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Fuzzy Quotient -3 Cordial Labeling on Generalized Petersen Graph

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Abstract

Objectives: To analyze the existence of fuzzy quotient-3 cordial labeling on the generalized Petersen graph. **Methods:** The method involves mathematically defining how to label the vertex of a generalized Petersen graph and demonstrating that these formulations produce fuzzy quotient-3 cordial labeling. **Findings:** In this study, we proved that the generalized Petersen graph $GP(\eta,m)\,, 1 \leq m < \left\lfloor \frac{\eta}{2} \right\rfloor$ is fuzzy quotient-3 cordial, except for $\eta \equiv 0(\bmod 6)$. **Novelty:** Here, we give fuzzy quotient-3 cordial labeling to some families of graphs, namely Durer graph, Desargues graph, Dodecahedral graph, Mobius Kantor graph, Petersen graph, Cubical graph, Cubic symmetric graph, Nauru graph and the generalized Petersen graph $GP(\eta,m)$.

Keywords: Labeling; Fuzzy Quotient -3 Cordial; Petersen Graph; Generalized Petersen Graph; Mobius Kantor Graph

1 Introduction

Under certain conditions, graph labelling is the allocation of values to vertices, edges, or both. A Dynamic Survey of Graph Labeling [(1)] provides a comprehensive report on graph labelling. Many scholars worked on the generalised Petersen graph and shown the occurrence of many types of labelling, such as Minimum Coprime Labelings [(2)], Prime Cordial Labelings [(3)], Narayana Prime Cordial Labelings [(4)], Difference Cordial Labelings [(5)], Vertex-magic total labelings [(6)], and so on. We explored the existence of Fuzzy Quotient-3 Cordial Labeling and demonstrated that the generalised Petersen graph is Fuzzy Quotient-3 Cordial. we introduced the new labelling called fuzzy quotient -3 cordial labeling [(4)] as a function σ from the vertex set $V\ (G)$ to the interval $[0,1]$ defined by $\sigma(v) = \frac{r}{10}\ ,\ r \in Z_4 - \{0\}$. Such that the induced function $\mu\!: E\ (G) \to [0,1]$ defined by$\mu(uv) = \frac{1}{10}\left\lceil \frac{3\sigma(u)}{\sigma(v)} \right\rceil$ where $\sigma\ (u) \leq\ \sigma(v)$ assigns labels to the edges. If the number of vertices labelled with i and the number of vertices labelled with j differ by atleast one and If the number of edges labelled with i and the number of edges labelled with j differ by atleast one, the function is called fuzzy quotient-3 cordial labelling of G. where $i\ \neq j\ \in\ \left\{\frac{r}{10}\ ,\ r\ \in\ Z_4 - (0)\right\}$. The number of vertices having label i denotes $v_\sigma(i)$ and the number of edges having label i denotes $e_\mu(i)$. In our earlier research, we established that some duplication of graphs, unicyclic graphs, trees with a diameter less than six, ladder graphs, and generalized Jahangir graphs are fuzzy quotient-3 cordial. In this study we showed that the generalized Petersen graph is a fuzzy quotient -3 cordial graph.

2 Methodology

Definition: 2.1 Generalized Petersen graph $GP(\eta,k)$

The generalized Petersen graph $GP(\eta,k)$ has its vertex and edge set as
$V(GP(\eta,k)) = \{x_i : 0 \le i \le \eta-1\} \cup \{y_i : 0 \le i \le \eta-1\}$ and
$E(GP(\eta,k)) = \{x_i y_i : 0 \le i \le \eta-1\} \cup \{x_i x_{i+1} : 0 \le i \le \eta-1\} \cup$
$\{y_i y_{i+k} : 0 \le i \le \eta-1\}$ where the subscripts are taken modulo η, for a positive integer $n \ge 3$ and $1 \le k < \lfloor \frac{\eta}{2} \rfloor$.
i) Durer graph is a $GP(n,k)$ graph with n=6 and k=2
ii) Cubical graph is a $GP(n,k)$ graph with n=4 and k=1
iii) Desargues graph is a $GP(n,k)$ graph with n=10 and k=3
iv) Mobius kantor is a $GP(n,k)$ graph with η=8 and k=3
v) Nauru is a $GP(n,k)$ graph with η=12 and k=5
vi) Dodecahedral is a $GP(n,k)$ with η=10 and k=2
vii) Cubic Symmetrical Graph is a $GP(n,k)$ graph with η=24 and k=5
viii) Petersen graph is a $GP(n,k)$graph with η=5 and k=2

3 Results and Discussion

Theorem: 3.1 The generalized Petersen graph $GP(\eta,m), 1 \le m < \lfloor \frac{\eta}{2} \rfloor$ is fuzzy quotient-3 cordial, except for $n \equiv 3(mod\ 6)$.

Proof:

Let G be a generalized Petersen graph $GP(n,m)$.
$V(G) = \{x_k : 0 \le k \le \eta-1\} \cup \{y_k : 0 \le k \le \eta-1\}$ and $E(G) = \{x_k y_k : 0 \le k \le \eta-1\} \cup \{x_k x_{k+1} : 0 \le k \le \eta-1\}$ $\cup \{y_k y_{k+m} : 0 \le k \le \eta-1\}$, where the subscription taken modulo η $|V(G)| = 2\eta$, $|E(G)| = 3\eta$.
We define $\sigma : V(G) \to [0,1]$ by $\sigma(v) = \frac{r}{10}$, $r \in Z_4 - (0)$
For $0 \le k \le \eta-1$, the labeling of y_k's and x_k's are as follows.
Case 1: $n \equiv 0(mod\ 6)$
Subcase 1.1: $m \equiv 0(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,3(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.2 & if\ \kappa \equiv 1,\ 4,\ 5(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.3 & if\ \kappa \equiv 2(mod\ 6) & for\ 0 \le k \le \eta-1 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2,\ 4(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.2 & if\ \kappa \equiv 0(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.3 & if\ \kappa \equiv 1,\ 3,\ 5(mod\ 6) & for\ 0 \le k \le \eta-1 \end{cases}$$

Subcase 1.2: $m \equiv 1(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,1(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.2 & if\ \kappa \equiv 3,\ 4(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.3 & if\ \kappa \equiv 2,\ 5(mod\ 6) & for\ 0 \le k \le \eta-1 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 5(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.2 & if\ \kappa \equiv 2,\ 3(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.3 & if\ \kappa \equiv 1,\ 4(mod\ 6) & for\ 0 \le k \le \eta-1 \end{cases}$$

Subcase 1.3: $m \equiv 2(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 1,\ 5(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.2 & if\ \kappa \equiv 2,\ 4(mod\ 6) & for\ 0 \le k \le \eta-1 \\ 0.3 & if\ \kappa \equiv 0,\ 3(mod\ 6) & for\ 0 \le k \le \eta-1 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 0,5(\bmod 6) & \text{for } 0 \le k \le \eta - 1 \\ 0.2 & \text{if } \kappa \equiv 2,3(\bmod 6) & \text{for } 0 \le k \le \eta - 1 \\ 0.3 & \text{if } \kappa \equiv 1,4(\bmod 6) & \text{for } 0 \le k \le \eta - 1 \end{cases}$$

Subcase 1.4: $m \equiv 3(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.2 & if\ \kappa \equiv 0,\ 1(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.3 & if\ \kappa \equiv 3,\ 4,\ 5(mod\ 6) & for\ 0 \le k \le \eta - 1 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2,\ 4,\ 5(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.2 & if\ \kappa \equiv 0,\ 1(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.3 & if\ \kappa \equiv 3(mod\ 6) & for\ 0 \le k \le \eta - 1 \end{cases}$$

Subcase 1.5: $m \equiv 4(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.2 & if\ \kappa \equiv 0,\ 1(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.3 & if\ \kappa \equiv 3,\ 4,\ 5(mod\ 6) & for\ 0 \le k \le \eta - 1 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 3,\ 4,\ 5(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.2 & if\ \kappa \equiv 0,\ 1(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.3 & if\ \kappa \equiv 2(mod\ 6) & for\ 0 \le k \le \eta - 1 \end{cases}$$

Subcase 1.6: $m \equiv 5(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 5(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.2 & if\ \kappa \equiv 2,\ 3(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.3 & if\ \kappa \equiv 1,\ 4(mod\ 6) & for\ 0 \le k \le \eta - 1 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 1(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.2 & if\ \kappa \equiv 3,\ 4(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.3 & if\ \kappa \equiv 2,\ 5(mod\ 6) & for\ 0 \le k \le \eta - 1 \end{cases}$$

Case 2: $\eta \equiv 1(mod\ 6)$ Subcase 2.1: $m \equiv 0(mod\ 6)$ Subcase 2.1.1: $If\ \eta = 13$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 1,\ 5(mod\ 6) & for\ 0 \le k \le 5 \\ 0.2 & if\ \kappa \equiv 0,\ 2,\ 4(mod\ 6) & for\ 0 \le k \le 5 \\ 0.3 & if\ \kappa \equiv 3(mod\ 6) & for\ 0 \le k \le 5 \end{cases}$$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 1,\ 4,\ 5(mod\ 6) & for\ 6 \le k \le 11 \\ 0.2 & if\ \kappa \equiv 0,\ 2,\ 3(mod\ 6) & for\ 6 \le k \le 11 \end{cases}$$

$$\sigma(y_{12}) = 0.2$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2,\ 3(mod\ 6) & for\ 0 \le k \le 5 \\ 0.2 & if\ \kappa \equiv 0(mod\ 6) & for\ 0 \le k \le 5 \\ 0.3 & if\ \kappa \equiv 1,\ 4,\ 5(mod\ 6) & for\ 0 \le k \le 5 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 3(mod\ 6) & for\ 6 \le k \le 11 \\ 0.2 & if\ \kappa \equiv 1,\ 2,\ 4,\ 5(mod\ 6) & for\ 6 \le k \le 11 \end{cases}$$

$$\sigma(x_{12}) = 0.2$$

Subcase 2.1.2: $If\ \eta > 13$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 1(mod\ 6) & for\ 0 \le k \le 5 \\ 0.2 & if\ \kappa \equiv 3,\ 4(mod\ 6) & for\ 0 \le k \le 5 \\ 0.3 & if\ \kappa \equiv 2,\ 5(mod\ 6) & for\ 0 \le k \le 5 \end{cases}$$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 3(mod\ 6) & for\ 6 \le k \le \eta - 8 \\ 0.2 & if\ \kappa \equiv 1,\ 4,\ 5(mod\ 6) & for\ 6 \le k \le \eta - 8 \\ 0.3 & if\ \kappa \equiv 2(mod\ 6) & for\ 6 \le k \le \eta - 8 \end{cases}$$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0(mod\ 6) & for\ n - 7 \le k \le \eta - 2 \\ 0.2 & if\ \kappa \equiv 1,\ 4,\ 5(mod\ 6) & for\ n - 7 \le k \le \eta - 2 \\ 0.3 & if\ \kappa \equiv 2,\ 3(mod\ 6) & for\ n - 7 \le k \le \eta - 2 \end{cases}$$

$$\sigma(y_{\eta-1}) = 0.2$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2,\ 5(mod\ 6) & for\ 0 \le k \le 5 \\ 0.2 & if\ \kappa \equiv 3,\ 4(mod\ 6) & for\ 0 \le k \le 5 \\ 0.3 & if\ \kappa \equiv 0,\ 1(mod\ 6) & for\ 0 \le k \le 5 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2,\ 4(mod\ 6) & for\ 6 \le k \le \eta - 8 \\ 0.2 & if\ \kappa \equiv 0(mod\ 6) & for\ 6 \le k \le \eta - 8 \\ 0.3 & if\ \kappa \equiv 1,\ 3,\ 5(mod\ 6) & for\ 6 \le k \le \eta - 8 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 3(mod\ 6) & for\ \eta - 7 \le k \le \eta - 2 \\ 0.2 & if\ \kappa \equiv 1(mod\ 6) & for\ \eta - 7 \le k \le \eta - 2 \\ 0.3 & if\ \kappa \equiv 2,\ 4,\ 5(mod\ 6) & for\ \eta - 7 \le k \le \eta - 2 \end{cases}$$

$$\sigma(x_{\eta-1}) = 0.1$$

Subcase 2.2: $m \equiv 1(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 1,\ 2(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.2 & if\ \kappa \equiv 4,\ 5(mod\ 6) & for\ 0 \le k \le \eta - 1 \\ 0.3 & if\ \kappa \equiv 0,\ 3(mod\ 6) & for\ 0 \le k \le \eta - 1 \end{cases}$$

$\sigma(x_0) = 0.2$, $\sigma(x_1) = 0.1$, $\sigma(x_2) = 0.3$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2,\ 3(mod\ 6) & for\ 3 \le k \le \eta - 1 \\ 0.2 & if\ \kappa \equiv 0,\ 5(mod\ 6) & for\ 3 \le k \le \eta - 1 \\ 0.3 & if\ \kappa \equiv 1,\ 4(mod\ 6) & for\ 3 \le k \le \eta - 1 \end{cases}$$

Subcase 2.3: $m \equiv 2(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2(mod\ 6) & for\ 0 \le k \le \eta - 2 \\ 0.2 & if\ \kappa \equiv 0,\ 1(mod\ 6) & for\ 0 \le k \le \eta - 2 \\ 0.3 & if\ \kappa \equiv 3,\ 4,\ 5(mod\ 6) & for\ 0 \le k \le \eta - 2 \end{cases}$$

$$\sigma\left(y_{\eta-1}\right)=0.1$$

$$\sigma\left(x_{k}\right)=\begin{cases}0.1 & if\ \kappa\equiv 3,\ 4,\ 5(mod\ 6) & for\ 0\leq k\leq \eta-1\\ 0.2 & if\ \kappa\equiv 0,\ 1(mod\ 6) & for\ 0\leq k\leq \eta-1\\ 0.3 & if\ \kappa\equiv 2(mod\ 6) & for\ 0\leq k\leq \eta-1\end{cases}$$

Subcase 2.4: $m\equiv 3(mod\ 6)$

$$\sigma\left(y_{k}\right)=\begin{cases}0.1 & if\ \kappa\equiv 3(mod\ 6) & for\ 0\leq k\leq \eta-2\\ 0.2 & if\ \kappa\equiv 0,\ 1(mod\ 6) & for\ 0\leq k\leq \eta-2\\ 0.3 & if\ \kappa\equiv 2,\ 4,\ 5(mod\ 6) & for\ 0\leq k\leq \eta-2\end{cases}$$

$$\sigma\left(y_{\eta-1}\right)=0.1$$

$$\sigma\left(x_{k}\right)=\begin{cases}0.1 & if\ \kappa\equiv 2,\ 4,\ 5(mod\ 6) & for\ 0\leq k\leq \eta-1\\ 0.2 & if\ \kappa\equiv 0,\ 1(mod\ 6) & for\ 0\leq k\leq \eta-1\\ 0.3 & if\ \kappa\equiv 3(mod\ 6) & for\ 0\leq k\leq \eta-1\end{cases}$$

Subcase 2.5: $m\equiv 4(mod\ 6)$

$$\sigma\left(x_{k}\right)=\begin{cases}0.1 & if\ \kappa\equiv 3,\ 4,\ 5(mod\ 6) & for\ 0\leq k\leq \eta-1\\ 0.2 & if\ \kappa\equiv 0,\ 1(mod\ 6) & for\ 0\leq k\leq \eta-1\\ 0.3 & if\ \kappa\equiv 2(mod\ 6) & for\ 0\leq k\leq \eta-1\end{cases}$$

$$\sigma\left(y_{\eta-1}\right)=0.3$$

$$\sigma\left(x_{k}\right)=\begin{cases}0.1 & if\ \kappa\equiv 3,\ 4,\ 5(mod\ 6) & for\ 0\leq k\leq \eta-1\\ 0.2 & if\ \kappa\equiv 0,\ 1(mod\ 6) & for\ 0\leq k\leq \eta-1\\ 0.3 & if\ \kappa\equiv 2(mod\ 6) & for\ 0\leq k\leq \eta-1\end{cases}$$

Subcase 2.6: $m\equiv 5(mod\ 6)$

$$\sigma\left(y_{k}\right)=\begin{cases}0.1 & \text{if } \kappa\equiv 4,5(\bmod 6) & \text{for } 0\leq k\leq \eta-2\\ 0.2 & \text{if } \kappa\equiv 1,2(\bmod 6) & \text{for } 0\leq k\leq \eta-2\\ 0.3 & \text{if } \kappa\equiv 0,3(\bmod 6) & \text{for } 0\leq k\leq \eta-2\end{cases}$$

$$\sigma\left(y_{\eta-1}\right)=0.2$$

$$\sigma\left(x_{k}\right)=\begin{cases}0.1 & \text{if } \kappa\equiv 0,5(\bmod 6) & \text{for } 0\leq k\leq \eta-2\\ 0.2 & \text{if } \kappa\equiv 2,3(\bmod 6) & \text{for } 0\leq k\leq \eta-2\\ 0.3 & \text{if } \kappa\equiv 1,4(\bmod 6) & \text{for } 0\leq k\leq \eta-2\end{cases}$$

$$\sigma\left(x_{\eta-1}\right)=0.2$$

Case 3: $n\equiv 2(mod\ 6)$

Subcase 3.1: $m\equiv 0(mod\ 6)$

Subcase 3.1.1: $If\ \eta=14$

$$\sigma\left(y_{k}\right)=\begin{cases}0.1 & if\ \kappa\equiv 1(mod\ 6) & for\ 0\leq k\leq 5\\ 0.2 & if\ \kappa\equiv 3,\ 4(mod\ 6) & for\ 0\leq k\leq 5\\ 0.3 & if\ \kappa\equiv 0,\ 2,\ 5(mod\ 6) & for\ 0\leq k\leq 5\end{cases}$$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0(mod\ 6) & for\ 6 \le k \le 11 \\ 0.2 & if\ \kappa \equiv 1,\ 4,\ 5(mod\ 6) & for\ 6 \le k \le 11 \\ 0.3 & if\ \kappa \equiv 2,\ 3(mod\ 6) & for\ 6 \le k \le 11 \end{cases}$$

$$\sigma(y_{12}) = 0.1,\ \sigma(y_{13}) = 0.2$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 2,\ 5(mod\ 6) & for\ 0 \le k \le 5 \\ 0.2 & if\ \kappa \equiv 3,\ 4(mod\ 6) & for\ 0 \le k \le 5 \\ 0.3 & if\ \kappa \equiv 1(mod\ 6) & for\ 0 \le k \le 5 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 3,\ 5(mod\ 6) & for\ 6 \le k \le 11 \\ 0.2 & if\ \kappa \equiv 1(mod\ 6) & for\ 6 \le k \le 11 \\ 0.3 & if\ \kappa \equiv 2,\ 4(mod\ 6) & for\ 6 \le k \le 11 \end{cases}$$

$$\sigma(x_{12}) = 0.3,\ \sigma(x_{13}) = 0.2$$

Subcase 3.1.2: $If\ \eta > 14$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 1(mod\ 6) & for\ 0 \le k \le 5 \\ 0.2 & if\ \kappa \equiv 3,\ 4(mod\ 6) & for\ 0 \le k \le 5 \\ 0.3 & if\ \kappa \equiv 0,\ 2,\ 5(mod\ 6) & for\ 0 \le k \le 5 \end{cases}$$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 3(mod\ 6) & for\ 6 \le k \le \eta - 9 \\ 0.2 & if\ \kappa \equiv 1,\ 4,\ 5(mod\ 6) & for\ 6 \le k \le \eta - 9 \\ 0.3 & if\ \kappa \equiv 2(mod\ 6) & for\ 6 \le k \le \eta - 9 \end{cases}$$

$$\sigma(y_k) = \begin{cases} 0.1 & if\ \kappa \equiv 3(mod\ 6) & for\ \eta - 8 \le k \le \eta - 3 \\ 0.2 & if\ \kappa \equiv 1,\ 4,\ 5(mod\ 6) & for\ \eta - 8 \le k \le \eta - 3 \\ 0.3 & if\ \kappa \equiv 0,\ 2(mod\ 6) & for\ \eta - 8 \le k \le \eta - 3 \end{cases}$$

$$\sigma(y_{\eta-2}) = 0.2,\ \sigma(y_{\eta-1}) = 0.3$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 2,\ 5(mod\ 6) & for\ 0 \le k \le 5 \\ 0.2 & if\ \kappa \equiv 3,\ 4(mod\ 6) & for\ 0 \le k \le 5 \\ 0.3 & if\ \kappa \equiv 1(mod\ 6) & for\ 0 \le k \le 5 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 2,\ 4(mod\ 6) & for\ 6 \le k \le \eta - 9 \\ 0.2 & if\ \kappa \equiv 0(mod\ 6) & for\ 6 \le k \le \eta - 9 \\ 0.3 & if\ \kappa \equiv 1,\ 3,\ 5(mod\ 6) & for\ 6 \le k \le \eta - 9 \end{cases}$$

$$\sigma(x_k) = \begin{cases} 0.1 & if\ \kappa \equiv 0,\ 3,\ 5(mod\ 6) & for\ \eta - 8 \le k \le \eta - 3 \\ 0.2 & if\ \kappa \equiv 1,\ 4(mod\ 6) & for\ \eta - 8 \le k \le \eta - 3 \\ 0.3 & if\ \kappa \equiv 2(mod\ 6) & for\ \eta - 8 \le k \le \eta - 3 \end{cases}$$

$$\sigma(x_{\eta-2}) = 0.3,\ \sigma(x_{\eta-1}) = 0.1$$

Subcase 3.2: $m \equiv 1(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 1,2(\bmod 6) & \text{for } 0 \le k \le \eta - 3 \\ 0.2 & \text{if } \kappa \equiv 4,5(\bmod 6) & \text{for } 0 \le k \le \eta - 3 \\ 0.3 & \text{if } \kappa \equiv 0,3(\bmod 6) & \text{for } 0 \le k \le \eta - 3 \end{cases}$$

$$\sigma\left(y_{\eta-2}\right)=0.2, \sigma\left(y_{\eta-1}\right)=0.2$$

$$\sigma\left(x_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 0,1(\bmod 6) \quad \text{for } 0 \leq k \leq n-3 \\ 0.2 & \text{if } \kappa \equiv 3,4(\bmod 6) \quad \text{for } 0 \leq k \leq n-3 \\ 0.3 & \text{if } \kappa \equiv 2,5(\bmod 6) \quad \text{for } 0 \leq k \leq n-3\end{cases}$$

$$\sigma\left(x_{\eta-2}\right)=0.1, \sigma\left(x_{\eta-1}\right)=0.3$$

Subcase 3.3: $m \equiv 2(mod\ 6)$

$$\sigma\left(y_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 5(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.2 & \text{if } \kappa \equiv 0,2(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.3 & \text{if } \kappa \equiv 1,3,4(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3\end{cases}$$

$$\sigma\left(y_{\eta-2}\right)=0.1, \sigma\left(y_{\eta-1}\right)=0.2$$

$$\sigma\left(x_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 1,3,5(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.2 & \text{if } \kappa \equiv 0,2(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.3 & \text{if } \kappa \equiv 4(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3\end{cases}$$

$$\sigma\left(x_{\eta-2}\right)=0.3, \sigma\left(x_{\eta-1}\right)=0.2$$

Subcase 3.4: $m \equiv 3(mod\ 6)$

$$\sigma\left(y_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 0,1(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.2 & \text{if } \kappa \equiv 2(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.3 & \text{if } \kappa \equiv 3,4,5(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3\end{cases}$$

$$\sigma\left(y_{\eta-2}\right)=0.1, \sigma\left(y_{\eta-1}\right)=0.2$$

$$\sigma\left(x_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 2,4,5(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.3 & \text{if } \kappa \equiv 3(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3\end{cases}$$

$$\sigma\left(x_{\eta-2}\right)=0.3, \sigma\left(x_{\eta-1}\right)=0.2$$

Subcase 3.5: $m \equiv 4(mod\ 6)$

$$\sigma\left(y_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 2,3,4(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3 \\ 0.3 & \text{if } \kappa \equiv 5(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-3\end{cases}$$

$$\sigma\left(y_{\eta-2}\right)=0.3, \sigma\left(y_{\eta-1}\right)=0.2$$

$$\sigma(x_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 5(\bmod 6) & \text{for } 0 \leq k \leq \eta - 3 \\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \leq k \leq \eta - 3 \\ 0.3 & \text{if } \kappa \equiv 2,3,4(\bmod 6) & \text{for } 0 \leq k \leq \eta - 3 \end{cases}$$

$$\sigma(x_{\eta-2}) = 0.1, \sigma(x_{\eta-1}) = 0.2$$

Subcase 3.6: $m \equiv 5(mod\ 6)\sigma(y_0) = 0.3$, $\sigma(y_1) = 0.1$

$$\sigma(y_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 4,5(\bmod 6) & \text{for } 2 \leq k \leq \eta - 3 \\ 0.2 & \text{if } \kappa \equiv 1,2(\bmod 6) & \text{for } 2 \leq k \leq \eta - 3 \\ 0.3 & \text{if } \kappa \equiv 0,3(\bmod 6) & \text{for } 2 \leq k \leq \eta - 3 \end{cases}$$

$$\sigma(y_{\eta-2}) = 0.2, \sigma(y_{\eta-1}) = 0.2$$

$$\sigma(x_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 0,5(\bmod 6) & \text{for } 0 \leq k \leq \eta - 3 \\ 0.2 & \text{if } \kappa \equiv 2,3(\bmod 6) & \text{for } 0 \leq k \leq \eta - 3 \\ 0.3 & \text{if } \kappa \equiv 1,4(\bmod 6) & \text{for } 0 \leq k \leq \eta - 3 \end{cases}$$

$$\sigma(x_{\eta-2}) = 0.2, \sigma(x_{\eta-1}) = 0.3$$

Case 4: $n \equiv 4(mod\ 6)$
Subcase 4.1: i$m \equiv 0(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 0,3(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \\ 0.2 & \text{if } \kappa \equiv 1,4,5(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \\ 0.3 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \end{cases}$$

$$\sigma(y_{\eta-4}) = 0.3,\ \sigma(y_{\eta-3}) = 0.1,\ \sigma(y_{\eta-2}) = 0.1,\ \sigma(y_{\eta-1}) = 0.2$$

$$\sigma(x_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 2,4(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \\ 0.2 & \text{if } \kappa \equiv 0(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \\ 0.3 & \text{if } \kappa \equiv 1,3,5(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \end{cases}$$

$$\sigma(x_{\eta-4}) = 0.3,\ \sigma(x_{\eta-3}) = 0.3,\ \sigma(x_{\eta-2}) = 0.1,\ \sigma(x_{\eta-1}) = 0.2$$

Subcase 4.2: $m \equiv 1(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 1,2(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \\ 0.2 & \text{if } \kappa \equiv 4,5(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \\ 0.3 & \text{if } \kappa \equiv 0,3(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \end{cases}$$

$$\sigma(y_{\eta-4}) = 0.3,\ \sigma(y_{\eta-3}) = 0.1,\ \sigma(y_{\eta-2}) = 0.3,\ \sigma(y_{\eta-1}) = 0.2$$

$$\sigma(x_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \\ 0.2 & \text{if } \kappa \equiv 3,4(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \\ 0.3 & \text{if } \kappa \equiv 2,5(\bmod 6) & \text{for } 0 \leq k \leq \eta - 5 \end{cases}$$

$$\sigma\left(x_{\eta-4}\right)=0.1,\ \sigma\left(x_{\eta-3}\right)=0.1,\ \sigma\left(x_{\eta-2}\right)=0.3,\ \sigma\left(x_{\eta-1}\right)=0.2$$

Subcase 4.3: $m \equiv 2(mod\ 6)$

$$\sigma\left(y_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.3 & \text{if } \kappa \equiv 3,4,5(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\end{cases}$$

$$\sigma\left(y_{\eta-4}\right)=0.1, \sigma\left(y_{\eta-3}\right)=0.1, \sigma\left(y_{\eta-2}\right)=0.3, \sigma\left(y_{\eta-1}\right)=0.1$$

$$\sigma\left(x_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 3,4,5(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.3 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\end{cases}$$

$$\sigma\left(y_{\eta-4}\right)=0.1, \sigma\left(y_{\eta-3}\right)=0.1, \sigma\left(y_{\eta-2}\right)=0.1, \sigma\left(y_{\eta-1}\right)=0.3$$

Subcase 4.4: $m \equiv 3(mod\ 6)$

$$\sigma\left(y_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 0,5(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.2 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.3 & \text{if } \kappa \equiv 1,3,4(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\end{cases}$$

$$\sigma\left(y_{\eta-4}\right)=0.1, \sigma\left(y_{\eta-3}\right)=0.1, \sigma\left(y_{\eta-2}\right)=0.1, \sigma\left(y_{\eta-1}\right)=0.3$$

$$\sigma\left(x_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 0,4(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.2 & \text{if } \kappa \equiv 1,2,3(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.3 & \text{if } \kappa \equiv 5(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\end{cases}$$

$$\sigma\left(x_{\eta-4}\right)=0.2, \sigma\left(x_{\eta-3}\right)=0.2, \sigma\left(x_{\eta-2}\right)=0.3, \sigma\left(x_{\eta-1}\right)=0.3$$

Subcase 4.5: $m \equiv 4(mod\ 6)$

$$\sigma\left(y_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.3 & \text{if } \kappa \equiv 3,4,5(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\end{cases}$$

$$\sigma\left(y_{\eta-4}\right)=0.2, \sigma\left(y_{\eta-3}\right)=0.1, \sigma\left(y_{\eta-2}\right)=0.3, \sigma\left(y_{\eta-1}\right)=0.1$$

$$\sigma\left(x_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 3,4,5(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.3 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\end{cases}$$

$$\sigma\left(x_{\eta-4}\right)=0.1, \sigma\left(x_{\eta-3}\right)=0.2, \sigma\left(x_{\eta-2}\right)=0.3, \sigma\left(x_{\eta-1}\right)=0.3$$

Subcase 4.6: $m \equiv 5(mod\ 6)$

$$\sigma\left(y_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 1,2(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.2 & \text{if } \kappa \equiv 4,5(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.3 & \text{if } \kappa \equiv 0,3(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\end{cases}$$

$$\sigma\left(y_{\eta-4}\right)=0.3, \sigma\left(y_{\eta-3}\right)=0.1, \sigma\left(y_{\eta-2}\right)=0.1, \sigma\left(y_{\eta-1}\right)=0.2$$

$$\sigma\left(x_k\right)=\begin{cases}0.1 & \text{if } \kappa \equiv 2,3(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.2 & \text{if } \kappa \equiv 0,5(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\\ 0.3 & \text{if } \kappa \equiv 1,4(\bmod 6) & \text{for } 0 \leq k \leq \eta-5\end{cases}$$

$$\sigma\left(x_{\eta-4}\right)=0.2, \sigma\left(x_{\eta-3}\right)=0.1, \sigma\left(x_{\eta-2}\right)=0.3, \sigma\left(x_{\eta-1}\right)=0.3$$

Case 5: $n \equiv 5(mod\ 6)$

Subcase 5.1: $m \equiv 0(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 0,3(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.2 & \text{if } \kappa \equiv 1,4,5(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.3 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \end{cases}$$

$$\sigma(y_{\eta-5}) = 0.2, \sigma(y_{\eta-4}) = 0.1, \sigma(y_{\eta-3}) = 0.1, \sigma(y_{\eta-2}) = 0.1, \sigma(y_{\eta-1}) = 0.2$$

$$\sigma(x_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 2,4(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.2 & \text{if } \kappa \equiv 0(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.3 & \text{if } \kappa \equiv 1,3,5(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \end{cases}$$

$$\sigma(x_{\eta-5}) = 0.3, \sigma(x_{\eta-4}) = 0.3, \sigma(x_{\eta-3}) = 0.3, \sigma(x_{\eta-2}) = 0.3, \sigma(x_{\eta-1}) = 0.2$$

Subcase 5.2: $m \equiv 1(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.2 & \text{if } \kappa \equiv 3,4(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.3 & \text{if } \kappa \equiv 2,5(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \end{cases}$$

$$\sigma(y_{\eta-5}) = 0.1, \sigma(y_{\eta-4}) = 0.3, \sigma(y_{\eta-3}) = 0.3, \sigma(y_{\eta-2}) = 0.1, \sigma(y_{\eta-1}) = 0.1$$

$$\sigma(x_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 0,5(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.2 & \text{if } \kappa \equiv 2,3(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.3 & \text{if } \kappa \equiv 1,4(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \end{cases}$$

$$\sigma(x_{\eta-5}) = 0.2, \sigma(x_{\eta-4}) = 0.2, \sigma(x_{\eta-3}) = 0.2, \sigma(x_{\eta-2}) = 0.3, \sigma(x_{\eta-1}) = 0.3$$

Subcase 5.3: $m \equiv 2(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.3 & \text{if } \kappa \equiv 3,4,5(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \end{cases}$$

$$\sigma(y_{\eta-5}) = 0.1, \sigma(y_{\eta-4}) = 0.1, \sigma(y_{\eta-3}) = 0.3, \sigma(y_{\eta-2}) = 0.1, \sigma(y_{\eta-1}) = 0.1$$

$$\sigma(x_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 3,4,5(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.3 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \end{cases}$$

$$\sigma(x_{\eta-5}) = 0.2, \sigma(x_{\eta-4}) = 0.2, \sigma(x_{\eta-3}) = 0.3, \sigma(x_{\eta-2}) = 0.3, \sigma(x_{\eta-1}) = 0.2$$

Subcase 5.4: $m \equiv 3(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 2,5(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.3 & \text{if } \kappa \equiv 3,4(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \end{cases}$$

$$\sigma(y_{\eta-5}) = 0.1, \sigma(y_{\eta-4}) = 0.3, \sigma(y_{\eta-3}) = 0.3, \sigma(y_{\eta-2}) = 0.1, \sigma(y_{\eta-1}) = 0.1$$

$$\sigma(x_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 3,4(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.3 & \text{if } \kappa \equiv 2,5(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \end{cases}$$

$$\sigma(x_{\eta-5}) = 0.2.\sigma(x_{\eta-4}) = 0.2, \sigma(x_{\eta-3}) = 0.2, \sigma(x_{\eta-2}) = 0.3, \sigma(x_{\eta-1}) = 0.1$$

Subcase 5.5: $m \equiv 4(mod\ 6)$

$$\sigma(y_k) = \begin{cases} 0.1 & \text{if } \kappa \equiv 2(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \\ 0.3 & \text{if } \kappa \equiv 3,4,5(\bmod 6) & \text{for } 0 \le k \le \eta - 6 \end{cases}$$

$$\sigma\left(y_{\eta-5}\right)=0.3, \sigma\left(y_{\eta-4}\right)=0.1, \sigma\left(y_{\eta-3}\right)=0.3, \sigma\left(y_{\eta-2}\right)=0.1, \sigma\left(y_{\eta-1}\right)=0.1$$
$$\sigma\left(x_k\right)=\begin{cases} 0.1 & \text{if } \kappa \equiv 3,4,5(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-6 \\ 0.2 & \text{if } \kappa \equiv 0,1(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-6 \\ 0.3 & \text{if } \kappa \equiv 2(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-6 \end{cases}$$
$$\sigma\left(x_{\eta-5}\right)=0.2, \sigma\left(x_{\eta-4}\right)=0.2, \sigma\left(x_{\eta-3}\right)=0.3, \sigma\left(x_{\eta-2}\right)=0.3, \sigma\left(x_{\eta-1}\right)=0.2$$

Subcase 5.6: $m \equiv 5(mod\ 6)$

$$\sigma\left(y_k\right)=\begin{cases} 0.1 & \text{if } \kappa \equiv 1,2(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-6 \\ 0.2 & \text{if } \kappa \equiv 4,5(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-6 \\ 0.3 & \text{if } \kappa \equiv 0,3(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-6 \end{cases}$$
$$\sigma\left(y_{\eta-5}\right)=0.3 . \sigma\left(y_{\eta-4}\right)=0.3, \sigma\left(y_{\eta-3}\right)=0.1, \sigma\left(y_{\eta-2}\right)=0.2, \sigma\left(y_{\eta-1}\right)=0.2$$
$$\sigma\left(x_k\right)=\begin{cases} 0.1 & \text{if } \kappa \equiv 2,3(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-6 \\ 0.2 & \text{if } \kappa \equiv 0,5(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-6 \\ 0.3 & \text{if } \kappa \equiv 1,4(\bmod 6) \quad \text{for } 0 \leq k \leq \eta-6 \end{cases}$$
$$\sigma\left(x_{\eta-5}\right)=0.2, \sigma\left(x_{\eta-4}\right)=0.1, \sigma\left(x_{\eta-3}\right)=0.1, \sigma\left(x_{\eta-2}\right)=0.3, \sigma\left(x_{\eta-1}\right)=0.1$$

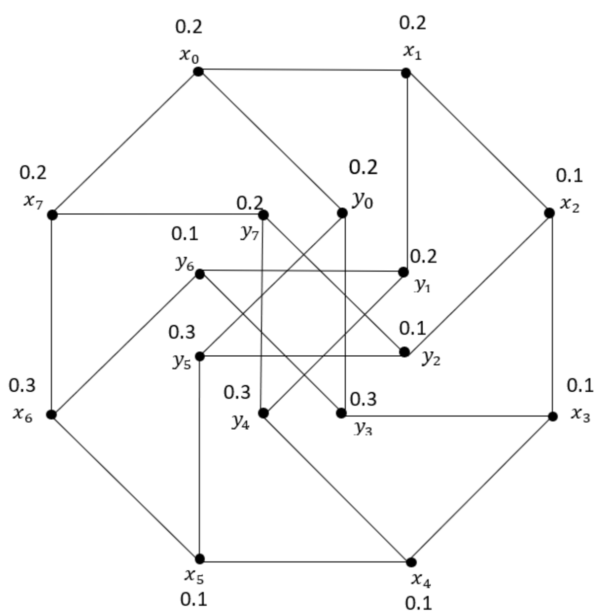
The number of vertices labeled with 0.1, 0.2 and 0.3 are tabulated below.

Table 1. $v_\sigma(i)$ for the generalized Petersen graph $GP(\eta, m), i \in \left\{\frac{r}{10}, r \in Z_4-\{0\}\right\}$

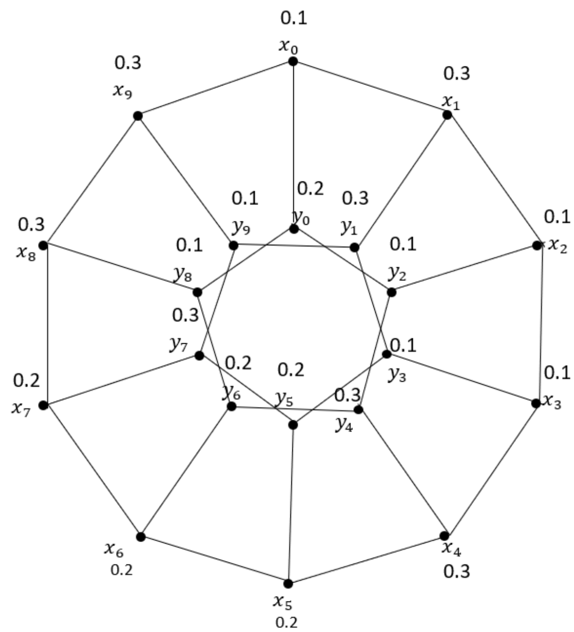
Nature of n	$v_\sigma(0.1)$	$v_\sigma(0.2)$	$v_\sigma(0.3)$
$\eta \equiv 0(mod\ 6)$	$2\eta/3$	$2\eta/3$	$2\eta/3$
$\eta \equiv 1(mod\ 6)$	$(2\eta+1)/3$	$(2\eta+1)/3$	$(2\eta+1)/3$
$\eta \equiv 2(mod\ 6)$	$[(2\eta-1)/3]$	$(2\eta-1)/3+1$	$(2\eta-1)/3$
$\eta \equiv 4(mod\ 6)$	$(2\eta+1)/3$	$(2\eta+1)/3-1$	$(2\eta+1)/3$
$\eta \equiv 5(mod\ 6)$	$(2\eta-1)/3+1$	$(2\eta-1)/3$	$(2\eta-1)/3$

From the above Table 1 it is clear that the number of vertices labeled with i and the number of vertices labeled with j differ by at most 1, where $i \neq j \in \left\{\frac{r}{10}, r \in Z_4-\{0\}\right\}$. By the induced function $\mu: E(G) \rightarrow[0,1]$ defined by $\mu(uv)=\frac{1}{10}\left\lceil\frac{3\sigma(u)}{\sigma(v)}\right\rceil$ we find that n edges receives the label 0.1, n edges receives the label 0.2 and n edges receives the label 0.3. Thus the number of edges labeled with i and the number of edges labeled with j differ by at most 1, where $i \neq j \in \left\{\frac{r}{10}, r \in Z_4-\{0\}\right\}$. Hence concluded that the generalized Petersen graph $GP(n, m), 1 \leq m<\left\lfloor\frac{\eta}{2}\right\rfloor$ is fuzzy quotient-3 cordial, except for $n \equiv 0(mod\ 6)$.

Example 3.1.2: The Mobius Kantor graph $GP(8,3)$ is fuzzy quotient-3 cordial graph.



Example 3.1.3: The Dodecahedral graph $GP(10,2)$ is fuzzy quotient-3 cordial.



4 Conclusion

In this study we give the mathematical formulation for labeling the vertices of generalized Petersen graphGP$(\eta,\mathrm{m}), 1 \leq \mathrm{m} < \left\lfloor \frac{\eta}{2} \right\rfloor$. We proved that generalized Petersen graphGP$(\eta,\mathrm{m}), 1 \leq \mathrm{m} < \left\lfloor \frac{\eta}{2} \right\rfloor$, except for $\eta \equiv 0(mod\ 6)$ and some special cases of Generalized Petersen Graph namely Durer graph, Desargues graph, Dodecahedral graph, Mobius Kantor graph, Petersen graph, Cubical graph, Cubic symmetric graph, Nauru graph are fuzzy quotient-3 cordial. Future research will focus on the existence of Fuzzy quotient-3 cordial labelling of various graph families.

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