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RESEARCH ARTICLE

New Odoma Distribution with Application to Cancer Data

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Abstract

Objective: To find appropriate model more flexible than classical models for fitting the survival data in health research especially in cancer blood (Leukemia) studies by introducing some additional parameters to the basic models. **Methods:** Based on the Alpha Power transformation technique and using Two-Parameter Odoma distribution, the new distribution called the Alpha Power Two- Parameter Odoma distribution is introduced and studied. The maximum likelihood estimation procedure is employed to estimate the unknown parameters. A simulation study is carried out to evaluate the performance of the maximum likelihood estimators. **Finding:** Common statistical properties such as quantile function, rth moments, moment generating function, characteristic function, incomplete moments, entropy and order statistics are derived. The practical importance of the proposed model is illustrated by using goodness-of-fit criterias based on real data set. **Novelty:** The APTPO distribution is the most appropriate model for fitting survival data of cancer studies comparing with other competitive distributions.

Keywords: TwoParameter Odoma Distribution; Alpha Power Transformation; Quantile; Moments; Maximum Likelihood Estimation

1 Introduction

Over the years, researchers have proposed a lot of new lifetime distributions to analysis and modeling lifetime data sets where the new lifetime distributions achieved the better fit to lifetime data sets over many fields of engineering, finance, insurance, etc. see [1–4]. One-Parameter Odoma (OPO) distribution is one of the new lifetime distributions introduced by [5] as a mixing three-components of an exponential distribution (with scale(with scale and shape 3), and gamma distribution (with scale and shape 5) Akash,… distributions. Since OPO distribution depends on one parameter and as such cannot be flexible in modeling varieties of data, we proposed Odoma (TPO) distribution was introduced by [6]. (TPO) distribution is more flexible than the one parameter Odoma distribution has a close form distributional expression. The probability density function (pdf) and its corresponding cumulative distribution function (cdf) are defined as follows

$$g(x) = \frac{\theta^5}{2\left(\theta^5\beta+\theta^3+24\right)}\left(2x^4+\theta x^2+2\theta\beta\right)e^{-\theta x}, x>0, \theta, \beta>0 \qquad (1)$$

and

$$G(x) = 1 - \left[1 + \frac{2\theta^2 x^2 \left(\theta^2 x^2 + 4\theta x + 12\right) + \theta x^2 \left(\theta^4 x + 2\theta^3 + 48\right)}{2\left(\theta^5 \beta + \theta^3 + 24\right)}\right] e^{-\theta x} \tag{2}$$

where θ and $\beta$are the scale and shape parameters. For $\beta = 1$we get the (OPO) distribution. The Odoma distribution is commenly used in fields of behavioral and emotional sciences. Odoma distribution to get some new distributions such as[7] proposed a new extension of Odoma distribution known as weighted Odoma distribution and discussed its properties and applications.[8] introduced Odoma distribution with various statistical properties and its application.[9] discussed Poisson Odoma distribution. The new version of Two-Parameter Odoma distribution namely area biased Two-Odoma distribution and its application of survival time is studied by[10].

The Alpha Power Transformation (APT) of the baseline (cdf) is proposed by[11] by adding a new parameter to obtain new flexible distribution for modeling lifetime data. The cdf of the APT is given by:

$$F_{APT}(x) = \begin{cases} \dfrac{\alpha^{G(x)} - 1}{\alpha - 1} & , \quad \alpha > 0, \alpha \neq 1 \\ G(x) & , \quad \alpha = 1 \end{cases} \tag{3}$$

In the literature, a lot of work has been done by using APT technique, such as[12] defined Alpha Power transformed power lindely distribution,[13] obtained Alpha Power transformed inverse lindely distribution,[14] suggested Alpha Power inverse Weibull distribution,[15] introduced Alpha Power transformed Frechet distribution,[16] proposed Alpha Power transformed inverse lomax distribution,[17] discussed Alpha Power transformed Pareto distribution,[18] obtained Alpha Power transformed Aradhana distribution,[19] studied Alpha Power transformation of lomax distribution,[20] introduced Alpha Power Two-Parameter Pranav distribution.The aim of this article is to introduce and study a new three parameter alpha power transformation of (TPO) distribution called the alpha power two parameter odoma (APTPO) distribution and proposed some special cases of it. Some properties of the new distribution including the shapes of density function and hazard rate function, quantile function, moments and moment generating function, incomplete moment, and Lorenz and Bonferroni curves are discussed. The parameter estimation for the proposed distribution is performed using maximum likelihood method. comparison the performance of the proposed distribution with other compitive distributions is discussed by using two real data sets, the first data consists of 40 blood cancer patients and second data set represents the survival time (in weeks) of 40 rates.

2 Methodology

2.1 The new distribution

The cdf and its corresponding pdf of the (APTPO) distribution can be obtained by using (2) and (3) as follows:

$$F_{AP2PO}(x) = \frac{\alpha^{1-\left[1+\frac{2\theta^2 x^2 \left(\theta^2 x^2 + 4\theta x + 12\right) + \theta x^2 \left(\theta^4 x + 2\theta^3 + 48\right)}{2\left(\theta^5 \beta + \theta^3 + 24\right)}\right] e^{-\theta x}} - 1}{\alpha - 1}, x > 0, \alpha, \theta, \beta > 0, \alpha \neq 1, \tag{4}$$

and

$$f_{AP2PO}(x) = \left(\frac{\theta^5 \log(\alpha)\left(2x^4 + \theta x^2 + 2\theta\beta\right)}{2(\alpha - 1)\left(\theta^5 \beta + \theta^3 + 24\right)}\right) \alpha^{1-\left[1+\frac{2\theta^2 x^2 \left(\theta^2 x^2 + 4\theta x + 12\right) + \theta x^2 \left(\theta^4 x + 2\theta^3 + 48\right)}{2\left(\theta^5 \beta + \theta^3 + 24\right)}\right] e^{-\theta x}} e^{-\theta x}, \quad x > 0, \alpha, \theta, \beta > 0, \alpha \neq 1, \tag{5}$$

where θ and $\alpha, \beta$are the scale and shape parameters.Odoma distribution as[6]. Figure 1 represents some plots of the pdf of the APTPO distribution for different parameter values which is positively sekewed and tends to normal distribution as sample size increases as is clear as follows:

Some sub models of the APTPO distribution are given in Table 1. The associated survival and hazard functions for the APTPO distribution are respectively defined as follows:

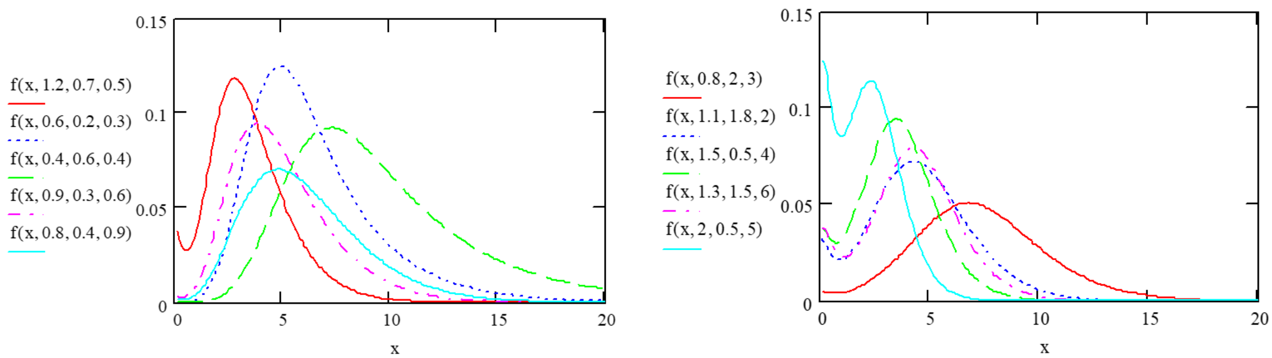

Fig 1. The pdf of the APTPO distribution

Table 1. Some sub models of the AP2PO distribution

α	θ	β	Reduced Model	Authors
-	-	1	Alpha Power One-Parameter Odoma(APOPO)	New
1	-	-	Two-Parameter Odoma (TPO)	[6]
1	-	1	Odoma (O)	[5]

$$S_{AP2PO}(x) = \frac{\alpha - \alpha^{1-\left[\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]\right]^{-\theta x}}}{\alpha-1}, x>0, \alpha, \theta, \beta > 0, \alpha \neq 1,$$

and

$$H_{AP2PO}(x) = \frac{\alpha^{-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]e^{-\theta x}}\theta^5\log(\alpha)}{1-\alpha^{-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x++\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]e^{-\theta x}}}\frac{\left(2x^4+\theta x^2+2\theta\beta\right)}{2\left(\theta^5\beta+\theta^3+24\right)}e^{-\theta x}$$

$x > 0, \alpha, \theta, \beta > 0, \alpha \neq 1,$

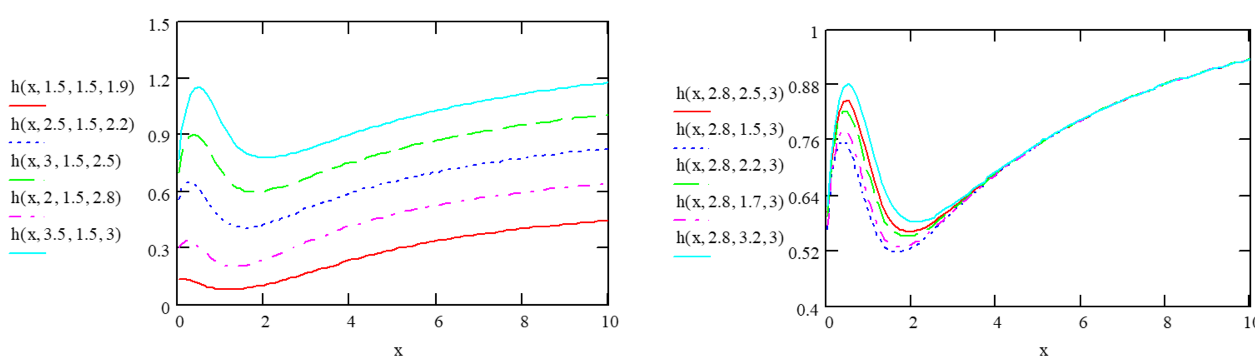

Fig 2. The hazard function of the APTPO distribution

The graph of the hazard rate function for the proposed distribution can be observed in many real life situations, since it is increasing, decreasing as well as showing constant behavior for different parameter values as shown as in Figure 2.

2.2 Quantile Function

The statistical distributions can be simulated by using quantile function; let q be a random variable having the uniform distribution on interval [0,1]. Using (4), the qth quantile function of the APTPO distribution is:

$$x_q = \frac{-1}{\theta}\ln\left[\frac{2\left(\theta^5\beta+\theta^3+24\right)\frac{\ln[\alpha/((\alpha-1)q+1)]}{\ln\alpha}}{2\left(\theta^5\beta+\theta^3+24\right)+2\theta^2x_q^2\left(\theta^2x_q^2+4\theta x_q+12\right)+\theta x_q^2\left(\theta^4x_q+2\theta^3+48\right)}\right]$$

The last equation can be solved numerically to get x_q . Also, by setting $\mathrm{q}=0.25, 0.5,$ and 0.75 we get the first, second (median), and third quartiles of the APTPO distribution, respectively. The quantile function can be used also in obtaining some statistical measures such as skewness and kurtosis measures when the moments of the distribution do not exit and estimation based on percentiles and quantile regression method.

2.3 Moments

Let X be a APTPO random variable distributed with parameters (α,θ,β), the r$^{\text{th}}$ non central moments of X can be obtained as follows

$$\mu_r' = \int_0^\infty x^r f_{AP2PO}(x)dx, r = 1,2,3,\ldots\ldots$$

$$= \int_0^\infty x^r\left(\frac{\theta^5\log(\alpha)\left(2x^4+\theta x^2+2\theta\beta\right)}{2(\alpha-1)\left(\theta^5\beta+\theta^3+24\right)}\right)\alpha^{1-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]e^{-\theta x}} e^{-\theta x}dx$$

$$\mu_r' = \left[\frac{\alpha\theta^5\log(\alpha)}{2(\alpha-1)\left(\theta^5\beta+\theta^3+24\right)}\right]\times$$

$$\int_0^\infty\left(2x^{r+4}+\theta x^{r+2}+2\theta\beta x^r\right)\alpha^{-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]e^{-\theta x}} e^{-\theta x}dx \tag{6}$$

Using the following series representation in equation (6)

$$a^{-z} = \sum_{i=0}^{\infty}\left(\frac{(-\log a)^i}{i!}z^i\right)$$

we get

$$\alpha^{-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4(\theta x+12)+\theta x^2\left(\theta^4x+2\theta^3+48\right)\right.}{2\left(\theta^5\beta+\theta^3+24\right)}\right]e^{-\theta x}} =$$

$$\sum_{i=0}^{\infty}\frac{(-\log\alpha)^i}{i!}\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]^i e^{-\theta i\,x}$$

and

$$\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]^i =$$

$$\sum_{j=0}^{i}\binom{i}{j}\left(\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right)^j,$$

So, the rth moment can be written as follows:

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\left[\frac{(-1)^i\alpha\theta^5(\log\alpha)^{i+1}}{2(\alpha-1)\left(\theta^5\beta+\theta^3+24\right)j!(i-j)!}\right]\times$$
$$\int_0^{\infty}\left[\begin{array}{l}\left(2x^{r+4}+\theta x^{r+2}+2\theta\beta x^r\right)\times\\ \left(\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right)^j e^{-\theta(i+1)x}dx\end{array}\right]$$

After simplifications, the rth moment of the APTPO distribution is given by:

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\tau_{i,j,k,l,n,m,s}\left[\begin{array}{c}\frac{2\Gamma(r+2j-k-l-m-s+5)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+5}}+\\ \frac{\theta^3\Gamma(r+2j-k-l-m-s+3)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+3}}\\ +\frac{2\beta\theta^5\Gamma(r+2j-k-l-m-s+1)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+1}}\end{array}\right], \tag{7}$$

where

$$\tau_{i,j,k,l,n,m,s} = \frac{(-1)^i\alpha(\log\alpha)^{i+1}2^{-k+l+2m-1}\left(\theta^5\beta+\theta^3+24\right)^{-(j+1)}3^s24^n}{(\alpha-1)(i-j)!(k-l)!n!(l-n)!(j-k-m)!s!(m-s)!}.$$

By setting $r=1$ and 2in equation (7) we obtain the first two non central moments, the mean of the APTPO distribution is expressed as:

$$E(x) = \mu_1' = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\tau_{i,j,k,l,n,m,s}\left[\begin{array}{c}\frac{2\Gamma(2j-k-l-m-s+6)}{\theta^{1-2j+3n}(i+1)^{2j-k-l-m-s+6}}+\\ \frac{\theta^3\Gamma(2j-k-l-m-s+4)}{\theta^{1-2j+3n}(i+1)^{2j-k-l-m-s+4}}\\ +\frac{2\beta\theta^5\Gamma(2j-k-l-m-s+2)}{\theta^{1-2j+3n}(i+1)^{2j-k-l-m-s+2}}\end{array}\right],$$

Additionally, the variance of the APTPO distribution can be given by

$$\mathrm{Var}(x) = \mu_2' - \left(\mu_1'\right)^2$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\tau_{i,j,k,l,n,m,s}\left[\begin{array}{c}\frac{2\Gamma(2j-k-l-m-s+7)}{\theta^{2-2j+3n}(i+1)^{2j-k-l-m-s+7}}+\\ \frac{\theta^3\Gamma(2j-k-l-m-s+5)}{\theta^{2-2j+3n}(i+1)^{2j-k-l-m-s+5}}\\ +\frac{2\beta\theta^5\Gamma(2j-k-l-m-s+3)}{\theta^{2-2j+3n}(i+1)^{2j-k-l-m-s+3}}\end{array}\right],$$

By using non central moments, we can calculate the central moments of the APTPO distribution by the following relation:

$$\mu_r = \sum_{i=0}^{r}\left(\begin{array}{c}r\\ i\end{array}\right)\left(-\mu_1'\right)^{r-i}\mu_r'.$$

The moment generating function of X distributed APTPO is:

$$M_X(t) = E\left(e^{tx}\right) = \sum_{r=0}^{\infty}\mu_r'\frac{t^r}{r!},$$

So

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\frac{t^r}{r!}\tau_{i,j,k,l,,m,s}\left[\begin{array}{c}\frac{2\Gamma(r+2j-k-l-m-s+5)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+5}}+\\ \frac{\theta^3\Gamma(r+2j-k-l-m-s+3)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+3}}\\ +\frac{2\beta\theta^5\Gamma(r+2j-k-l-m-s+1)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+1}}\end{array}\right].$$

The characteristic function of the APTPO distribution is given by:

$$\phi_X(it) =$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\tau_{i,j,k,l,n,m,s}\left[\begin{array}{c}\frac{2\Gamma(r+2j-k-l-m-s+5)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+5}}+\\ \frac{\theta^3\Gamma(r+2j-k-l-m-s+3)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+3}}\\ +\frac{2\beta\theta^5\Gamma(r+2j-k-l-m-s+1)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+1}}\end{array}\right].$$

The s$^{\text{th}}$ incomplete moments of X having APTPO distribution is obtained as follows: median .

$$\varphi_s(t) = \int_0^t x^s f_{AP2PO}(x)dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\tau_{i,j,k,l,n,m,s}\left[\begin{array}{c}\frac{2\gamma(r+2j-k-l-m-s+5,t)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+5}}+\\ \frac{\theta^3\gamma(r+2j-k-l-m-s+3,t)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+3}}\\ +\frac{2\beta\theta^5\gamma(r+2j-k-l-m-s+1,t)}{\theta^{r-2j+3n}(i+1)^{r+2j-k-l-m-s+1}}\end{array}\right],$$

where

$\gamma(s,t) = \int_0^t x^{s-1}e^{-x}dx$ is the lower incomplete gamma function.By using incomplete moments, we can calculate some statistical measures as the mean deviations about mean μ_1' and median M. The mean deviation about mean δ_1 and mean deviation about median δ_2 can be, respectively, expressed as:

$$\delta_1 = 2\mu_1' F\left(\mu_1'\right) - 2\varphi_1\left(\mu_1'\right), \text{ and } \delta_2 = \mu_1' - 2\varphi_1(M).$$

2.4 Entropy

Entropy or information theory deals with the study of processing, utilization, transmission and extraction of information. Entropy measures quantify the diversity, uncertainty or randomness of a system. Also it has many applications in different areas such as probability and statistics, physics and economics.

2.4.1 ***Renyi Entropy***

The Renyi entropy is as the index of diversity in statistics and ecology. The Renyi entropy of order τAPTPO random variable X is given by:

$$R_X(\tau) = \frac{1}{1-\tau}\log\left[\int_0^\infty f_{AP2PO}^\tau(x)dx\right], \tau > 0, \tau \neq 0.$$

$$=\frac{1}{1-\tau}\log\left[\begin{array}{l}\left(\frac{\theta^5\alpha\log(\alpha)}{2(\alpha-1)\left(\theta^5\beta+\theta^3+24\right)}\right)^\tau\times\\ \int_0^\infty\left(2x^4+\theta x^2+2\theta\beta\right)^\tau\alpha^{-\tau}\left(\left[\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)++x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]e^{-\theta x}\right)e^{-\theta\tau x}dx\end{array}\right]$$

$$=\frac{1}{1-\tau}\log\left[\begin{array}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\binom{j}{k}\binom{k}{l}\binom{l}{n}\binom{j-k}{m}\binom{m}{s}2^{j-k-\tau+l+2m}.3^s.24^n\\ \frac{(-\tau)^i\alpha^\tau(\log\alpha)^{i+\tau}\theta^{4j+5\tau-k-l-3n-m-s}}{j!(i-j)!(\alpha-1)^\tau\left(\theta^5\beta+\theta^3+24\right)^\tau}\left[\begin{array}{l}2\frac{\Gamma(2j-k-l-m-s+5)}{[\theta(i+\tau)]^{2j-k-l-m-s+5}}\\ +\theta^3\frac{\Gamma(r+2j-k-l-m-s+3)}{[\theta(i+\tau)]^{2j-k-l-m-s+3}}\\ +2\theta\beta\frac{\Gamma(2j-k-l-m-s+1)}{[\theta(i+\tau)]^{2j-k-l-m-s+1}}\end{array}\right]\end{array}\right].$$

2.4.2 *Tsallis Entropy*

The Tasllis entropy for a continous random variable X as follows

$$S_X(\lambda) = \frac{1}{\lambda - 1}\left[1 - \int_0^\infty f^\lambda(x)dx\right]$$

For the APTPO distribution,

$$= \frac{1}{\lambda-1}\left[\begin{array}{l} 1-\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\binom{j}{k}\binom{k}{l}\binom{l}{n}\binom{j-k}{m}\binom{m}{s}2^{j-k-\lambda+l+2m}.3^s.24^n \times \\ \frac{(-\lambda)^i\alpha^\lambda(\log\alpha)^{i+\lambda}\theta^{4j+5\lambda-k-l-3n-m-s}}{j!(i-j)!(\alpha-1)^\lambda(\theta^5\beta+\theta^3+24)^\lambda}\left[\begin{array}{l} 2\frac{\Gamma(2j-k-l-m-s+5)}{[\theta(i+\lambda)]^{2j-k-l-m-s+5}} \\ +\theta^3\frac{\Gamma(r+2j-k-l-m-s+3)}{[\theta(i+\lambda)]^{2j-k-l-m-s+3}} \\ +2\theta\beta\frac{\Gamma(2j-k-l-m-s+1)}{[\theta(i+\lambda)]^{2j-k-l-m-s+1}} \end{array}\right] \end{array}\right].$$

2.5 Lorenz and Bonferroni curves

The curves of Lorenz and Bonferroni are very useful in economics, insurance, reliability, demography, and medicine. These curves can be computed using the incomplete moments. Let X having APTPO distribution cdf and pdf as defined in (4) and (5) the Lorenz curve and Bonferroni curve are respectively defined as

$$L(x) = \frac{\varphi_1(x)}{\mu_1'}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\tau_{i,j,k,l,n,m,s}\left[\begin{array}{c} \frac{2\gamma(2j-k-l-m-s+6,t)}{\theta^{1-2j+3n}(i+1)^{2j-k-l-m-s+6}}+ \\ \frac{\theta^3\gamma(2j-k-l-m-s+4,t)}{\theta^{1-2j+3n}(i+1)^{2j-k-l-m-s+4}} \\ +\frac{2\beta\theta^5\gamma(2j-k-l-m-s+2,t)}{\theta^{1-2j+3n}(i+1)^{2j-k-l-m-s+2}} \end{array}\right](\mu_1')^{-1},$$

and

$B(x) = \frac{L(x)}{F_{AP2PO}(x)}$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{n=0}^{l}\sum_{m=0}^{j-k}\sum_{s=0}^{m}\tau_{i,j,k,l,n,m,s}\left[\begin{array}{c} \frac{2\gamma(2j-k-l-m-s+6,t)}{\theta^{1-2j+3n}(i+1)^{2j-k-l-m-s+6}}+ \\ \frac{\theta^3\gamma(2j-k-l-m-s+4,t)}{\theta^{1-2j+3n}(i+1)^{2j-k-l-m-s+4}} \\ +\frac{2\beta\theta^5\gamma(2j-k-l-m-s+2,t)}{\theta^{1-2j+3n}(i+1)^{2j-k-l-m-s+2}} \end{array}\right]\times \left[\frac{\alpha^{1-\left[1+\frac{2\theta^2x^2(\theta^2x^2+4\theta x+12)+\theta x^2(\theta^4x+2\theta^3+48)}{2(\theta^5\beta+\theta^3+24)}\right]e^{-\theta x}}-1}{\alpha-1}\mu_1'\right]^{-1},$$

where

$$\tau_{i,j,k,l,n,m,s} = \frac{(-1)^i\alpha(\log\alpha)^{i+1}2^{-k+l+2m-1}\left(\theta^5\beta+\theta^3+24\right)^{-(j+1)}3^s24^n}{(\alpha-1)(i-j)!(k-l)!n!(l-n)!(j-k-m)!s!(m-s)!}.$$

2.6 Order Statistics

We obtain the expression of the pdf of the rth order statistics $X_{(r)}$ of the APTPO distribution cdf and pdf given by (4) and (5), respectively. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be an ordered random sample drawn from the proposed distribution, The pdf of $X_{(r)}$ is given by:

$$f_r(x) = \sum_{d=0}^{n-r}\frac{(-1)^d}{B(r,n-r+1)}\binom{n-r}{d}[F_{APTPO}(x)]^{r+d-1}f_{APTPO}(x), \tag{8}$$

By substituting (4) and (5) in (8), we get:

$$f_r(x)=\left(\frac{\theta^5\log(\alpha)\left(2x^4+\theta x^2+2\theta\beta\right)e^{-\theta x}}{2\left(\theta^5\beta+\theta^3+24\right)B(r,n-r+1)}\right)\alpha^{1-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]e^{-\theta x}}$$

$$\times\sum_{d=0}^{n-r}\frac{(-1)^{r+2d-1}}{(\alpha-1)^{r+d}}\binom{n-r}{d}\left[1-\alpha^{1-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^324\right)}\right]}\right]^{r+d-1}, \quad (9)$$

where B(., .) is the beta function. by applying power series expantion given by:

$$(1-z)^{t-1}=\sum_{l=0}^{\infty}\binom{t-1}{l}(-z)^l,$$

For t positive real non-integer and $z\leq l$ in equation (9), we have

$$f_r(x)=\sum_{d=0}^{n-r}\sum_{l=0}^{\infty}\frac{\eta(-1)^{r+2d+l-1}}{(\alpha-1)^{r+d}}\binom{n-r}{d}\binom{r+d-1}{l}\left[\alpha^{1-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^324\right)}\right]e^{-\theta *}}\right]^{l+1}, \quad (10)$$

Where

$$\eta=\left(\frac{\theta^5\log(\alpha)\left(2x^4+\theta x^2+2\theta\beta\right)e^{-\theta x}}{2\left(\theta^5\beta+\theta^3+24\right)B(r,n-r+1)}\right)$$

In particular, we define the smallest order statistics $X_{(1)}$ and the largest order statistics $X_{(n)}$ by substituting and and in (10), respectively, as follows:

$$f_1(x)=n\sum_{d=0}^{n-1}\sum_{l=0}^{\infty}\frac{\eta_1(-1)^{2d+l}}{(\alpha-1)^{d+1}}\binom{n-1}{d}\binom{d}{l}\left[\alpha^{1-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^324\right)}\right]e^{-\theta x}}\right]^{l+1},$$

and

$$f_n(x)=n\sum_{l=0}^{\infty}\frac{\eta_1(-1)^{n+l-1}}{(\alpha-1)^{n}}\binom{n-1}{l}\left[\alpha^{1-\left[1+\frac{2\theta^2x^2\left(\theta^2x^2+4\theta x+12\right)+\theta x^2\left(\theta^4x+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^324\right)}\right]}\right]^{l+1},$$

where

$$\eta_1=\left(\frac{\theta^5\log(\alpha)\left(2x^4+\theta x^2+2\theta\beta\right)e^{-\theta x}}{2\left(\theta^5\beta+\theta^3+24\right)}\right).$$

2.7 Parameters Estimation

The maximum likelihood estimates (MLEs) of the unknown parameters for the APTPO distribution are derived by taken random sample $x_1, x_2, \ldots, x_n$ having APTPO pdf (5) with parameters(α,θ,β) . The log-likelihood function is given by:

$$\log L=$$

$$n\left[\log(\log\alpha)-\log(\alpha-1)+5\log(\theta)-\log\left(2\theta^5\beta+2\theta^3+48\right)\right]+\sum_{i=1}^{n}\log\left(2x_i^4+\theta x_i^2+2\theta\beta\right)$$

$$-\theta\sum_{i=1}^{n}x_i+\log(\alpha)\sum_{i=1}^{n}\left[1-\left(1+\frac{2\theta^2x_i^2\left(\theta^2x_i^2+4\theta x_i+12\right)+\theta x_i^2\left(\theta^4x_i+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right]e^{-\theta x_i}\right].$$

To get the nonlinear likelihood equations, we differentiate the last equation with respect to (α,θ,β)as follows:

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} =&\frac{1}{\alpha}\sum_{i=1}^{n}\left[1-\left(1+\frac{2\theta^2x_i^2\left(\theta^2x_i^2+4\theta x_i+12\right)+\theta x_i^2\left(\theta^4x_i+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right)e^{-\theta x_i}\right]\\&+\frac{n}{\alpha\log\alpha}-\frac{n}{(\alpha-1)},\\ \frac{\partial \log L}{\partial \theta} =&\frac{5n}{\theta}-\left[\frac{\left(10\beta\theta^4+6\theta^2\right)n}{\left(2\theta^5\beta+2\theta^3+48\right)}\right]+\sum_{i=1}^{n}\left[\frac{\left(x_i^2+2\beta\right)}{\left(2x_i^4+\theta x_i^2+2\theta\beta\right)}\right]-\sum_{i=0}^{n}x_i+\log(\alpha)\\&\sum_{i=1}^{n}\left(1+\frac{2\theta^2x_i^2\left(\theta^2x_i^2+4\theta x_i+12\right)+\theta x_i^2\left(\theta^4x_i+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right)x_ie^{-\theta x_i}-\log(\alpha)\\&\sum_{i=1}^{n}\left(\frac{2\theta x_i^2\left(4\theta^2x_i^2+12\theta x_i+24\right)+x_i^2\left(5\theta^4x_i+8\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right)e^{-\theta x_i}-\log(\alpha)\\&\sum_{i=1}^{n}\left[\left(\frac{2\theta_i^2x_i^2\left(\theta^2x_i^2+4\theta x_i+12\right)+\theta x_i^2\left(\theta^4x_i+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)}\right)\left(5\beta\theta^4+3\theta^2\right)\right]e^{-\theta x_i},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \beta} =&-\left[\frac{2n\theta^5}{\left(2\theta^5\beta+2\theta^3+48\right)}\right]+2\sum_{i=1}^{n}\left[\frac{\theta}{\left(2x_i^4+\theta x_i^2+2\theta\beta\right)}\right]\\&+\log(\alpha)\sum_{i=1}^{n}\left[\left(\frac{2\theta^2x_i^2\left(\theta^2x_i^2+4\theta x_i+12\right)+\theta x_i^2\left(\theta^4x_i+2\theta^3+48\right)}{2\left(\theta^5\beta+\theta^3+24\right)^2}\right)\theta^5e^{-\theta x_i}\right].\end{aligned}$$

The MLEs $(\hat{\alpha},\hat{\theta},\hat{\beta})$ of (α,θ,β) of the APTPO distribution are obtained by setting the nonlinear likelihood equations equal to zero. Since the equations does not have a closed form, but can be found numerically by using Mathcad package.

2.8 Simulation Study

The Monte Carlo simulation is performed to evaluate the performance of the MLEs for the APTPO distribution parameters. A simulation study is carried out by generating 1000 random samples of size 80, 100, and 150 from the APTPO distribution. Three sets of parameters values are selected as: $(\theta=0.2,\alpha=0.4,\beta=0.1)(\theta=0.2,\alpha=0.3,\beta=0.1)$ and $(\theta=0.1,\alpha=0.4,\beta=0.1)$. . Compute the biases, mean square errors (MSEs), and means of MLEs of the unknown parameters. The numerical results are summarized in Table 2.

Table 2. Bias, MSEs, and Means for the MLEs

Actual Values	n	Bias			MSE			Mean		
		(θ)	(α)	(β)	(θ)	(α)	(β)	(θ)	(α)	(β)
θ = 0.2	4.238	1.507	0.425	5.016	1.437	0.051	7.819	1.107	0.225	80
α = 0.4	7.857	0.492	0.412	4.936	0.105	0.045	5.457	0.212	0.212	100
β = 0.1	2.421	0.612	0.413	2.067	0.071	0.045	2.321	0.092	0.212	150
θ = 0.2	6.864	0.852	0.421	4.784	0.429	0.049	8.764	0.552	0.221	80
α = 0.3	3.776	0.799	0.418	4.751	0.345	0.048	6.099	0.499	0.218	100
β = 0.1	1.091	0.624	0.414	4.322	0.153	0.046	3.676	0.324	0.214	150
θ = 0.1	2.761	7.053	0.511	7.702	7.619	0.168	2.761	6.653	0.411	80
α = 0.4	2.578	6.689	0.499	6.710	6.836	0.160	2.578	6.324	0.399	100
β = 0.1	2.432	6.724	0.492	5.955	2.308	0.154	2.432	6.289	0.392	150

We observe from Table 2 that for different choices of sample sizes, the estimates are performed quite well and MSE and bias decreases as the sample size increases. Generally the results of the proposed model are better in the sense of MSE, and Bias.

3 Results and Discussion

3.1 Application

The APTPO has been applied the real data set to show that the proposed distribution which can be a better model than the Beta Weibull (BW), Exponentiated Generalized Weibull (EGW), Kumuraswamy Weibull (KW), Two Parameter Odoma (TPO), Alpha Power Weibull (APW), Exponentiated Kumuraswamy Weibull (EKW), Alpha Power One Parameter Odoma (APOPO), and Weibull (W) models.

3.2 Data set

This data represent 40 patients suffering from blood cancer (leukemia) from one of Ministry of Health Hospitals in Saudi Arabia as [21].

Table 3. The lifetimes (in days) of 40 patients suffering from blood cancer (leukemia)

0.315	0.496	0.616	1.145	1.208	2.805	3.767	4.647
2.162	2.211	2.370	2.532	2.698	3.534	4.397	2.036
3.263	3.348	3.348	3.427	3.499	4.392	2.025	3.192
3.986	4.049	4.244	4.323	4.381	1.414	2.912	3.858
4.929	4.973	5.074	5.381	1.263	2.91	3.751	4.753

In order to compare the APTPO distribution with Beta-Weibull (BW), Exponentiated Generalized Weibull (EGW), Kumaraswamy Weibull (KW), Two Parameter Odoma (TPO), Alpha Power Weibull (APW), Exponentiated Kumaraswamy Weibull (EKW), Alpha Power One Parameter Odoma (APOPO), and Weibull (W) distributions.

To investigate the goodness of fit for the compared distribution using the following criterias like -2 log-likelihood function (-2lnL), Akaike Information Criterion (AIC), Akaike Information Criterion Corrected (AICC), Bayesian information criterion (BIC), Kolmogorov statistic (K-S), and p-value. The better distribution corresponds to lower values of criteria. The MLEs of the parameters for the proposed model is $\hat{\alpha} = 0.145, \hat{\theta} = 1.215,$ and $\hat{\beta} = 1.575$

and for data set. The corresponding -2lnL, AIC, AICC, BIC, K-S, and p-value are displayed in Table 4 for data set.

Table 4. Analytical results for the data set

Model	-2lnL	AIC	AICC	BIC	KS	p-value
BW	177.748	185.748	186.891	192.503	0.317	0.0006
EGW	141.756	149.756	150.898	156.512	0.149	0.3330
KW	138.738	146.738	147.881	153.493	0.097	0.841
TPO	142.102	146.102	146.426	149.481	0.214	0.05
APW	137.364	143.364	144.031	148.431	0.092	0.888
EKW	132.262	142.262	144.027	150.706	0.103	0.780
APOPO	138.276	142.276	142.655	145.654	0.331	0.00012
W	139.116	143.116	143.440	146.494	0.118	0.628
APTPO	**135.496**	**141.496**	**142.163**	**145.563**	**0.081**	**0.901**

Results in Table 4 indicate that the APTPO distribution has the smallest values of -2lnL, AIC, AICC, and BIC as compared to other competitive distributions with respect to cancer data set. Hence, the APTPO distribution leads to the better fit than the other competitive distributions.

4 Conclusion

By using Alpha power transformation, we obtained a new distribution called the alpha power two parameter odoma APTPO distribution. It has different special cases which have been presented in the paper. Various important mathematical properties of the new distribution are derived, including hazard function, quantile function, moments, moment generating function, entropy, and order statistics. It has been observed that from graph of the probability density function the proposed distribution is positively sekewed and tends to normal distribution as sample size increases. The inference of unknown parameters for the proposed distribution are obtained by using the maximum likelihood estimation method. A simulation study is performed to examine the estimated parameters. The application of the proposed distribution with its competitive statistical distributions are essential done for medical research especially for cancer patients real data set. The results indicate the superior performance of the APTPO distribution compared to other competing distributions depending on different goodness of fit criteria.

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