INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Check for updates

OPEN ACCESS

Received: 19-11-2022

Accepted: 26-02-2023

Published: 28-03-2023

Citation: Kishore J, Verma V (2023) Some Identities on Sums of Finite Products of the Pell, Fibonacci, and Chebyshev Polynomials . Indian Journal of Science and Technology 16(12): 941-955. https://doi.org/10.17485/IJST/v16i12.2213

* **Corresponding author**.

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Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Some Identities on Sums of Finite Products of the Pell, Fibonacci, and Chebyshev Polynomials

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Abstract

Objectives: This study will introduce some new identities for sums of finite products of the Pell, Fibonacci, and Chebyshev polynomials in terms of derivatives of Pell polynomials. Similar identities for Fibonacci and Lucas numbers will be deduced. **Methods:** Results are obtained by using differential calculus, combinatory computations, and elementary algebraic computations. **Findings:** In terms of derivatives of Pell polynomials, identities on sums of finite products of the Fibonacci numbers, Lucas numbers, Pell and Fibonacci polynomials, and Chebyshev polynomials of third and fourth kinds are obtained. **Novelty:** Existing research has identified identities on sums of finite products of the Fibonacci numbers, Lucas numbers, Pell and Fibonacci polynomials, and Chebyshev polynomials of the third and fourth kinds in terms of derivatives of Fibonacci polynomials or Chebyshev polynomials; identities on sums of finite products in terms of Pell polynomials, however, have not been investigated, so identities primarily in terms of Pell polynomials are obtained.

Keywords: Fibonacci Numbers; Pell Numbers; Lucas Numbers; Pell Polynomials; Chebyshev Polynomials

1 Introduction

This section will introduce the basic definitions and symbols that are going to be used throughout the paper and the central theme of the manuscript through the citation of previously deduced results by various authors on sums of products of some special polynomials.

To start with, the Fibonacci numbers ($\mathscr{F}_\alpha$) and Lucas numbers ($\mathscr{L}_\alpha$) are respectively defined by the linear recursive relations

$$\mathscr{F}_\alpha = \mathscr{F}_{\alpha-1} + \mathscr{F}_{\alpha-2}, \alpha \geq 2, \mathscr{F}_0 = 0 \quad \text{and } \mathscr{F}_1 = 1$$

and

$$\mathscr{L}_\alpha = \mathscr{L}_{\alpha-1} + \mathscr{L}_{\alpha-2}, \alpha \geq 2, \mathscr{L}_0 = 2 \text{ and } \mathscr{L}_1 = 1$$

Similarly, the Fibonacci polynomials are defined recursively as follows:

$$\mathscr{F}_\alpha(z) = z\mathscr{F}_{\alpha-1}(z) + \mathscr{F}_{\alpha-2}(z), \alpha \geq 2, \mathscr{F}_0(z) = 0 \quad \text{and } \mathscr{F}_1(z) = 1$$

The Pell numbers ($\mathscr{P}_\alpha$) and Pell polynomials ($\mathscr{P}_\alpha(z)$)[1] are defined recursively as

$$\mathscr{P}_\alpha = 2\mathscr{P}_{\alpha-1} + \mathscr{P}_{\alpha-2}, \alpha \geq 2, \mathscr{P}_0 = 0 \quad \text{and } \mathscr{P}_1 = 1$$

and

$$\mathscr{P}_\alpha(z) = 2\mathscr{P}_{\alpha-1}(z) + \mathscr{P}_{\alpha-2}(z), \alpha \geq 2, \mathscr{P}_0(z) = 0 \text{ and } \mathscr{P}_1(z) = 1$$

The Chebyshev polynomials of the first ($\mathscr{T}_\alpha(z)$), second ($\mathscr{U}_\alpha(z)$), third ($\mathscr{V}_\alpha(z)$), and fourth kind ($\mathscr{W}_\alpha(z)$)[2–4] are defined recursively for $\alpha \geq 1$, as

$$\mathscr{T}_\alpha(z) = 2z\,\mathscr{T}_{\alpha-1}(z) - \mathscr{T}_{\alpha-2}(z), \mathscr{T}_0(z) = 1 \text{ and } \quad \mathscr{T}_1(z) = z$$

$$\mathscr{U}_\alpha(z) = 2z\,\mathscr{U}_{\alpha-1}(z) - \mathscr{U}_{\alpha-2}(z), \mathscr{U}_0(z) = 1 \quad \text{and} \quad \mathscr{U}_1(z) = 2z$$

$$\mathscr{V}_\alpha(z) = 2z\,\mathscr{V}_{\alpha-1}(z) - \mathscr{V}_{\alpha-2}(z), \quad \mathscr{V}_0(z) = 1 \text{ and } \quad \mathscr{V}_1(z) = 2z - 1$$

$$\mathscr{W}_\alpha(z) = 2z\,\mathscr{W}_{\alpha-1}(z) - \mathscr{W}_{\alpha-2}(z), \mathscr{W}_0(z) = 1 \text{ and } \mathscr{W}_1(z) = 2z + 1$$

These linear recurrence sequences will in turn lead to the following general formulae[5,6]

$$\mathscr{T}_\alpha(z) = \frac{1}{2}\left[\left(z + \sqrt{z^2-1}\right)^\alpha + \left(z - \sqrt{z^2-1}\right)^\alpha\right]$$
$$\mathscr{U}_\alpha(z) = \frac{1}{2\sqrt{z^2-1}}\left[\left(z + \sqrt{z^2-1}\right)^{\alpha+1} - \left(z - \sqrt{z^2-1}\right)^{\alpha+1}\right]$$
$$\mathscr{P}_\alpha(z) = \frac{1}{2\sqrt{z^2+1}}\left[\left(z + \sqrt{z^2+1}\right)^\alpha - \left(z - \sqrt{z^2+1}\right)^\alpha\right]$$

With the help of the above-discussed recurrence relations and formulae, one can easily observe[6] that for any integer, $\alpha \geq 0$,

$$\mathscr{V}_\alpha(z) = \mathscr{U}_\alpha(z) - \mathscr{U}_{\alpha-1}(z) \tag{1}$$

$$\mathscr{W}_\alpha(z) = \mathscr{U}_\alpha(z) + \mathscr{U}_{\alpha-1}(z) \tag{2}$$

$$\mathscr{W}_\alpha(z) = (-1)^\alpha \quad \mathscr{V}_\alpha(-z) \tag{3}$$

$$\mathscr{U}_\alpha(iz) = i^\alpha \mathscr{P}_{\alpha+1}(z) \tag{4}$$

Many authors have studied the properties of Chebyshev polynomials. For example, Zhang investigated the sums of finite products of the second kind of Chebyshev polynomials[7] and derived many identities, particularly

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{U}_{d_1}(z) \cdot \mathscr{U}_{d_2}(z) \cdots \mathscr{U}_{d_{r+1}}(z) = \frac{1}{2^r r!}\mathscr{U}^r_{\alpha+r}(z) \tag{5}$$

where $U^r_\alpha(z)$ denotes the r^{th} derivative of $U_\alpha(z)$ w.r.t z and the sum runs over all the *r+1*- dimensional non-negative integral coordinates $d_1, d_2, \cdots, d_{r+1}$ such that $d_1 + d_2 + \cdots + d_{r+1} = \alpha$. Similar results were observed by T. Kim, D. S. Kim, D.V. Dolgy, and J. Kwon[8] for finite products of Chebyshev polynomials of the first kind and Lucas polynomials.

In[9], T. Kim, D. S. Kim, D.V. Dolgy, and D. Kim have observed the sums of finite products of Chebyshev polynomials of the third $(V_\alpha(z))$ and fourth kind $(W_\alpha(z))$ as follows:

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{V}_{d_1}(z)\cdot v_{d_2}(z)\cdots v_{d_{r+1}}(z) = \frac{1}{2^r r!}\sum_{j=0}^{\alpha}(-1)^j \left(\begin{array}{c} r+1 \\ j \end{array}\right) \mathscr{U}^r_{\alpha-j+r}(z) \tag{6}$$

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{W}_{d_1}(z)\cdot \mathscr{W}_{d_2}(z)\cdots \mathscr{W}_{d_{r+1}}(z) = \frac{1}{2^r r!}\sum_{j=0}^{\alpha} \left(\begin{array}{c} r+1 \\ j \end{array}\right) \mathscr{U}^r_{\alpha-j+r}(z) \tag{7}$$

where all sums in eqns. (6)-(7) runs over all non-negative integers $d_1, d_2, \cdots, d_{r+1}$ such that $d_1+d_2+\cdots+d_{r+1}=\alpha$ with $\left(\begin{array}{c} r+1 \\ j \end{array}\right)$ for $j > r+1$. Here the authors have developed the results on sums of finite products of Chebyshev polynomials of the third and fourth kind in terms of Chebyshev polynomials of the second kind and hypergeometric functions. Similar results were studied by D. Han and L. Xinging[10] for sums of finite products of Chebyshev polynomials of the first and second kind, Lucas and Fibonacci polynomials in terms of Chebyshev polynomials of the first and second kind, and Lucas polynomials. A. Patra and G.K. Panda[11] also developed similar identities on sums of finite products of Pell polynomials in terms of orthogonal polynomials, including Chebyshev polynomials.

According to the preceding literature, previous work has been done to develop identities on the sums of finite products of Fibonacci and Lucas numbers, Pell, Lucas, and Fibonacci polynomials, and Chebyshev polynomials of third and fourth kinds in terms of derivatives of Fibonacci polynomials, Lucas polynomials, or Chebyshev polynomials, but the identities on the sums of finite products in terms of Pell polynomials have not been investigated. So, in this paper, some more identities on sums of finite products of the Fibonacci and Lucas numbers and Pell, Fibonacci, and third and fourth kinds of Chebyshev polynomials, primarily in terms of derivatives of the Pell polynomials, are obtained.

2 Methodology

Results are obtained by using differential calculus, combinatory computations, and elementary algebraic computations.

3 Results and Discussion

In this section, the main results of the paper on the sums of finite products of the Pell polynomials, Chebyshev polynomials of third and fourth kind, Fibonacci and Lucas numbers, and the derivative of the Pell polynomials are obtained using elementary computations. These results are established along the lines of the sums of finite products in Eqns (5)-(7) and are encapsulated in the following theorems.

Lemma 1

For any non-negative integer α, the following identities hold:

i). $\mathscr{P}_{\alpha+1}\left(-\frac{3}{2}i\right) = i^{-\alpha}\,\mathscr{F}_{2(\alpha+1)}$.

ii). $\mathscr{P}_{\alpha+1}\left(\frac{3}{2}i\right) = i^{\alpha}\,\mathscr{F}_{2(\alpha+1)}$.

iii). $\mathscr{P}_{\alpha+1}(-2) = \frac{i^\alpha}{2}\mathscr{F}_{3(\alpha+1)}$.

iv). $\mathscr{V}_\alpha\left(\frac{3}{2}\right) = F_{2\alpha+1}$.

v). $\mathscr{W}_\alpha\left(\frac{3}{2}\right) = L_{2\alpha+1}$.

Proof

The lemma can be easily established by taking $z=-\frac{3i}{2}, \frac{3i}{2}, -2$, in eq. (4), $z=\frac{3}{2}$ in Eq.1 and Eq. 2 and using the fact $\mathscr{U}_\alpha\left(\frac{3}{2}\right)=\mathscr{F}_{2(\alpha+1)}$, $\mathscr{U}_\alpha\left(-\frac{3}{2}\right)=(-1)^\alpha\,\mathscr{F}_{2(\alpha+1)}$, $\mathscr{U}_\alpha(-2i)=\frac{(-1)^\alpha}{2}\mathscr{F}_{3(\alpha+1)}$, $\mathscr{U}_\alpha\left(\frac{3}{2}\right)=\mathscr{F}_{2\alpha+2}$ respectively.

Lemma 2

For any integer $\alpha \geq 0$, the following identity holds

$$\mathscr{P}'_{\alpha+1}(z) = \frac{(\alpha+1)}{(1+z^2)}\mathscr{P}_\alpha(z) + \frac{\alpha z}{(1+z^2)}\mathscr{P}_{\alpha+1}(z)$$

Proof

From[12],

$$\left(1-z^2\right)\mathscr{U}'_\alpha(z) = (\alpha+1)\mathscr{U}_{\alpha-1}(z) - \alpha z\,\mathscr{U}_\alpha(z) \tag{8}$$

Replacing z with iz in Eq. 8 gives

$$\left(1+z^2\right)\mathscr{U}'_\alpha(iz)=(\alpha+1)\mathscr{U}_{\alpha-1}(iz)-\alpha i\,z u_\alpha(iz) \tag{9}$$

Differentiating Eq. 4 yields

$$\mathscr{U}'_\alpha(iz)=i^{\alpha-1}\mathscr{P}'_{\alpha+1}(z) \tag{10}$$

Using Eq. 10 in Eq. 9 yields

$$\left(1+z^2\right)i^{\alpha-1}\,\mathscr{P}'_\alpha(z)=(\alpha+1)\;i^{\alpha-1}\,\mathscr{P}_\alpha(z)-\alpha\;i\;z\;i^\alpha\;\mathscr{P}_\alpha(z)$$

$$\left(1+z^2\right)\mathscr{P}'_{\alpha+1}(z)=(\alpha+1)\mathscr{P}_\alpha(z)+\alpha z\;\mathscr{P}_{\alpha+1}(z)$$

$$\mathscr{P}'_{\alpha+1}(z)=\frac{(\alpha+1)}{(1+z^2)}\mathscr{P}_\alpha(z)+\frac{\alpha z}{(1+z^2)}\mathscr{P}_{\alpha+1}(z)$$

This proves the lemma.

Lemma 3

For any integer $\alpha\geq 0$, the following identity holds

$$\mathscr{P}''_\alpha(z)=\frac{\alpha(\alpha+2)}{(1+z^2)}\mathscr{P}_{\alpha+1}(z)-\frac{3z}{(1+z^2)}\mathscr{P}'_{\alpha+1}(z)\,.$$

Proof

From [12],

$$\left(1-z^2\right)\mathscr{U}''_\alpha(z)=3z\,\mathscr{U}'_\alpha(z)-\alpha(\alpha+2)\mathscr{U}_\alpha(z) \tag{11}$$

Replacing z with iz in Eq. 11 gives

$$\left(1+z^2\right)\mathscr{U}''_\alpha(iz)=3z\,i\mathscr{U}'_\alpha(iz)-\alpha(\alpha+2)\mathscr{U}_\alpha(iz) \tag{12}$$

Differentiating Eq. 4 gives

$$\mathscr{U}'_\alpha(iz)=i^{\alpha-1}\mathscr{P}'_{\alpha+1}(z) \tag{13}$$

$$\mathscr{U}''_\alpha(iz)=-i^\alpha\,\mathscr{P}''_{\alpha+1}(z) \tag{14}$$

Using Eqns. 13 and 14 in Eq. 12 and proceeding as above in lemma 2, yields

$$\left(1+z^2\right)\mathscr{P}''_\alpha(z)=\alpha(\alpha+2)\mathscr{P}_{\alpha+1}(z)-3z\;\mathscr{P}'_{\alpha+1}(z)$$

$$\therefore\;\mathscr{P}''_\alpha(z)=\frac{\alpha(\alpha+2)}{(1+z^2)}\mathscr{P}_{\alpha+1}(z)-\frac{3z}{(1+z^2)}\mathscr{P}'_{\alpha+1}(z)$$

This proves the lemma.

Lemma 4

For any integer $\alpha\;\geq r>0$, the following identity holds

$$\mathscr{P}^r_{\alpha+1}(z)=-\frac{1}{(1+z^2)}(2r-1)z\;\;\mathscr{P}^{r-1}_{\alpha+1}(z)+\;((r-2)r-\alpha(\alpha+2))\;\;\mathscr{P}^{r-2}_{\alpha+1}(z)\,,$$

Proof

From lemma 2 and lemma 3,

$$\left(1+z^2\right)\mathscr{P}'_{\alpha+1}(z)=(\alpha+1)\mathscr{P}_\alpha(z)+\alpha z\;\mathscr{P}_{\alpha+1}(z) \tag{15}$$

$$\left(1+z^2\right)\mathscr{P}''_{\alpha}(z)=\alpha(\alpha+2)\mathscr{P}_{\alpha+1}(z)-3z\,\mathscr{P}'_{\alpha+1}(z) \tag{16}$$

Differentiating Eq. 16 $(r-2)$ times and using Eq. 15, yields

$$\left(1+z^2\right)\mathscr{P}^{r}_{\alpha+1}(z)=-(2r-1)z\,\mathscr{P}^{r-1}_{\alpha+1}(z)-((r-2)r-\alpha(\alpha+2))\mathscr{P}^{r-2}_{\alpha+1}(z)$$

$$\mathscr{P}^{r}_{\alpha+1}(z)=-\frac{1}{(1+z^2)}\left[(2r-1)z\,\mathscr{P}^{r-1}_{\alpha+1}(z)+((r-2)r-\alpha(\alpha+2))\mathscr{P}^{r-2}_{\alpha+1}(z)\right]$$

This proves the lemma.

Lemma 5

For any non-negative integer α and r,

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{P}_{d_1+1}(z)\cdot\mathscr{P}_{d_2+1}(z)\cdots\mathscr{P}_{d_{r+1}+1}(z)=\frac{1}{2^r r!}\mathscr{P}^{r}_{\alpha+r+1}(z)$$

where sum runs over all non-negative integers $d_1,d_2,\cdots,d_{r+1}$ such that $d_1+d_2+\cdots+d_{r+1}=\alpha$.

Proof

Using Eq. 4 in Eq. 5 easily establishes the lemma.

Theorem 1

For any non-negative integer$\alpha\ \geq r>0$,

$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{P}_{d_1+1}(z)\cdot\mathscr{P}_{d_2+1}(z)\cdots\mathscr{P}_{d_{r+1}+1}(z)$

$$=-\frac{1}{2^r r!\,(1+z^2)}\left[(2r-1)z\,\mathscr{P}^{r-1}_{\alpha+r+1}(z)+((r-2)r-(\alpha+r)(\alpha+r+2))\mathscr{P}^{r-2}_{\alpha+r+1}(z)\right]$$

where sum runs over all non-negative integers $d_1,d_2,\cdots,d_{r+1}$ such that $d_1+d_2+\cdots+d_{r+1}=\alpha$.

Proof

By virtue of lemma 4 and lemma 5,

$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{P}_{d_1+1}(z)\cdot\mathscr{P}_{d_2+1}(z)\cdots\mathscr{P}_{d_{r+1}+1}(z)$

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{P}_{d_1+1}(z)\cdot\mathscr{P}_{d_2+1}(z)\cdots\mathscr{P}_{d_{r+1}+1}(z)$$

$$=-\frac{1}{2^r\ r!\,(1+z^2)}\left((2r-1)z\ \ \mathscr{P}^{r-1}_{\alpha+r+1}(z)+\ ((r-2)\,r-(\alpha+r)\,(\alpha+r+2))\ \ \mathscr{P}^{r-2}_{\alpha+r+1}(z)\right]$$

Theorem 2

For any non-negative integers$\alpha\ \geq r>0$,

$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{F}_{d_1+1}(z)\cdot\mathscr{F}_{d_2+1}(z)\cdots\mathscr{F}_{d_{r+1}+1}(z)$

$$=\frac{(-1)^{\alpha}}{2^{r-1}\ r!\,(z^2+4)}\left((2r-1)\,z\ \ \mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{z}{2}\right)-\ 2\,((r-2)\,r-(\alpha+r)\,(\alpha+r+2))\ \ \mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{z}{2}\right)\right],$$

where sum runs over all non-negative integers $d_1,d_2,\cdots,d_{r+1}$ such that $d_1+d_2+\cdots+d_{r+1}=\alpha$.

Proof

Replacing z with $-\frac{z}{2}$ in theorem 1, gives

$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{P}_{d_1+1}\left(-\frac{z}{2}\right)\cdot\mathscr{P}_{d_2+1}\left(-\frac{z}{2}\right)\cdots\mathscr{P}_{d_{r+1}+1}\left(-\frac{z}{2}\right)$

$$=-\frac{1}{2^r r!\left(1+\left(-\frac{z}{2}\right)^2\right)}\left[(2r-1)\left(-\frac{z}{2}\right)\mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{z}{2}\right)+((r-2)r-(\alpha+r)(\alpha+r+2))\mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{z}{2}\right)\right] \tag{17}$$

Now, replacing z with $\frac{z}{2}$ in eq. (4) and using $F_\alpha(z) = i^{\alpha-1}\ U_{\alpha-1}\ \left(-\frac{zi}{2}\right)$ yields

$$\mathscr{P}_{\alpha+1}\left(-\frac{Z}{2}\right) = (-1)^\alpha \mathscr{F}_{\alpha+1}(z) \tag{18}$$

Using Eq. 18 in Eq. 17 yields
$\therefore \sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{F}_{d_1+1}(z)\cdot\mathscr{F}_{d_2+1}(z)\cdots\mathscr{F}_{d_{r+1}+1}(z)$

$$= \frac{(-1)^{\alpha+1}}{2^r r!\left(1+\left(-\frac{z}{2}\right)^2\right)}\left[(2r-1)\left(-\frac{z}{2}\right)\mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{z}{2}\right)+((r-2)r-(\alpha+r)(\alpha+r+2))\mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{z}{2}\right)\right]$$
$$= \frac{(-1)^{\alpha+2}2^2}{2^{r+1}r!\,(4+z^2)}\left[(2r-1)z\,\mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{z}{2}\right)-2((r-2)r-(\alpha+r)(\alpha+r+2))\mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{z}{2}\right)\right]$$
$$= \frac{(-1)^{\alpha}}{2^{r-1}r!\,(z^2+4)}\left[(2r-1)z\,\mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{z}{2}\right)-2((r-2)r-(\alpha+r)(\alpha+r+2))\mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{z}{2}\right)\right]$$

$\therefore \sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{F}_{d_1+1}(z)\cdot\mathscr{F}_{d_2+1}(z)\cdots\mathscr{F}_{d_{r+1}+1}(z)$ $\therefore \sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{F}_{d_1+1}(z)\cdot\mathscr{F}_{d_2+1}(z)\cdots\mathscr{F}_{d_{r+1}+1}(z)$

$$= \frac{(-1)^{\alpha}}{2^{r-1}\ r!\,(z^2+4)}\left((2r-1)\,z\ \ \mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{z}{2}\right)-\ 2\,((r-2)\,r-(\alpha+r)\,(\alpha+r+2))\ \ \mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{z}{2}\right)\right]$$

This establishes the theorem.

Theorem 3

For any non-negative integer$\alpha\ \geq r > 0$,

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{F}_{2(d_1+1)}\cdot\mathscr{F}_{2(d_2+1)}\cdots\mathscr{F}_{2(d_{r+1}+1)}$$
$$= -\frac{i^\alpha}{2^{r-1}\cdot 5\cdot r!}\left[3i(2r-1)\mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{3i}{2}\right)+2((\alpha+r)(\alpha+r+2)-(r-2)r)\mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{3i}{2}\right)\right]$$
$$= \frac{1}{2^{r-1}\cdot 5\cdot i^\alpha\cdot r!}\left[3i(2r-1)\mathscr{P}^{r-1}_{\alpha+r+1}\left(\frac{3i}{2}\right)+2((r-2)r-(\alpha+r)(\alpha+r+2))\mathscr{P}^{r-2}_{\alpha+r+1}\left(\frac{3i}{2}\right)\right]$$

where sum runs over all non-negative integers $(d_1,d_2,\cdots,d_{r+1})$ such that $d_1+d_2+\cdots+d_{r+1}=\alpha$ with $\begin{pmatrix} r+1 \\ j \end{pmatrix}$for $j > r+1$.

Proof

Taking $z = -\frac{3i}{2}$ in theorem 1 gives
$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{P}_{d_1+1}\left(-\frac{3i}{2}\right)\cdot\mathscr{P}_{d_2+1}\left(-\frac{3i}{2}\right)\cdots\mathscr{P}_{d_{r+1}+1}\left(-\frac{3i}{2}\right)$

$$= -\frac{1}{2^r\ r!\left(1+\left(-\frac{3i}{2}\right)^2\right)}\ \left(\ (2r-1)\left(-\frac{3i}{2}\right)\ \ \mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{3i}{2}\right)\ +\ ((r-2)\,r-(\alpha+r)\,(\alpha+r+2))\ \ \mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{3i}{2}\right)\ \right]$$
$$+((r-2)r-(\alpha+r)(\alpha+r+2))\mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{3i}{2}\right)\Bigg]$$
$$= -\frac{1}{2^{r-1}r!5}\left[3i(2r-1)\mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{3i}{2}\right)-2((r-2)r-(\alpha+r)(\alpha+r+2))\mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{3i}{2}\right)\right]$$

Now using lemma 1(i) yields

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{F}_{2(d_1+1)}\cdot\mathscr{F}_{2(d_2+1)}\cdots\mathscr{F}_{2(d_{r+1}+1)}$$

$$= -\frac{i^\alpha}{2^{r-1}.5\ .r!}\left(3i\ \ (2r-1)\ \ \mathscr{P}^{r-1}_{\alpha+r+1}\left(-\frac{3i}{2}\right)+\ 2\,((\alpha+r)\,(\alpha+r+2)-(r-2)\,r)\ \ \mathscr{P}^{r-2}_{\alpha+r+1}\left(-\frac{3i}{2}\right)\right]$$

Again taking $z = \frac{3i}{2}$ in theorem 1 and using lemma 1(ii) and proceeding as above yields
$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{F}_{2(d_1+1)} \cdot \mathscr{F}_{2(d_2+1)} \cdots \mathscr{F}_{2(d_{r+1}+1)}$

$$= \frac{1}{2^{r-1} \cdot 5 \cdot i^{\alpha} \cdot r!} \left[3i(2r-1) \mathscr{P}^{r-1}_{\alpha+r+1} \left(\frac{3i}{2} \right) + 2((r-2)r - (\alpha+r)(\alpha+r+2)) \mathscr{P}^{r-2}_{\alpha+r+1} \left(\frac{3i}{2} \right) \right]$$

Thus, the theorem is established.

Theorem 4

For any non-negative integer $\alpha \geq r > 0$,
$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{F}_{3(d_1+1)} \cdot \mathscr{F}_{3(d_2+1)} \cdots \mathscr{F}_{3(d_{r+1}+1)}$

$$= \frac{2}{i^n 5.r!} \left(2\,(2r-1)\ \mathscr{P}^{r-1}_{\alpha+r+1} (-2) + \ ((\alpha+r)\,(\alpha+r+2) - (r-2)\,r)\ \mathscr{P}^{r-2}_{\alpha+r+1} (-2) \right],$$

where sum runs over all non-negative integers $(d_1, d_2, \cdots, d_{r+1})$ such that $d_1 + d_2 + \cdots + d_{r+1} = \alpha$ with $\begin{pmatrix} r+1 \\ j \end{pmatrix}$ for $j > r+1$.

Proof

Taking $z = -2$ in theorem 1 gives
$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{P}_{d_1+1} (-2) \cdot \mathscr{P}_{d_2+1} (-2) \cdots \mathscr{P}_{d_{r+1}+1} (-2)$

$$= -\frac{1}{2^r r!\,(1+(-2)^2)} \left[(2r-1)(-2) \mathscr{P}^{r-1}_{\alpha+r+1}(-2) + ((r-2)r - (\alpha+r)(\alpha+r+2)) \mathscr{P}^{r-2}_{\alpha+r+1}(-2) \right]$$
$$= \frac{1}{2^r\ r!5} \left(2\,(2r-1)\ \mathscr{P}^{r-1}_{\alpha+r+1} (-2) + ((\alpha+r)\,(\alpha+r+2) - (r-2)\,r)\ \mathscr{P}^{r-2}_{\alpha+r+1} (-2) \right]$$

Now using lemma 1(iii) yields
$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{F}_{3(d_1+1)} \cdot \mathscr{F}_{3(d_2+1)} \cdots \mathscr{F}_{3(d_{r+1}+1)}$

$$= \frac{2}{i^n 5.r!} \left(2\,(2r-1)\ \mathscr{P}^{r-1}_{\alpha+r+1} (-2) + \ ((\alpha+r)\,(\alpha+r+2) - (r-2)\,r)\ \mathscr{P}^{r-2}_{\alpha+r+1} (-2) \right]$$

Hence, the theorem is established.

Theorem 5

For any non-negative integer $\alpha \geq r > 0$,
$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{V}_{d_1} (iz) \cdot \mathscr{V}_{d_2} (iz) \cdots \mathscr{V}_{d_{r+1}} (iz)$

$$= \frac{1}{2^r\ r!\,(1+z^2)} \sum_{j=0}^{\alpha} (-1)^{j+1} i^{\alpha-j} \binom{r+1}{j} \left((2r-1)\,z\ \mathscr{P}^{r-1}_{\alpha-j+r+1} (z) - (\alpha-j+2)\,(\alpha-j+2r)\ \mathscr{P}^{r-2}_{\alpha-j+r+1} (z) \right],$$

where sum runs over all non-negative integers $(d_1, d_2, \cdots, d_{r+1})$ such that $d_1 + d_2 + \cdots + d_{r+1} = \alpha$ with $\begin{pmatrix} r+1 \\ j \end{pmatrix}$ for $j > r+1$ and $i = \sqrt{-1}$.

Proof

Replacing z with iz in Eq. 6, gives

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{V}_{d_1}(iz) \cdot \mathscr{V}_{d_2}(iz) \cdots \mathscr{V}_{d_{r+1}}(iz) = \frac{1}{2^r r!} \sum_{j=0}^{\alpha} (-1)^j \begin{pmatrix} r+1 \\ j \end{pmatrix} \mathscr{U}^r_{\alpha-j+r}(iz) \tag{19}$$

Differentiating Eq. 4 w.r.t z yields

$$\mathscr{U}^r_{\alpha}(iz) = i^{\alpha-r} \mathscr{P}_{\alpha+1}(z) \tag{20}$$

Using Eq. 20 in Eq. 19 gives

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{V}_{d_1}(iz) \cdot \mathscr{V}_{d_2}(iz) \cdots \mathscr{V}_{d_{r+1}}(iz) = \frac{1}{2^r r!} \sum_{j=0}^{\alpha} (-1)^j i^{\alpha-j} \begin{pmatrix} r+1 \\ j \end{pmatrix} \mathscr{P}^r_{\alpha-j+r+1}(z) \tag{21}$$

Using lemma 4 in Eq. 21 yields

$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{V}_{d_1}(iz)\cdot\mathscr{V}_{d_2}(iz)\cdots\mathscr{V}_{d_{r+1}}(iz)$

$= \frac{1}{2^r\, r!\left(1+z^2\right)}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\binom{r+1}{j}\left[(2r-1)z\ \mathscr{P}^{r-1}_{\alpha-j+r+1}(z)+\left((r-2)r-(\alpha-j+2)(\alpha-j+r+2)\right)\ \mathscr{P}^{r-2}_{\alpha-j+r+1}(z)\right]$

$= \frac{1}{2^r\, r!\left(1+z^2\right)}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\binom{r+1}{j}\left((2r-1)z\ \mathscr{P}^{r-1}_{\alpha-j+r+1}(z)-(\alpha-j+2)(\alpha-j+2r)\ \mathscr{P}^{r-2}_{\alpha-j+r+1}(z)\right]$

$$\therefore \sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{V}_{d_1}(iz)\cdot\mathscr{V}_{d_2}(iz)\cdots\mathscr{V}_{d_{r+1}}(iz)$$

$$= \frac{1}{2^r r!\,(1+z^2)}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\begin{pmatrix} r+1 \\ j\end{pmatrix}\Big[(2r-1)z\,\mathscr{P}^{r-1}_{\alpha-j+r+1}(z)$$

$$-(\alpha-j+2)(\alpha-j+2r)\mathscr{P}^{r-2}_{\alpha-j+r+1}(z)\Big]$$

which establishes the theorem.

Theorem 6

For any non-negative integer $\alpha\ \geq r>0$,

$$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{W}_{d_1}(iz)\cdot\mathscr{W}_{d_2}(iz)\cdots\mathscr{W}_{d_{r+1}}(iz)$$

$$= -\frac{1}{2^r r!\,(1+z^2)}\sum_{j=0}^{\alpha}i^{\alpha-j}\begin{pmatrix} r+1 \\ j\end{pmatrix}\Big[(2r-1)z\,\mathscr{P}^{r-1}_{\alpha-j+r+1}(z)-(\alpha-j+2)(\alpha-j+2r)\mathscr{P}^{r-2}_{\alpha-j+r+1}(z)\Big],$$

where all sum runs over all non-negative integers $(d_1,d_2,\cdots,d_{r+1})$ such that $d_1+d_2+\cdots+d_{r+1}=\alpha$ with $\begin{pmatrix} r+1 \\ j\end{pmatrix}$ for $j>$ *r+1* and $i=\sqrt{-1}$.

Proof

Using Eq. 7 and proceeding as above in Theorem 5 establishes the theorem.

Theorem 7

For any non-negative integer $\alpha\ \geq r>0$,

$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{F}_{2d_1+1}\cdot\mathscr{F}_{2d_2+1}\cdots\mathscr{F}_{2d_{r+1}+1}$

$$= \frac{1}{2^{r-1}.5.r!}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\left(\frac{r+1}{j}\right)\left(3i(2r-1)\,\mathscr{P}^{r-1}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right)-2(\alpha-j+2)(\alpha-j+2r)\ \mathscr{P}^{r-2}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right)\right],$$

where sum runs over all non-negative integers $(d_1,d_2,\cdots,d_{r+1})$ such that $d_1+d_2+\cdots+d_{r+1}=\alpha$ with $\begin{pmatrix} r+1 \\ j\end{pmatrix}$ for $j>$ *r+1* and $i=\sqrt{-1}$.

Proof

Replacing z by $z=-\frac{3}{2}i$ in theorem 5, gives

$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{V}_{d_1}\left(\frac{3}{2}\right)\cdot\mathscr{V}_{d_2}\left(\frac{3}{2}\right)\cdots\mathscr{V}_{d_{r+1}}\left(\frac{3}{2}\right)$

$$= \frac{1}{2^r r!\left(1+\left(-\frac{3}{2}i\right)^2\right)}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\left(\frac{r+1}{j}\right)\left((2r-1)\left(-\frac{3}{2}i\right)\ \mathscr{P}^{r-1}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right)+(\alpha-j+2)(\alpha-j+2r)\ \mathscr{P}^{r-2}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right)\right]$$

$$= \frac{1}{2^{r-1}.5.r!}\sum_{j=0}^{\alpha}(-1)^{j+1}\quad i^{\alpha-j}\left(\frac{r+1}{j}\right)\left(3i\ (2r-1)\ \mathscr{P}^{r-1}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right)-2\ (\alpha-j+2)\ (\alpha-j+2r)\ \mathscr{P}^{r-2}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right.\right.$$

Using lemma 1(iv) yields

$\therefore \sum_{d_1+d_2+\cdots+d_{r+1}=\alpha}\mathscr{F}_{2d_1+1}\cdot\mathscr{F}_{2d_2+1}\cdots\mathscr{F}_{2d_{r+1}+1}$

$$= \frac{1}{2^{r-1}.5.r!}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\left(\frac{r+1}{j}\right)\left(3i(2r-1)\,\mathscr{P}^{r-1}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right)-2(\alpha-j+2)(\alpha-j+2r)\ \mathscr{P}^{r-2}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right)\right]$$

Hence, the theorem is established.

Theorem 8

For any non-negative integer $\alpha \geq r > 0$,

$\sum_{d_1+d_2+\cdots+d_{r+1}=\alpha} \mathscr{L}_{2d_1+1} \cdot \mathscr{L}_{2d_2+1} \cdots \mathscr{L}_{2d_{r+1}+1}$

$$= -\frac{1}{2^{r-1}.5.r!} \sum_{j=0}^{\alpha} i^{\alpha-j} \binom{r+1}{j} \left(3i\,(2r-1)\,\mathscr{P}^{r-1}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right) - 2\,(\alpha-j+2)\,(\alpha-j+2r)\ \mathscr{P}^{r-2}_{\alpha-j+r+1}\left(-\frac{3}{2}i\right)\right],$$

where all sum runs over all non-negative integers $(d_1, d_2, \cdots, d_{r+1})$ such that $d_1+d_2+\cdots+d_{r+1} = \alpha$ with $\binom{r+1}{j}$ for $j >$ *r+1* and $i = \sqrt{-1}$.

Proof

In theorem 6, replacing z with $z = -\frac{3}{2}i$ and proceeding as in theorem 7 yields the desired result.

Corollary 1

For any non-negative integer α, the following identities hold:

$$\sum_{a+b+c=\alpha} \mathscr{P}_{a+1}(z) \cdot \mathscr{P}_{b+1}(z) \,.\, \mathscr{P}_{c+1}(z) = A_\alpha(z)\ \mathscr{P}_{\alpha+3}(z) - B_\alpha(z)\,\mathscr{P}_{\alpha+2}(z),$$

$$\sum_{a+b+c+d=\alpha} \mathscr{P}_{a+1}(z) \cdot \mathscr{P}_{b+1}(z) \,.\, \mathscr{P}_{c+1}(z) \,.\, \mathscr{P}_{d+1}(z) = C_\alpha(z)\ \mathscr{P}_{\alpha+3}(z) + D_\alpha(z)\ \mathscr{P}_{\alpha+4}(z),$$

where

$$A_\alpha(z) = \frac{(\alpha+2)}{8(1+z^2)^2}\left((\alpha+1)\,z^2 + (\alpha+4)\right], \quad B_\alpha(z) = \frac{3z\,(\alpha+3)}{8(1+z^2)^2},$$

$$C_\alpha(z) = \frac{(\alpha+4)}{48(1+z^2)^3}\left(\left(\alpha^2+8\alpha+27\right)z^2 + \left(\alpha^2+8\alpha+12\right)\right],$$

$$D_\alpha(z) = \frac{(\alpha+3)\,z}{48(1+z^2)^3}\ \left(\left(\alpha^2+3\alpha+2\right)z^2 + \left(\alpha^2+3\alpha-13\right)\right].$$

Proof

Taking $r = 2$ in theorem 1 gives

$\sum_{a+b+c=\alpha} \mathscr{P}_{a+1}(z) \cdot \mathscr{P}_{b+1}(z) \,.\, \mathscr{P}_{c+1}(z)$

$$= -\frac{1}{8\,(1+z^2)}\left(3z\mathscr{P}'_{\alpha+3}(z) - (\alpha+2)\,(\alpha+4)\,\mathscr{P}_{\alpha+3}(z)\right]$$

$$= \frac{1}{8\,(1+z^2)}\left((\alpha+2)\,(\alpha+4)\,\mathscr{P}_{\alpha+3}(z) - 3z\mathscr{P}'_{\alpha+3}(z)\right]$$

$$= \frac{1}{8\,(1+z^2)}(\alpha+2)\,(\alpha+4)\,\mathscr{P}_{\alpha+3}(z) - \frac{3z}{8\,(1+z^2)}\left(\frac{(\alpha+3)}{(1+z^2)}\mathscr{P}_{\alpha+2}(z) + \frac{(\alpha+2)}{(1+z^2)}\,z\,\mathscr{P}_{\alpha+3}(z)\right]$$

$$= \frac{(\alpha+2)}{8\,(1+z^2)}\left((\alpha+4) - \frac{3z^2}{(1+z^2)}\right]\ \mathscr{P}_{\alpha+3}(z) - \frac{3z\,(\alpha+3)}{8\,(1+z^2)^2}\mathscr{P}_{\alpha+2}(z)$$

$$= \frac{(\alpha+2)}{8\left(1+z^2\right)^2}\left((\alpha+1)z^2+(\alpha+4)\right]\ \mathscr{P}_{\alpha+3}(z) - \frac{3z(\alpha+3)}{8\left(1+z^2\right)^2}\mathscr{P}_{\alpha+2}(z)$$

$$\therefore \sum_{a+b+c=\alpha} \mathscr{P}_{a+1}(z)\cdot\mathscr{P}_{b+1}(z)\cdot\mathscr{P}_{c+1}(z) = A_\alpha(z)\mathscr{P}_{\alpha+3}(z) - B_\alpha(z)\mathscr{P}_{\alpha+2}(z)$$

$$where\ A_\alpha(z) = \frac{(\alpha+2)}{8(1+z^2)^2}\left((\alpha+1)z^2+(\alpha+4)\right],\quad B_\alpha(z) = \frac{3z(\alpha+3)}{8(1+z^2)^2}.$$

Taking $r = 3$ in Theorem 1(i) gives
$\sum_{a+b+c+d=\alpha} \mathscr{P}_{a+1}(z)\cdot\mathscr{P}_{b+1}(z).\mathscr{P}_{c+1}(z).\mathscr{P}_{d+1}(z)$

$$= -\frac{1}{48(1+z^2)}\left(5z\mathscr{P}''_{\alpha+4}(z) + (3-(\alpha+3)(\alpha+5))\,\mathscr{P}'_{\alpha+4}(z)\right]$$

$$= \frac{1}{48(1+z^2)}\left((\alpha+3)(\alpha+5)-3\right]\mathscr{P}'_{\alpha+4}(z) - 5z\mathscr{P}''_{\alpha+4}(z)$$

$$= \frac{1}{48(1+z^2)}\left(((\alpha+3)(\alpha+5)-3) + \frac{15z^2}{(1+z^2)}\right]\mathscr{P}'_{\alpha+4}(z) - \frac{5z\ (\alpha+3)(\alpha+5)}{48\ (1+z^2)^2}\mathscr{P}_{\alpha+4}(z)$$

$$= \frac{1}{48(1+z^2)}\left((\alpha+3)(\alpha+5)-3\right]\mathscr{P}'_{\alpha+4}(z) - \frac{5z}{48(1+z^2)}\left(\frac{(\alpha+3)(\alpha+5)}{(1+z^2)}\mathscr{P}_{\alpha+4}(z) - \frac{3z}{(1+z^2)}\ \mathscr{P}'_{\alpha+4}(z)\right]$$

$$= \frac{1}{48(1+z^2)}\left(((\alpha+3)(\alpha+5)-3) + \frac{15z^2}{(1+z^2)}\right]\left(\frac{(\alpha+4)}{(1+z^2)}P_{\alpha+3}(z) + \frac{(\alpha+3)z}{(1+z^2)}\ P_{\alpha+4}(z)\right] - \frac{5z\ (\alpha+3)(\alpha+5)}{48\ (1+z^2)^2}P_{\alpha+4}(z)$$

$$= \frac{(\alpha+4)}{48(1+z^2)^3}\left(((\alpha+3)(\alpha+5)-3)\left(1+z^2\right)+15z^2\right]\ \mathscr{P}_{\alpha+3}(z) + \frac{(\alpha+3)z}{48(1+z^2)^3}\ \left(((\alpha+3)(\alpha+5)-3)\left(1+z^2\right)+15z^2-5\ (\alpha+5)\left(1+z^2\right)\right.$$

$$= \frac{(\alpha+4)}{48(1+z^2)^3}\left(\left(\alpha^2+8\alpha+27\right)z^2+\left(\alpha^2+8\alpha+12\right)\right]\ \mathscr{P}_{\alpha+3}(z) + \frac{(\alpha+3)z}{48(1+z^2)^3}\ \left(\left(\alpha^2+3\alpha+2\right)z^2+\left(\alpha^2+3\alpha-13\right)\right]\mathscr{P}_{\alpha+4}(z)$$

$$\therefore \sum_{a+b+c+d=\alpha} \mathscr{P}_{a+1}(z)\cdot\mathscr{P}_{b+1}(z).\mathscr{P}_{c+1}(z).\mathscr{P}_{d+1}(z) = C_\alpha(z)\ \mathscr{P}_{\alpha+3}(z) + D_\alpha(z)\ \mathscr{P}_{\alpha+4}(z)$$

$$where\ C_\alpha(z) = \frac{(\alpha+4)}{48(1+z^2)^3}\left(\left(\alpha^2+8\alpha+27\right)z^2+\left(\alpha^2+8\alpha+12\right)\right],$$

$$D_\alpha(z) = \frac{(\alpha+3)z}{48(1+z^2)^3}\ \left(\left(\alpha^2+3\alpha+2\right)z^2+\left(\alpha^2+3\alpha-13\right)\right]$$

Hence, the Corollary is established.

Corollary 2. For any non-negative integer $\alpha > 0$, the following identities hold:
$\sum_{a+b+c=\alpha} \mathscr{F}_{a+1}(z)\cdot\mathscr{F}_{b+1}(z).\mathscr{F}_{c+1}(z)$

$$= \frac{(\alpha+2)}{2\left(z^2+4\right)^2}\left((\alpha+1)z^2+4(\alpha+4)\right]\mathscr{F}_{\alpha+3}(z) - \frac{3z(\alpha+3)}{\left(z^2+4\right)^2}\ \mathscr{F}_{\alpha+2}(z)\ .$$

$\sum_{a+b+c=\alpha} \mathscr{F}_{a+1}(z) \cdot \mathscr{F}_{b+1}(z) .\mathscr{F}_{c+1}(z) \mathscr{F}_{d+1}(z)$.

$$= \frac{(\alpha+4)}{3(z^2+4)^3}\left[\left(\alpha^2+8\alpha+27\right)z^2+4\left(\alpha^2+8\alpha+12\right)\right]\left(\alpha^2+8\alpha+27\right)z^2+4\left(\alpha^2+8\alpha+12\right)\mathscr{F}_{\alpha+3}(z)+\frac{(\alpha+3)\,z}{6(z^2+4)^3}\left(\left(\alpha^2+3\alpha+2\right)z\right.$$

Proof

Taking $r=2$ in theorem 2, gives

$\sum_{a+b+c=a} \mathscr{F}_{a+1}(z) \cdot \mathscr{F}_{b+1}(z) \cdot \mathscr{F}_{c+1}(z)$

$$= \frac{(-1)^{\alpha}}{4(z^2+4)}\left[\ 3z \quad \mathscr{P}'_{\alpha+3}\left(-\frac{z}{2}\right)+2(\alpha+2)(\alpha+4)\mathscr{P}_{\alpha+3}\left(-\frac{z}{2}\right)\right]\ \Big] \tag{22}$$

Using lemma 2 yields

$$\mathscr{P}'_{\alpha+3}\left(-\frac{z}{2}\right)=\frac{4(\alpha+3)}{(z^2+4)}\mathscr{P}_{\alpha+2}\left(-\frac{z}{2}\right)-\frac{2(\alpha+2)z}{(z^2+4)}\mathscr{P}_{\alpha+3}\left(-\frac{z}{2}\right) \tag{23}$$

By using Eq. 23 in 22 yields

$\sum_{a+b+c=a} \mathscr{F}_{a+1}(z) \cdot \mathscr{F}_{b+1}(z) \cdot \mathscr{F}_{c+1}(z)$

$$= \frac{(-1)^{\alpha}}{4\ \ (z^2+4)}\left(3z\left(\frac{4\ (\alpha+3)}{(z^2+4)}\mathscr{P}_{\alpha+2}\left(-\frac{z}{2}\right)-\frac{2(\alpha+2)z}{(z^2+4)}\ \mathscr{P}_{\alpha+3}\left(-\frac{z}{2}\right)\right)\ +\ 2(\alpha+2)(\alpha+4)\ \ \mathscr{P}_{\alpha+3}\left(-\frac{z}{2}\right)\right]$$

$$= \frac{(-1)^{\alpha}}{4\ \ (z^2+4)}\left(\frac{12z\ (\alpha+3)}{(z^2+4)}\mathscr{P}_{\alpha+2}\left(-\frac{z}{2}\right)-2(\alpha+2)\left(\frac{3z^2}{(z^2+4)}-(\alpha+4)\ \right)\ \ \mathscr{P}_{\alpha+3}\left(-\frac{z}{2}\right)\right]$$

$$= \frac{(-1)^{\alpha}}{4\ \ (z^2+4)}\left(\frac{12z\ (\alpha+3)}{(z^2+4)}\mathscr{P}_{\alpha+2}\left(-\frac{z}{2}\right)-2(\alpha+2)\left(\frac{3z^2}{(z^2+4)}-(\alpha+4)\ \right)\ \ \mathscr{P}_{\alpha+3}\left(-\frac{z}{2}\right)\right]$$

Now using Eq. 18 gives

$$= -\frac{1}{2\ (z^2+4)^2}\left(6z\ (\alpha+3)\ \mathscr{F}_{\alpha+2}(z)-(\alpha+2)\left((\alpha+1)z^2+4(\alpha+4)\right]\mathscr{F}_{\alpha+3}(z)\right\}$$

$$= \frac{(\alpha+2)}{2(z^2+4)^2}\left((\alpha+1)z^2+4(\alpha+4)\right]\mathscr{F}_{\alpha+3}(z)-\frac{3z(\alpha+3)}{(z^2+4)^2}\mathscr{F}_{\alpha+2}(z)$$

Again taking $r=3$ in theorem 2, and using lemma 2 and lemma 3 and using eq. (18) and proceeding as above yields

$$\sum_{a+b+c=\alpha} \mathscr{F}_{a+1}(z) \cdot \mathscr{F}_{b+1}(z) .\mathscr{F}_{c+1}(z) \cdot \mathscr{F}_{d+1}(z)$$

$$= \frac{(\alpha+4)}{3(z^2+4)^3}\left(\left(\alpha^2+8\alpha+27\right)z^2+4\left(\alpha^2+8\alpha+12\right)\right]\mathscr{F}_{\alpha+3}(z)+\frac{(\alpha+3)\ z}{6(z^2+4)^3}\left(\left(\alpha^2+3\alpha+2\right)z^2+4\left(\alpha^2+3\alpha-13\right)\right]\ \mathscr{F}_{\alpha+4}(z)$$

This establishes the corollary.

Corollary 3

For any non-negative integer $\alpha\ >0$, the following identities hold:

$$\sum_{a+b+c=\alpha} \mathscr{F}_{2(a+1)} \cdot \mathscr{F}_{2(b+1)} .\mathscr{F}_{2(c+1)} = \frac{1}{50}\left(18\ (\alpha+3)\mathscr{F}_{2\alpha+4}+(\alpha+2)(5\alpha-7)\ \mathscr{F}_{2\alpha+6}\right] .$$

$$\sum_{a+b+c+d=\alpha} \mathscr{F}_{2(a+1)} \cdot \mathscr{F}_{2(b+1)} .\mathscr{F}_{2(c+1)} .\mathscr{F}_{2(d+1)} = \frac{1}{150}\ \left(3(\alpha+3)\left(\alpha^2+3\alpha+14\right)\ \mathscr{F}_{2\alpha+8}-2(\alpha+4)\left(\alpha^2+8\alpha+39\right)\mathscr{F}_{2\alpha+6}\right] .$$

Proof

Taking $r=2$ in theorem 3 gives
$\sum_{a+b+c=\alpha}\mathscr{F}_{2(a+1)}\cdot\mathscr{F}_{2(b+1)}\cdot\mathscr{F}_{2(c+1)}$

$$=-\frac{i^{\alpha}}{20}\left[9i\,\mathscr{P}'_{\alpha+3}\left(-\frac{3}{2}i\right)+2(\alpha+2)(\alpha+4)\mathscr{P}_{\alpha+3}\left(-\frac{3}{2}i\right)\right] \tag{24}$$

Using lemma 1,

$$\mathscr{P}'_{\alpha+3}\left(-\frac{3}{2}i\right)=-\frac{4(\alpha+3)}{5}\mathscr{P}_{\alpha+2}\left(-\frac{3}{2}i\right)+\frac{6i}{5}(\alpha+2)\mathscr{P}_{\alpha+3}\left(-\frac{3}{2}i\right) \tag{25}$$

From Eq. 25 and Eq. 24,
$\sum_{a+b+c=\alpha}\mathscr{F}_{2(a+1)}\cdot\mathscr{F}_{2(b+1)}\cdot\mathscr{F}_{2(c+1)}$

$$=-\frac{i^{\alpha}}{20}\left[9i\left(-\frac{4(\alpha+3)}{5}\mathscr{P}_{\alpha+2}\left(-\frac{3}{2}i\right)+\frac{6i}{5}(\alpha+2)\mathscr{P}_{\alpha+3}\left(-\frac{3}{2}i\right)\right)+2(\alpha+2)(\alpha+4)\mathscr{P}_{\alpha+3}\left(-\frac{3}{2}i\right)\right]$$
$$=-\frac{i^{\alpha}}{50}\left(-18i\,(\alpha+3)\;\mathscr{P}_{\alpha+2}\left(-\frac{3}{2}\,i\right)+(\alpha+2)\,(5\alpha-7)\,\mathscr{P}_{\alpha+3}\left(-\frac{3}{2}\,i\right)\right]$$

Using lemma 1(i) leads to
$\sum_{a+b+c=\alpha}\mathscr{F}_{2(a+1)}\cdot\mathscr{F}_{2(b+1)}\cdot\mathscr{F}_{2(c+1)}$

$$=-\frac{i^{\alpha}}{50}\left(-18i\,(\alpha+3)\;\frac{\mathscr{F}_{2\alpha+4}}{i^{\alpha+1}}+(\alpha+2)\,(5\alpha-7)\;\frac{\mathscr{F}_{2\alpha+6}}{i^{\alpha+2}}\right]$$

$$=\frac{1}{50}\,(18\;(\alpha+3)\,\mathscr{F}_{2\alpha+4}+(\alpha+2)\,(5\alpha-7)\;\mathscr{F}_{2\alpha+6}]$$

$\therefore\sum_{a+b+c=\alpha}\mathscr{F}_{2(a+1)}\cdot\mathscr{F}_{2(b+1)}\cdot\mathscr{F}_{2(c+1)}$

$$=\frac{1}{50}\,(18\;(\alpha+3)\,\mathscr{F}_{2\alpha+4}+(\alpha+2)\,(5\alpha-7)\;\mathscr{F}_{2\alpha+6}]$$

Again taking $r=3$ in theorem 3, and using lemma 2 and lemma 3, with $z=-\frac{3}{2}\,i$ and proceeding as above yields

$$\sum_{a+b+c+d=\alpha}\mathscr{F}_{2(a+1)}\cdot\mathscr{F}_{2(b+1)}\cdot\mathscr{F}_{2(c+1)}\cdot\mathscr{F}_{2(d+1)}=\frac{1}{150}\;\left(3\,(\alpha+3)\left(\alpha^2+3\alpha+14\right)\;\mathscr{F}_{2\alpha+8}-2\,(\alpha+4)\left(\alpha^2+8\alpha+39\right)\mathscr{F}_{2\alpha+6}\right]$$

Hence, the corollary is established.

Corollary 4

For any non-negative integer $\alpha\;>0$, the following identities hold:

$$\sum_{a+b+c=\alpha}\mathscr{F}_{3(a+1)}\cdot\mathscr{F}_{3(b+1)}\cdot\mathscr{F}_{3(c+1)}=-\frac{1}{50}\,((\alpha+2)\,(5\alpha+8)\,\mathscr{F}_{3\alpha+9}-6i\,(\alpha+3)\;\mathscr{F}_{3\alpha+6}]\,.$$

$$\sum_{a+b+c+d=\alpha}\mathscr{F}_{3(a+1)}\cdot\mathscr{F}_{3(b+1)}\cdot\mathscr{F}_{3(c+1)}\cdot\mathscr{F}_{3(d+1)}=-\frac{1}{150}\;\left((\alpha+4)\left(\alpha^2+8\alpha+24\right)\;\mathscr{F}_{3\alpha+9}-2i\,(\alpha+3)\left(\alpha^2+3\alpha-1\right)\mathscr{F}_{3\alpha+12}\right]\,.$$

Proof

Taking $r=2$ in theorem 4 implies
$\sum_{a+b+c=\alpha}\mathscr{F}_{3(a+1)}\cdot\mathscr{F}_{3(b+1)}\cdot\mathscr{F}_{3(c+1)}$

$$=\frac{1}{5i^{\alpha}}\left[6\mathscr{P}'_{\alpha+3}(-2)+(\alpha+2)(\alpha+4)\mathscr{P}_{\alpha+3}(-2)\right] \tag{26}$$

Using lemma 2

$$\mathscr{P}'_{\alpha+3}(-2) = \frac{(\alpha+3)}{5}\mathscr{P}_{\alpha+2}(-2) - \frac{2}{5}(\alpha+2)\mathscr{P}_{\alpha+3}(-2) \tag{27}$$

From Eq. 27 and Eq. 26 with lemma 1(iii) leads to

$$\sum_{a+b+c=\alpha} \mathscr{F}_{3(a+1)} \cdot \mathscr{F}_{3(b+1)} \cdot \mathscr{F}_{3(c+1)}$$

$$= \frac{1}{5\, i^{\alpha}} \left(6\left(\frac{(\alpha+3)}{5}\, \mathscr{P}_{\alpha+2}(-2) - \frac{2}{5}\, (\alpha+2)\, \mathscr{P}_{\alpha+3}(-2)\right) + (\alpha+2)(\alpha+4)\, \mathscr{P}_{\alpha+3}(-2)\right]$$

$$= \frac{1}{5\, i^{\alpha}} \left(6\left(\frac{(\alpha+3)}{5}\, \mathscr{P}_{\alpha+2}(-2) - \frac{2}{5}\, (\alpha+2)\, \mathscr{P}_{\alpha+3}(-2)\right) + (\alpha+2)(\alpha+4)\, \mathscr{P}_{\alpha+3}(-2)\right]$$

$$= -\frac{1}{50}\,\left(\,(\alpha+2)\,(5\alpha+8)\,\mathscr{F}_{3\alpha+9} + 6i\,(\alpha+3)\,\mathscr{F}_{3\alpha+6}\right]$$

$$\therefore \sum_{a+b+c=\alpha} \mathscr{F}_{3(a+1)} \cdot \mathscr{F}_{3(b+1)} \cdot \mathscr{F}_{3(c+1)} = -\frac{1}{50}\,\left(\,(\alpha+2)\,(5\alpha+8)\,\mathscr{F}_{3\alpha+9} - 6i\,(\alpha+3)\,\mathscr{F}_{3\alpha+6}\right]$$

Again taking $r=3$ in theorem 4, and using lemma 2 and lemma 3, with $z=-2$ and proceeding as above yields

$$\sum_{a+b+c+d=\alpha} \mathscr{F}_{3(a+1)} \cdot \mathscr{F}_{3(b+1)} \cdot \mathscr{F}_{3(c+1)} \cdot \mathscr{F}_{3(d+1)} = -\frac{1}{150}\,\left((\alpha+4)\left(\alpha^2+8\alpha+24\right)\mathscr{F}_{3\alpha+9} - 2i(\alpha+3)\left(\alpha^2+3\alpha-1\right)\mathscr{F}_{3\alpha+12}\right]$$

This establishes the corollary.

Corollary 5

For any non-negative integer $\alpha \geq r > 0$, the following identities hold:

$$\sum_{a+b+c=\alpha} \mathscr{V}_a(iz) \cdot \mathscr{V}_b(iz) \cdot \mathscr{V}_c(iz)$$

$$= \frac{1}{8(1+z^2)} \sum_{j=0}^{\alpha} (-1)^{j+1} i^{\alpha-j} \binom{3}{j} \left(3z\,(\alpha-j+3)\;P_{\alpha-j+2}(z) - (\alpha-j+2)\left((\alpha-j+1)z^2 + (\alpha-j+4)\right]\;P_{\alpha-j+3}(z)\right\}.$$

$$\therefore \sum_{a+b+c=\alpha} \mathscr{F}_{3(a+1)} \cdot \mathscr{F}_{3(b+1)} \cdot \mathscr{F}_{3(c+1)} = -\frac{1}{50}\,\left(\,(\alpha+2)\,(5\alpha+8)\,\mathscr{F}_{3\alpha+9} - 6i\,(\alpha+3)\,\mathscr{F}_{3\alpha+6}\right]$$

$$= \frac{1}{48\left(1+z^2\right)^3} \left. \begin{array}{l} \sum_{j=0}^{\alpha}(-1)^{j+1} i^{\alpha-j} \binom{4}{j} \left\{(\alpha-j+3)z\left[(5(\alpha-j+5)-(\alpha-j+2)(\alpha-j+6))\left(1+z^2\right)\right.\right. \\ \left.-15z^2\right]\mathscr{P}_{\alpha-j+4}(z) - (\alpha-j+4)\left[15z^2+(\alpha-j+2)(\alpha-j+6)\left(1+z^2\right)\right]\mathscr{P}_{\alpha-j+3}(z) \end{array} \right\}.$$

Proof

Taking $r=2$ in theorem 5 gives

$$\sum_{a+b+c=\alpha} \mathscr{V}_a(iz) \cdot \mathscr{V}_b(iz) \mathscr{V}_c(iz) = \frac{1}{8(1+z^2)} \sum_{j=0}^{\alpha} (-1)^{j+1} i^{\alpha-j} \binom{3}{j} 3z\left\{\mathscr{P}'_{\alpha-j+3}(z) - (\alpha-j+2)(\alpha-j+4)\mathscr{P}_{\alpha-j+3}(z)\right\} \tag{28}$$

From lemma 2,

$$\mathscr{P}'_{\alpha-j+3}(z)=\frac{(\alpha-j+3)}{(1+z^2)}\mathscr{P}_{\alpha-j+2}(z)+\frac{(\alpha-j+2)z}{(1+z^2)}\mathscr{P}_{\alpha-j+3}(z) \tag{29}$$

Using Eq. 29 in Eq. 28, leads to

$$\begin{aligned}
&\sum_{a+b+c=\alpha}\mathscr{V}_a(iz)\cdot\mathscr{V}_b(iz)\mathscr{V}_c(iz)\\
&=\frac{1}{8\,(1+z^2)}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\begin{pmatrix}3\\ j\end{pmatrix}\left\{3z\left[\frac{(\alpha-j+3)}{(1+z^2)}\mathscr{P}_{\alpha-j+2}(z)+\frac{(\alpha-j+2)z}{(1+z^2)}\mathscr{P}_{\alpha-j+3}(z)\right]\right.\\
&\left.-(\alpha-j+2)(\alpha-j+4)\mathscr{P}_{\alpha-j+3}(z)\right\}
\end{aligned}$$

$$=\frac{1}{8\,(1+z^2)}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\binom{3}{j}\left(\frac{3z\,(\alpha-j+3)}{(1+z^2)}\,P_{\alpha-j+2}(z)+\frac{(\alpha-j+2)}{(1+z^2)}\,\left(3z^2-(\alpha-j+4)\left(1+z^2\right)\right]\,P_{\alpha-j+3}(z)\right\}$$

$$=\frac{1}{8(1+z^2)^2}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\binom{3}{j}\left(3z\,(\alpha-j+3)\,P_{\alpha-j+2}(z)-(\alpha-j+2)\,\left((\alpha-j+1)z^2+(\alpha-j+4)\right]\,P_{\alpha-j+3}(z)\right\}$$

$\therefore \sum_{a+b+c=\alpha}\mathscr{V}_a(iz)\cdot\mathscr{V}_b(iz)\cdot\mathscr{V}_c(iz)$

$$=\frac{1}{8(1+z^2)^2}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\binom{3}{j}\left(3z\,(\alpha-j+3)\,P_{\alpha-j+2}(z)-(\alpha-j+2)\,\left((\alpha-j+1)z^2+(\alpha-j+4)\right]\,P_{\alpha-j+3}(z)\right\}$$

Now, taking $r=3$ in theorem 5 results
$\sum_{a+b+c+d=\alpha}\mathscr{V}_a(iz)\cdot\mathscr{V}_b(iz)\cdot\mathscr{V}_c(iz)\cdot\mathscr{V}_d(iz)$

$$=\frac{1}{48\,(1+z^2)}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\begin{pmatrix}4\\ j\end{pmatrix}\left\{5z\,\mathscr{P}''_{\alpha-j+4}(z)-(\alpha-j+2)(\alpha-j+6)\mathscr{P}'_{\alpha-j+3}(z)\right\} \tag{30}$$

From lemma 3,

$$\mathscr{P}''_{\alpha-j+4}(z)=\frac{(\alpha-j+3)(\alpha-j+5)}{(1+z^2)}\mathscr{P}_{\alpha-j+4}(z)-\frac{3z}{(1+z^2)}\mathscr{P}'_{\alpha-j+4}(z) \tag{31}$$

Using Eq. 31 in Eq. 30 yields
$\sum_{a+b+c+d=\alpha}\mathscr{V}_a(iz)\cdot\mathscr{V}_b(iz)\cdot\mathscr{V}_c(iz)\cdot\mathscr{V}_d(iz)$

$$\begin{aligned}
&=\frac{1}{48\,(1+z^2)}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\begin{pmatrix}4\\ j\end{pmatrix}\left\{5z\left[\frac{(\alpha-j+3)(\alpha-j+5)}{(1+z^2)}\mathscr{P}_{\alpha-j+4}(z)-\frac{3z}{(1+z^2)}\mathscr{P}'_{\alpha-j+4}(z)\right]\right.\\
&\left.-(\alpha-j+2)(\alpha-j+6)\mathscr{P}'_{\alpha-j+3}(z)\right\}\\
&=\frac{1}{48\,(1+z^2)}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\begin{pmatrix}4\\ j\end{pmatrix}\left\{\frac{5z(\alpha-j+3)(\alpha-j+5)}{(1+z^2)}\mathscr{P}_{\alpha-j+4}(z)\right.\\
&\left.-\left[\frac{15z^2}{(1+z^2)}+(\alpha-j+2)(\alpha-j+6)\right]\mathscr{P}'_{\alpha-j+4}(z)\right\}
\end{aligned}$$

$$\begin{aligned}
&=\frac{1}{48\,(1+z^2)^2}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\begin{pmatrix}4\\ j\end{pmatrix}\left\{5z(\alpha-j+3)(\alpha-j+5)\mathscr{P}_{\alpha-j+4}(z)\right.\\
&\left.-\left[15z^2+(\alpha-j+2)(\alpha-j+6)\left(1+z^2\right)\right]\mathscr{P}'_{\alpha-j+4}(z)\right\}
\end{aligned}$$

Again, from lemma 2,

$$\mathscr{P}'_{\alpha-j+4}(z) = \frac{(\alpha-j+4)}{(1+z^2)}\mathscr{P}_{\alpha-j+3}(z) + \frac{(\alpha-j+3)z}{(1+z^2)}\mathscr{P}_{\alpha-j+4}(z) \tag{32}$$

Using Eq. 32, leads to

$$\sum_{a+b+c+d=\alpha} \mathscr{V}_a(iz)\cdot\mathscr{V}_b(iz)\cdot\mathscr{V}_c(iz)\cdot\mathscr{V}_d(iz)$$
$$= \frac{1}{48(1+z^2)^3}\sum_{j=0}^{\alpha}(-1)^{j+1}i^{\alpha-j}\begin{pmatrix}4\\ j\end{pmatrix}\{(\alpha-j+3)z\left[(5(\alpha-j+5)-(\alpha-j+2)(\alpha-j+6))(1+z^2)\right.$$
$$\left.-15z^2\right]\mathscr{P}_{\alpha-j+4}(z) - (\alpha-j+4)\left[15z^2+(\alpha-j+2)(\alpha-j+6)(1+z^2)\right]\}\mathscr{P}_{\alpha-j+3}(z)$$

Hence, the corollary is established.

Corollary 6

For any non-negative integer $\alpha \geq r > 0$, the following identities hold:

$$\sum_{a+b+c=\alpha}\mathscr{W}_a(iz)\cdot\mathscr{W}_b(iz)\cdot\mathscr{W}_c(iz)$$
$$= -\frac{1}{8(1+z^2)}\sum_{j=0}^{\alpha} i^{\alpha-j}\binom{3}{j}\left(3z\ (\alpha-j+3)\ \ \mathscr{P}_{\alpha-j+2}(z) - (\alpha-j+2)\left((\alpha-j+1)z^2+(\alpha-j+4)\right]\ \mathscr{P}_{\alpha-j+3}(z)\right\}.$$

$$\sum_{a+b+c+d=\alpha}\mathscr{W}_a(iz)\cdot\mathscr{W}_b(iz)\cdot\mathscr{W}_c(iz)\cdot\mathscr{W}_d(iz)$$
$$= -\frac{1}{8(1+z^2)}\sum_{j=0}^{\alpha} i^{\alpha-j}\binom{3}{j}\left(3z\ (\alpha-j+3)\ \ \mathscr{P}_{\alpha-j+2}(z) - (\alpha-j+2)\left((\alpha-j+1)z^2+(\alpha-j+4)\right]\ \mathscr{P}_{\alpha-j+3}(z)\right\}.$$

Proof

Taking $r = 2, 3$ in theorem 6, and proceeding as in corollary 5, establishes this corollary.

4 Conclusion

In this paper, the sums of the finite products of the Fibonacci and Lucas numbers, Pell and Fibonacci polynomials, and Chebyshev polynomials of the third and fourth kind were considered as a linear combination of the derivatives of the Pell polynomials. Here, various lemmas using Pell polynomials are developed, and certain relations involving the Fibonacci and Lucas numbers, Pell and Fibonacci polynomials, and Chebyshev polynomials of the third and fourth kind are deduced and utilised to achieve the objectives. These results will certainly serve as a possibility for the prospective researchers to express such sums of finite products in terms of other orthogonal polynomials. Moreover, there is a possibility of studying sums of finite products of other orthogonal polynomials as well. The control theory is an intriguing area in which these findings are useful.

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