete graph and complete graph with complete graphs are taken and examined for the existence of d-lucky labeling for the same. **Methods:** The rooted product graphs are obtained of path with cycle, path with complete graph, cycle with complete graph and complete graph with itself and then proper d-lucky numbers are obtained for the above mentioned graphs. Construction and descriptive method are used to prove the results. **Findings:** Proper d-lucky labeling for the said graphs are verified and proper d-lucky numbers for the same are obtained. **Novelty:** It involves the mathematical formulations which involves labeling the vertices of a graph in such a manner that no two adjacent vertices have the same labeling and the neighborhood sums and degree sums of the adjacent vertices are different, which gives rise to proper d-lucky labeling.

Keywords: d-Lucky Labeling; d-Lucky Numbers; Proper d-Lucky Labeling; Proper d-Lucky Number; Rooted Product

1 Introduction

There are different types of labeling of graphs. According to one's requirements various types of labeling are originating in this field. Proper d-lucky labeling is one such labeling of graph, which was defined by Mirka Miller et al[1]. Recently a new labeling called d-lucky edge labeling was introduced and d-lucky edge number for path graph was determined by G. Rajini Ram et al[2]. Proper lucky labeling and lucky edge labeling for extended duplicate graph of quadrilateral snake graph were investigated by P. Indira et al[3]. The existence of an efficient zero ring labeling for some classes of trees and their disjoint union were studied by Dhenmar E. Chua et al[4]. Lucky and Proper Lucky Labeling of Quadrilateral Snake Graphs were studied by T. V. Sateesh Kumar et al[5]. d-lucky number for rooted product and corona product of certain graphs were obtained by Kujur C[6].

For a vertex u in a graph G, let $N(u) = \{v \in V(G)/uv \in E(G)$. Let $l : V(G) \rightarrow \{1, 2, \ldots, k\}$ be a labeling of vertices by positive integers. Define $C(u) = \sum_{v \in N(u)} l(v) + d(u)$, where $d(u)$ denotes the degree of u. Define a labeling l as d-lucky if $C(u) \neq C(v)$, for every pair of adjacent vertices u and v in G. The d-lucky number of a graph G, denoted by $\eta_{dl}(\mathrm{G})$, is the least positive k such that G has a d-lucky labeling with $\{1, 2, \ldots, k\}$ as the set of labels. Further if every pair adjacent vertices have different labels, then it is called proper d-lucky labeling and k is called proper d-lucky number, denoted by $\eta_{pdl}(G)$.

Rooted product of graphs is defined by C. D. Godsil as, "Given a graph G of order n and a graph H with root vertex v, the rooted product graph $G \circ H$ (copies of H in G, symbol used for rooted product of G with H) is defined as the graph obtained from G and H by taking one copy of G and n copies of H and identifying the vertex u_i of G with the vertex v in the i^{th} copy of H for every $1 \leq i \leq n$". In this paper rooted product of path with cycle, path with complete graph, cycle with complete graph and complete graph with complete graphs are taken and examined for the existence of d-lucky labeling for the same.

2 Result and Discussion

Theorem 1:

Rooted product of path with cycle,$P_n o C_n$ admits proper d-lucky labeling and $\eta_{pdl}\,(P_n o\, C_n) = 4, n \geq 3$.

Proof: Label the base vertices of path P_n with $1,4$ alternately. There are two cases while labeling the vertices of C_n.

Case 1: When n is even

If its base vertex has received a label as 1then label all other vertices of C_n in clockwise direction with $3,1$ alternately. If the base vertex of C_n has been labeled as 4, then label all other vertices in clockwise direction with $1,3$ alternately.

Case 2: When n is odd

If the base vertex of C_n has received label as 1 then label all other vertices in clockwise direction with $3,1$ alternately and label the last vertex with 2. If the base vertex of C_n has been labeled as 4, then label all other vertices in clockwise direction with $1,3$ alternately.

The neighborhood sums $s(u)$ and degree sums $c(u)$ are calculated as follows:- In cycle C_n the vertices which are labeled as 3, get $s\,(u) = 2$ and $c\,(u) = 4$, except the vertices which are adjacent to vertex with label as 2 get $s\,(u) = 3$ and $c\,(u) = 5$ and the vertices which are adjacent to the vertex with label as 4 get $s\,(u) = 5$ and $c\,(u) = 7$. The vertices which are labeled as 1 get $s\,(u) = 6$ and $c\,(u) = 8$ except the vertices which are adjacent to the vertex with label 4 get $s\,(u) = 7$ and $c\,(u) = 9$. The vertices which are labeled as 2 have $s\,(u) = 4$ and $c\,(u) = 6$.

In P_n the end vertex which are labeled as 1 have $s\,(u) = 10$ and $c\,(u) = 13$ and all other vertices with the same label have $s\,(u) = 14$ and $c\,(u) = 18$, when n is even.

$s\,(u) = 9$ and $c\,(u) = 12$ for the end vertices which are labeled as 1, and all other vertices with the same label have $s\,(u) = 13$ and $c\,(u) = 17$, when n is odd. The end vertices which are labeled as 4 have $s\,(u) = 3$ and $c\,(u) = 6$, when n is even.

When n is odd $s\,(u) = 5$ and $c\,(u) = 8$, for the end vertices which are labeled as 4. All other vertices which are labeled as 4 have $s\,(u) = 4$ and $c\,(u)$ =8, when n is even. $s\,(u) = 6$ and $c\,(u) = 10$, when n is odd (for illustration see Figure 1).

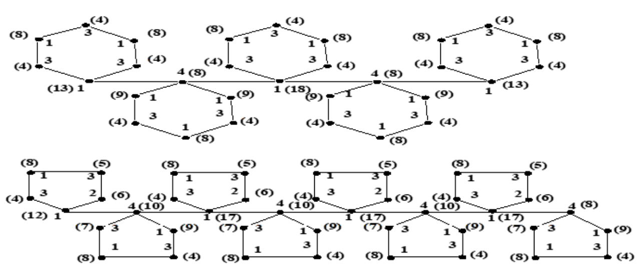

Fig 1. Proper d-lucky labeling with $c(u)'s$ of $P_5 o C_6$ and $P_8 o C_5$

It is observed that no two adjacent vertices have the same $c(u)'s$. Hence rooted product $P_n o\, C_n$ admits proper d-lucky labeling and $\eta_{pdl}\,(P_n o\, C_n) = 4, n \geq 3$.

Theorem 2:

Rooted product of path with complete graph, $P_m o K_n$ admits proper d-lucky labeling and

$\eta_{pdl}\,(P_m o\, K_n) = n, n \geq 3$.

Proof: Label the base vertices of the path P_m with 1,2,3 in cyclic order. The vertices of K_n are labelled as follows: if the base vertex is labeled as 1, then all other vertices receive the label as 2,3,4..., n. If the base vertex is labeled as 2, then all the other vertices are labeled as 3,4, 5…, n,1. If the base vertex is labeled as 3, the rest of the vertices are labeled as 4,5,6,…n,2,1, and so on (for illustration see Figure 2).

The neighborhood sums and degree sums for K_n are calculated as follows:

$$for\; i = 1,2,3,\ldots n,\; n \geq 3$$

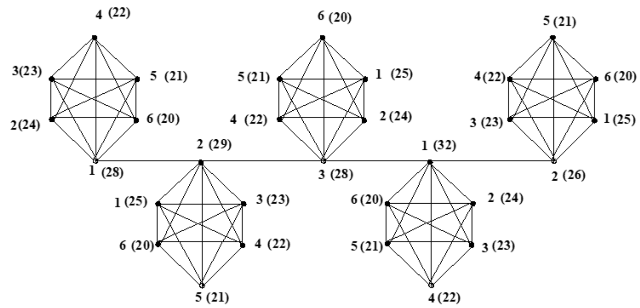

Fig 2. Proper d-lucky labeling with $c(u)'s$ of $P_5 o K_5$

$s(u_i) = \frac{n(n+1)-2i}{2}$ and

$c(u_i) = \frac{n(n+3)-2(i+1)}{2}$, for all the vertices except the base vertices of P_n.

Neighborhood sum and degree sums for P_m are calculated as follows:

If a vertex is labeled as 1 and is a corner vertex adjacent to vertex with labeled as 2,

then $s(u) = \frac{n(n+1)-2(i-2)}{2}$ and $c(u) = \frac{n(n+3)-2(i-2)}{2}$.

If the vertex is labeled as 1 and adjacent to the vertex with label as 3 which is corner vertex then, $s(u) = \frac{n(n+1)-2(i-3)}{2}$ and $c(u) = \frac{n(n+3)-2(i-3)}{2}$, all other vertices in P_n with label as 1 have $s(u) = \frac{n(n+1)-2(i-5)}{2}$ and $c(u) = \frac{n(n+3)-2(i-6)}{2}$.

If the vertex has label as 2 which is not a corner vertex then $s(u) = \frac{n(n+1)-2(i-4)}{2}$ and $c(u) = \frac{n(n+3)-2(i-1)}{2}$ and if it is corner vertex then, $s(u) = \frac{n(n+1)-2(i-1)}{2}$ *and* $c(u) = \frac{n(n+3)-2(i-1)}{2}$.

The corner vertex which has label 3 then $s(u) = \frac{n(n+1)-2(i-1)}{2}$ and $c(u) = \frac{n(n+3)-2(i-1)}{2}$, all other vertices with label as 3 have $s(u) = \frac{n(n+1)-2(i-3)}{2}$ and $c(u) = \frac{n(n+3)-2(i-4)}{2}$.

It is observed that no two adjacent vertices have the same $c(u)'s$.

Therefor the rooted product $P_m o K_n$ admits proper d-lucky labeling and $\eta_{pdl}(P_m o\, K_n) = n, n \geq 3$.

Theorem 3:

Rooted product of cycle with complete graph,$C_n o\, K_n$ admits proper d-lucky labeling and $\eta_{pdl}(C_n o\, K_n) = n, n \geq 3$.

Proof: The proof is given for two cases.

Case1: when n is even.

Label the vertices of the base of C_n with 1,2 alternately. All other vertices of K_n are labeled as follows: If one of the vertices of K_n in the base of C_n is labeled as 1, then all other vertices are labeled as 2,3,4,..n respectively. If it is labeled as 2 then all other vertices are labeled as 3,4, 5…, n,1, respectively and so on till all the vertices of the graph are labeled.

The neighborhood sum and degree sum for K_n except for the base vertices of C_n are calculated as given below

$$for\; i = 1, 2, 3, \ldots n,\; n \geq 3$$

$s(u_i) = \frac{n(n+1)-2i}{2}$ and $c(u_i) = \frac{n(n+3)-2(i+1)}{2}$

In the base cycle of C_n the vertices which are labeled as 1 have $s(u_i) = \frac{n(n+1)-2(i-4)}{2}$ and $c(u_i) = \frac{n(n+3)-2(i-5)}{2}$.

The vertices which are labeled as 2 have $s(u_i) = \frac{n(n+1)-2(i-2)}{2}$ and $c(u_i) = \frac{n(n+3)-2(i-3)}{2}$.

Case 2: when n is odd.

Label the base vertices of C_n with 1,2 alternately and the last vertex with 3. All vertices of K_n are labeled as in case 1, except the vertices which are labeled as 3 in the base of C_n, label all other vertices of K_n as 4,5,6,…n,2,1 and so on till all the vertices are labeled (for illustration see Figure 3).

The neighborhood sums and degree sums of C_n are calculated as given below:-

If the vertex is labeled as 1 which is adjacent to vertices with label as 2 and 3, then $s(u_i) = \frac{n(n+1)-2i-2(i-5)}{2}$ and $c(u_i) = \frac{n(n+3)-2(i-6)}{2}$. All other vertices with label as 1 have $s(u_i) = \frac{n(n+1)-2(i-4)}{2}$ and $c(u_i) = \frac{n(n+3)-2(i-5)}{2}$. If the vertex is labeled as 2 and is adjacent to vertices with label as 3 and 1, then $s(u_i) = \frac{n(n+1)-2(i-4)}{2}$ and $c(u_i) = \frac{n(n+3)-2(i-5)}{2}$. All other vertices

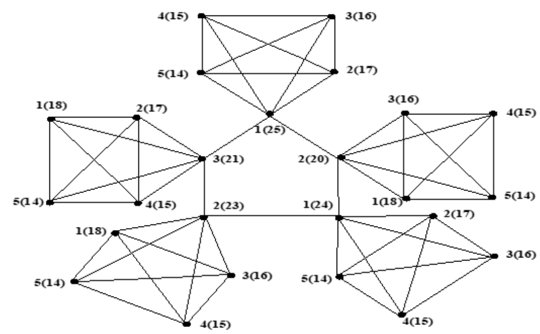

Fig 3. Proper d-lucky labeling with $c(u)'s$ of $C_5 o K_5$

with label as 2 have $s\left(u_i\right)=\frac{n(n+1)-2(i-2)}{2}$ and $c\left(u_i\right)=\frac{n(n+3)-2(i-3)}{2}$. If the vertex is labeled as 3 then $s\left(u_i\right)=\frac{n(n+1)-2(i-3)}{2}$ and $c\left(u_i\right)=\frac{n(n+3)-2(i-4)}{2}$.

It is seen that in both the cases no two adjacent vertices have the same $c(u_i)'s$. Hence the rooted product $C_n o\ K_n$ admits proper d-lucky labeling and $\mathrm{n}_{pdl}\left(C_n o\ K_n\right)=n, n \geq 3$.

Theorem 4:

Rooted product of complete graph with itself $K_n o\ K_n$, admits proper d-lucky labeling and $\eta_{pdl}\left(K_n o\ K_n\right)=n, n \geq 3$.

Proof: Label the base vertices of $K_n$with $1,2,3, \ldots n$ respectively. Label the outer vertices of K_n as given below: if one of its vertices are labeled as 1 in the base vertex then label rest of the vertices as $2,3,4, \ldots n$. If one of its vertices in the base have label as 2, then label rest of the vertices of K_n as $3,4,5, \ldots, n, 1$. Follow similar pattern of labeling for the rest of the vertices of outer $K_n's$ till all the vertices are labeled (for illustration see Figure 4).

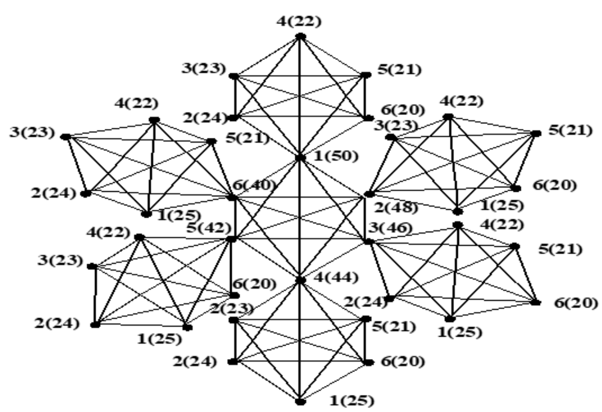

Fig 4. Proper d-lucky labeling with $c(u)'s$ of $K_6 o\ K_6$

The neighborhood sums and degree sums for outer vertices of $K_n$are calculated as given bellow:

$$for\ i=1,2,3,4 \ldots, n$$

$s\left(u_i\right)=\frac{n(n+1)-2 i}{2}$ and $c\left(u_i\right)=\frac{n(n+3)-2(i+1)}{2}$.

The neighborhood sum and degree sum for inner vertices of K_n are calculate as given bellow:

$$for\ i=1,2, \ldots n$$

$s\left(u_i\right)=n(n+1)-2 i$ and $c\left(u_i\right)=n(n+3)-2(i+1)$.

It is observed that no two adjacent vertices have the same $c(u)'s$.

Hence the rooted product $K_n o\ K_n$ admits proper d-lucky labeling and $\eta_{pdl}\left(K_n o\ K_n\right) = n, n \geq 3$.

(the operation $C_n o\ K_n$ means K_n copies added in C_n, which is read as rooted product of C_n with K_n, this notation is used for simplification)

3 Conclusion

Proper d-lucky numbers for rooted product graphs of $P_m o\ C_n$, $P_m o\ K_n$, $C_n o\ K_n$ and $K_n o\ K_n$ are obtained and found to be $\eta_{pdl}\left(P_m o\ C_n\right) = 4$, $n \geq 3$, $\eta_{pdl}\left(P_m o\ K_n\right) = n$, $n \geq 3$,$\eta_{pdl}\left(C_n o\ K_n\right) = n$, $n \geq 3$,$\eta_{pdl}\left(C_n o K_n\right) = n$, and $\eta_{pdl}\left(K_n o\ K_n\right) = n$, $n \geq 3$. The work could be continued for other rooted product graphs.

Acknowledgment

The author would like to acknowledge the reviewers for their valuable suggestions, which are incorporated in the article and thank them immensely for their contribution in making this paper better.

References

1) Miller M, Rajasingh I, Emilet DA, Jemilet DA. d-Lucky Labeling of Graphs. *Procedia Computer Science*. 2015;57:766–771. Available from: https://doi.org/10.1016/j.procs.2015.07.473.
2) Ram GR, Hemalatha S, Anitha K. D-lucky edge labeling of path families. *AIP Conference Proceedings*. 2020;2282:20032–20032. Available from: https://doi.org/10.1063/5.0028309.
3) Indira P, Selvam B, Thirusangu K. Proper Lucky Labeling and Lucky Edge Labeling for the Extended Duplicate Graph of Quadrilateral Snake. *International Journal of Mathematics Trends and Technology*. 2021;67(1):60–65. Available from: https://doi.org/10.14445/22315373/IJMTT-V67I1P509.
4) Dhenmar E, Chua F, Joseph H, Campena FA, Franco. Efficient Zero Ring Labeling of Graphs. *European Journal of Pure and Applied Mathematics*. 2020;13(3):674–696. Available from: https://doi.org/10.29020/nybg.ejpam.v13i3.3780.
5) Kumar TV, Meenakshi S. Lucky and Proper Lucky Labeling of Quadrilateral Snake Graphs. *IOP Conference Series: Materials Science and Engineering*. 2021. Available from: https://doi.org/10.1088/1757-899X/1085/1/012039.
6) Kujur C. Proper d-Lucky Labeling of Rooted Products and Corona Products of Certain Graphs. *Indian Journal Of Science And Technology*. 2022;15(39):2021–2026. Available from: https://doi.org/10.17485/IJST/v15i39.1154.