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RESEARCH ARTICLE

# Iterative Approach for Solving Variational Inequality Problems using Fixed Point Concept

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## Abstract

**Objectives:** The objective of the paper is to find the solutions of variational inequality problems via the concept of common fixed point of a sequence of nearly nonexpansive mappings. **Methods:** The present work uses three step iterative algorithm to get the solutions of variational inequality problems. **Findings:** By applying three step iterative algorithm, solutions of variational inequality problem has been obtained. **Novelty:** In the present work, a specific three step iterative algorithm has been deployed to get solution. Furthermore, Matlab programming has been utilised to eastablish the accuracy of the results. **Mathematics Subject Classification 2020:** 47H06, 47H09, 47H10, 47J25.

**Keywords:** Variational inequality; Fixed point; Nonexpansive mapping; Iterative algorithm; Matlab programming

## 1 Introduction

Let D be a nonempty convex and bounded subset of real Hilbert space M. Variational inequalities are used in various fields, i.e., economics, optimization, game theory etc. First of all it was used to study Stampacchia [1] variational inequality problem (VIP). In that problem, we have to find out $y^* \in D$ such that

$$\langle By^*, y^* - z\rangle \leq 0, \forall z \in D$$

where $B : D \to M$ be a linear operator.

The corresponding fixed point problem to the VIP (1) is to find the fixed point of the mapping $P_D(I - \chi B)$, for all $\chi > 0$, where $P_D$ is the projection mapping from M onto D. If B is strongly monotone and Lipschitzian mapping and $\chi > 0$ is a small number then the mapping $P_D(I - \chi B)$ is a strict contraction mapping. So, problem (1) has a unique solution $y^*$, by using Banach contraction principle, and the Picard iterations $\{y_m\}$ which are defined by $y_{m+1} = P_D(I - \chi B)y_m$ converge strongly to $y^*$.

To find out the solutions of VIPs and corresponding optimization problems, equivalence relations between fixed point problems and VIPs are developed. So, the prime objective is to find the common fixed point of the nonexpansive mappings.

Moudafi[2] introduced the viscosity approximation method to determine the fixed point of a nonexpansive mapping and that mapping was defined on a Hilbert space. If M is a Hilbert space and D be a closed and convex subset of M. Also, $F: D \rightarrow D$ be a nonexpansive self-mapping and $f: M \rightarrow D$ is a strongly nonexpansive mapping. Then he proved that the sequence $(y_m\}$ in D which is defined by the iterative scheme:

$$y_0 \in D,\ \ y_m = \frac{1}{1+\alpha_m} F(y_m) + \frac{\alpha_m}{1+\alpha_m} f(y_m),\ \ \forall m \geq 0,$$

strongly converges and has the unique solution $y^* \in Fix(F)$ of VI

$$\langle (I-f)y^*, y^*-y \rangle \leq 0, \forall y \in Fix(F)$$

,

where $(\alpha_m\}$ is a sequence of positive numbers such that $\alpha_m \rightarrow 0\ as\ m \rightarrow \infty$.

Marino and Xu[3] studied the viscosity approximation method for nonexpansive mappings. They proved that the sequence $(y_m\}$ is given by

$$y_0 \in M\ and\ y_{m+1} = \alpha_m \delta f(y_m) + (I - \alpha_m B) F y_m,\ m \geq 0$$

,

strongly converges and which has the unique solution of the corresponding VI

$$\langle (B-\delta f)y^*, y-y^* \rangle \geq 0, \forall y \in \mathrm{Fix}(F)$$

that is the optimization condition for the minimization problem

$$\min_{y \in Fix(F)} \frac{1}{2}\langle By, y \rangle - \phi(y)$$

where B is the strongly positive and bounded operator on M and $\phi$ is the potential function of $\delta f$.

Yao et al.[4] introduced the iterative scheme to find out the solutions of generalized VIP and fixed point problem (FPP). Akram et al.[5] worked on FPP and split variational inclusion problem. Many researchers introduced different iterative schemes to find out the solutions of VIPs[6–11]. Lamba and Panwar[12] and Panwar et al.[13] introduced iterative schemes for nonexpansive mappings. Sahu et al.[14] and Tuyen[15] determined the solutions of VIPs by finding the common fixed point of a sequence of nearly nonexpansive mappings. The result proved by Sahu et al.[14] is as follows.

**Theorem 1.1**[14] "Let E be a non-empty, closed and convex subset of a real Hilbert space M. Let $V: E \rightarrow M$ be an L-Lipschitzian and $G: E \rightarrow M$ be a k-Lipschitzian and $\theta$-strongly monotone operator. Let $(T_m\}$ be a sequence of nearly nonexpansive mappings from E into itself with respect to the sequence $(a_m\}$ such that $F = \cap Fix(T_m) \neq \phi$ and T be a self-mapping on E such that $Tu = \lim_{m\rightarrow\infty} T_m u,\ \forall u \in E$. Let $FixT = \cap Fix(T_m\},\ 0 < \omega < 2\theta/k^2$ and $0 < \delta L < \tau$, where $\tau = 1 - \sqrt{1-\omega(2\theta - \omega k^2)}$. For $u_0 \in E$, the sequence $(u_m\}$ is defined on E and is given by the iterative algorithm as

$$\begin{cases} u_0 \in E \\ u_{m+1} = P_E[\alpha_m \delta V u_m + (I - \alpha_m \omega G) v_m] \end{cases} \tag{2}$$

where $(\alpha_m\}$ be the sequence in (0,1) and satisfy the following conditions:

(a) $\lim_{m\rightarrow\infty} \alpha_m = 0,\ \sum_{m=1}^{\infty} \alpha_m = \infty$;

(b) either $\sum_{m=1}^{\infty} (\alpha_{m+1} - \alpha_m| < \infty$ or $\lim_{m\rightarrow\infty} \frac{\alpha_{m+1}}{\alpha_m} = 1$;

(c) either $\sum_{m=1}^{\infty} D_B(T_m, T_{m+1}) < \infty$ or $\lim_{m\rightarrow\infty} \frac{D_B(T_m, T_{m+1})}{\alpha_{m+1}} = 0$ for each $B \in E$;

(d) $\lim_{m\rightarrow\infty} \frac{a_m}{\alpha_m} = 0$.

Then the sequence $(u_m\}$ strongly converges to $u^* \in F$ and $u^*$ be the unique solution of VI

$$\langle (\omega G - \delta V)u^*, u^* - u \rangle \leq 0, \forall u\ \in F."$$

Chuadchawna et al.[16] proved the convergence theorem for generalized nonexpansive mappings in hyperbolic spaces. Lohawech et al.[17] worked on finding the solutions of VIPs for Hilbert spaces. The algorithm given by Lohawech et al.[17] is

$$\begin{cases} v_m = (1-\alpha_m) u_m + \alpha_m (I - \beta_m \omega G) T u_m \\ u_{m+1} = (I - \beta_m \omega G) T v_m \end{cases} \tag{3}$$

## 2 Methodology

Let the inner product and norm in the Hilbert space M are denoted by $\langle\ ,\ \rangle$ and $||.||$ respectively. Let D be the nonempty, convex and closed subset of M. The mapping $P_D : M \to D$ defined by

$$|(h - P_D h|| = \inf_{d\in D} ||h - d||$$

is known as metric projection from M on D.

Let D be a nonempty subset of M and $G_1, G_2 : D \to M$ be two mappings. We assign the collection of all bounded subsets of D as $D$. Then for $B \in D$, the deviation $D_B(G_1, G_2)$ between

$$D_B(G_1, G_2) = sup(|(G_1 b - G_2 b|| : b \in B\}.$$

A mapping $G : D \to M$ is called

- monotone is

$$\langle Gd_1 - Gd_2, d_1 - d_2\rangle \geq 0, \forall d_1, d_2 \in D$$

- $\eta$-strongly monotone if

$$\langle Gd_1 - Gd_2, d_1 - d_2\rangle \geq \eta \|d_1 - d_2\|^2, \forall d_1, d_2 \in D$$

- K-Lipschitzian if

$$\|Gd_1 - Gd_2\| \leq k \|d_1 - d_2\|, \forall d_1, d_2 \in D$$

For $0 \leq k < 1$, G is called strict contraction and for k=1, G is called nonexpansive.

If $(T_m\}$ is a sequence of mappings from D into M and $(a_m\} \subset [0,1)$ is a sequence such that $\lim_{m\to\infty} a_m = 0$ then $(T_m\}$ is known as sequence of nearly nonexpansive mappings with respect to sequence $(a_m\}$ if

$$|(T_m d_1 - T_m d_2|| \leq |(d_1 - d_2|| + a_m,\ \forall d_1, d_2 \in D.$$

**Definition:** (see [18]) "Let $(a_m\}$ and $(b_m\}$ are two real convergent sequences with limits a and b respectively. Then $(a_m\}$ converges faster than $(b_m\}$ if

$$\lim_{m\to\infty} \frac{a_m - a}{b_m - b} = 0."$$

The lemmas which will be used for proving the main result are as follow:

**Lemma 2.1** (see [3]): "Let M be a real Hilbert space. For all $h_1, h_2 \in M$ and $\varepsilon \in (0,1]$, we have

1. $\|h_1 + h_2\| \leq \|h_1\|^2 + 2\langle h_2, h_1 + h_2\rangle$;

2. $||(1-\varepsilon)h_1 + \varepsilon h_2||^2 = (1-\varepsilon)||h_1||^2 + \varepsilon||h_2||^2 - \varepsilon(1-\varepsilon)||h_1 - h_2||^2$."

**Lemma 2.2** (see [19]): "Let $V : D \to M$ be an L-Lipschitzian mapping and $G : D \to M$ be a k-Lipschitzian and $\theta$-strongly monotone operator. If $0 \leq \delta L < \omega\theta$, then

$$\langle h_1 - h_2, (\omega G - \delta V)h_1 - (\omega G - \delta V)h_2\rangle \geq (\omega\theta - \delta L)\|h_1 - h_2\|^2, \forall h_1, h_2 \in D,$$

that is, $\omega G-\delta V$ is strongly monotone having coefficient $\omega\theta-\delta L$."

**Lemma 2.3** (see [20]): "Let $\chi\in(0,1)$, $\omega>0$ and the mapping $J:D\to M$ defined by

$Jh_1=(1-\chi\omega G)h_1,\ \forall h_1\in D,$

is a strict contraction mapping if $\omega<2\theta/k^2$ or we can also say that, for $\omega\in(0,2\theta/k^2)$,

$$|(Jh_1-Jh_2||\le(1-\chi\tau)|(h_1-h_2||,\ \forall h_1,h_2\in D,$$

where $\tau=1-\sqrt{1-\omega(2\theta-\omega k^2)}$."

**Lemma 2.4** (see [21]):"Let J be a nonexpansive self-mapping defined on closed and convex subset D of the Hilbert space M. If J has a fixed point then (I-J) is demiclosed or we can say that if $(y_m\}$ is a sequence in D which converges weakly to some $y\in D$ and the sequence $((I-J)y_m\}$ converges strongly to z then $(I-J)y=z$."

**Lemma 2.5** (see [21,22]): "If $P_D$ is a metric projection mapping then it satisfies the following properties:

1. $P_Dh\in D,\forall h\in M$;
2. $\langle h-P_Dh,P_Dh-d\rangle\ge0,\forall h\in M$ and $d\in D$;
3. $|(h-d||^2\ge||h-P_Dh||^2+||d-P_Dh||^2,\ \ \forall h\in M\ and\ d\in D$;
4. $\langle P_Dh_1-P_Dh_2,h_1-h_2\rangle\ge\|P_Dh_1-P_Dh_2\|^2,\forall h_1,h_2\in M$."

**Lemma 2.6** (see [23]):" Let the sequence $(s_m\}$ of non-negative real numbers be such that

$$s_{m+1}\le(1-a_m)s_m+a_mb_m+c_m\ ,$$

where the sequences $(a_m\},\ (b_m\}\ \ and\ \ (c_m\}$ follow the conditions:

1. $a_m\subset[0,1]$ and $\sum_{m=1}^{\infty}a_m=\infty$;
2. $limsup_{m\to\infty}b_m\le0$;
3. $\sum_{m=1}^{\infty}c_m\le\infty\ or\ limsup_{m\to\infty}c_m/b_m\le0$.

Then $\lim\limits_{m\to\infty}s_m=0$."

## 3 Result and Discussion

Our main result is as follow:

**Theorem 3.1** Let E be a non-empty, closed and convex subset of a real Hilbert space M. Let $V:E\to M$ be an L-Lipschitzian and $G:E\to M$ be a k-Lipschitzian and $\theta$-strongly monotone operator. Let $(T_m\}$ be a sequence of nearly nonexpansive mappings from E into itself with respect to the sequence sequence $(a_m\}$ such that $F=\cap Fix(T_m)\neq\phi$ and T be a self-mapping on E such that $Tu=\lim\limits_{m\to\infty}T_mu,\ \forall u\in E$. Let $FixT=\cap Fix(T_m\}$, $0<\omega<2\theta/k^2$ and $0<\delta L<\tau$, where $\tau=1-\sqrt{1-\omega(2\theta-\omega k^2)}$. For $u_0\in E$, the sequence $(u_m\}$ is defined on E and is given by the iterative algorithm as

$$\begin{cases} w_m=\gamma_mu_m+(1-\gamma_m)T_mu_m\\ v_m=\beta_mw_m+(1-\beta_m)T_mw_m\\ u_{m+1}=P_E[\alpha_m\delta Vu_m+(I-\alpha_m\omega G)v_m]\end{cases}\tag{4}$$

where $(\alpha_m\}\ ,\ (\beta_m\}\ \ and\ \ (\gamma_m\}$ be the sequences in (0,1) and satisfy the following conditions:

(a) $\lim\limits_{m\to\infty}\alpha_m=0,\ \sum_{m=0}^{\infty}\alpha_m=\infty$;

(b) either $\sum_{m=0}^{\infty}(\alpha_{m+1}-\alpha_m|<\infty$ or $\lim\limits_{m\to\infty}\frac{\alpha_{m+1}}{\alpha_m}=1$;

(c) $\sum_{m=0}^{\infty}(\beta_{m+1}-\beta_m|<\infty$;

(d) $\sum_{m=0}^{\infty}(\gamma_{m+1}-\gamma_m|<\infty$;

(e) either $\sum_{m=0}^{\infty}D_B(T_m,T_{m+1})<\infty$ or $\lim\limits_{m\to\infty}\frac{D_B(T_m,T_{m+1})}{\alpha_{m+1}}=0$ for each $B\in E$;

(f) $\lim\limits_{m\to\infty}\frac{a_m}{\alpha_m}=0$.

Then the sequence $(u_m\}$ strongly converges to $u^*\in F$ and $u^*$ be the unique solution of VI

$$\langle(\omega G-\delta V)u^*,u^*-u\rangle\le0,\forall u\ \in F$$

Proof. Step 1. The sequence $(u_m\}$ is bounded.

Let $d \in F$, then using algorithm (4),

$$\begin{aligned}\|u_{m+1}-d\| &= \|P_E\left[\alpha_m \delta V u_m + (I-\alpha_m \omega G)\, v_m\right] - d\| \\ &\leq \|\left[\alpha_m \delta V u_m + (I-\alpha_m \omega G)\, v_m\right] - d\| \\ &= \|\alpha_m \delta V u_m - \alpha_m \omega G d + \alpha_m \omega G d + (I-\alpha_m \omega G)\, v_m - d\| \\ &= \|\alpha_m (\delta V u_m - \omega G d) + (I-\alpha_m \omega G)\, v_m - (I-\alpha_m \omega G)\, d\| \\ &= \|\alpha_m (\delta V u_m - \omega G d) - \alpha_m \delta V d + \alpha_m \delta V d + (I-\alpha_m \omega G)(v_m - d)\| \\ &= \|\alpha_m \delta V (u_m - d) + \alpha_m (\delta V - \omega G) d + (I-\alpha_m \omega G)(v_m - d)\| \\ &\leq \alpha_m \delta \|V(u_m - d)\| + \alpha_m \|(\delta V - \omega G) d\| + \|(I-\alpha_m \omega G)(v_m - d)\| \\ &\leq \alpha_m \delta L \|u_m - d\| + \alpha_m \|(\delta V - \omega G) d\| + (1-\alpha_m \tau)\|v_m - d\|\end{aligned} \tag{5}$$

Now,

$$\begin{aligned}\|v_m - d\| &= \|\beta_m w_m + (1-\beta_m)\, T_m w_m - d\| \\ &= \|\beta_m w_m - \beta_m\, \mathrm{d} + \beta_m\, \mathrm{d} + (1-\beta_m)\, T_m w_m - T_m d\| \\ &= \|\beta_m (w_m - \mathrm{d}) + (1-\beta_m)\, T_m w_m - T_m d + \beta_m T_m d\| \\ &= \|\beta_m (w_m - \mathrm{d}) + (1-\beta_m)\, T_m w_m - (1-\beta_m)\, T_m d\| \\ &= \|\beta_m (w_m - \mathrm{d}) + (1-\beta_m)(T_m w_m - T_m d)\| \\ &\leq \beta_m \|w_m - \mathrm{d}\| + (1-\beta_m)\|T_m w_m - T_m d\| \\ &\leq \beta_m \|w_m - \mathrm{d}\| + (1-\beta_m)\left[\|(w_m - \mathrm{d})\| + a_m\right] \\ &\leq \|w_m - \mathrm{d}\| + a_m\end{aligned} \tag{6}$$

Now,

$$\begin{aligned}\|w_m - d\| &= \|\gamma_m u_m + (1-\gamma_m)\, T_m u_m - d\| \\ &= \|\gamma_m u_m - \gamma_m\, \mathrm{d} + \gamma_m\, \mathrm{d} + (1-\gamma_m)\, T_m u_m - d\| \\ &= \|\gamma_m u_m - \gamma_m\, \mathrm{d} + (1-\gamma_m)\, T_m u_m - T_m d + \gamma_m T_m d\| \\ &= \|\gamma_m (u_m - \mathrm{d}) + (1-\gamma_m)\, T_m w_m - (1-\gamma_m)\, T_m d\| \\ &= \|\gamma_m (u_m - \mathrm{d}) + (1-\gamma_m)(T_m u_m - T_m d)\| \\ &\leq \gamma_m \|u_m - \mathrm{d}\| + (1-\gamma_m)\|T_m u_m - T_m d\| \\ &\leq \gamma_m \|\|u_m - \mathrm{d}\| + (1-\gamma_m)\left[\|u_m - \mathrm{d}\| + a_m\right] \\ &\leq \|u_m - \mathrm{d}\| + a_m\end{aligned} \tag{7}$$

From (5), using (6) and (7),

$$\begin{aligned}\|u_{m+1} - d\| &\leq \alpha_m \delta L \|u_m - d\| + \alpha_m \|(\delta \mathrm{V} - \omega \mathrm{G}) d\| + (1-\alpha_m \tau)\left[\|(u_m - \mathrm{d})\| + 2a_m\right] \\ &\leq (\alpha_m \delta L + 1 - \alpha_m \tau)\|(u_m - \mathrm{d})\| + \alpha_m \|(\delta \mathrm{V} - \omega \mathrm{G}) d\| + 2a_m\end{aligned} \tag{8}$$

$\because \lim\limits_{m\to\infty} \frac{a_m}{\alpha_m} = 0$, $\exists\, M_1$ such that

$$\frac{\alpha_m || (\delta V - \omega G)\, d|| + 2a_m}{\alpha_m} \leq M_1, \ \ \forall m \geq 0.$$

So, from (8),

$$||u_{m+1} - d|| \leq (1 - \alpha_m (\tau - \delta L)] \,|\, (u_m - d|| + \alpha_m M_1$$

$$\leq max\left( ||u_m - d||, \frac{M_1}{\tau - \delta L} \right\}, \forall m \in N.$$

$\therefore$ $(u_m\}$ is bounded and thus $(T_m u_m\}$, $(Gv_m\}$ *and* $(Vu_m\}$ are bounded.

Step 2. $||u_{m+1} - u_m|| \to 0$ *as* $m \to \infty$. Now,

$$\begin{aligned}
\|u_{m+2} - u_{m+1}\| &= \|P_E\left[\alpha_{m+1}\delta Vu_{m+1} + (I - \alpha_{m+1}\omega G)\, v_{m+1}\right] \\
&\quad - P_E\left[\alpha_m \delta Vu_m + (I - \alpha_m \omega G)\, v_m\right]\| \\
&\leq \|\alpha_{m+1}\delta Vu_{m+1} + (I - \alpha_{m+1}\omega G)\, v_{m+1} - \alpha_m \delta Vu_m - (I - \alpha_m \omega G)\, v_m\| \\
&\quad + \|\alpha_{m+1}\delta Vu_{m+1} - \alpha_{m+1}\delta Vu_m + \alpha_{m+1}\delta Vu_m - \alpha_m \delta Vu_m \\
&\quad + \| - \alpha_{m+1}\omega G)\, v_{m+1} - (I - \alpha_{m+1}\omega G + \alpha_{m+1}\omega G - \alpha_m \omega G)\, v_m\| \\
&\quad - (I - \alpha_{m+1}\delta\,(Vu_{m+1} - Vu_m) + \delta\,(\alpha_{m+1} - \alpha_m)\, Vu_m + (I - \alpha_{m+1}\omega G)\, v_{m+1} - (\alpha_{m+1} - \alpha_m)\,\omega G v_m\| \\
&\quad = \|\alpha_{m+1}\delta\,(Vu_{m+1} - Vu_m) + \delta\,(\alpha_{m+1} - \alpha_m)\, Vu_m \\
&\quad + + (I - \alpha_{m+1}\omega G)\,(v_{m+1} - v_m) - (\alpha_{m+1} - \alpha_m)\,\omega G v_m\| \\
&\quad = \|\alpha_{m+1}\delta\,(Vu_{m+1} - Vu_m) + (\alpha_{m+1} - \alpha_m)\,(\delta Vu_m - \omega G v_m) \\
&\quad + (I - \alpha_{m+1}\omega G)\,(v_{m+1} - v_m)\,\| \\
&\leq \alpha_{m+1}\delta L\,\|u_{m+1} - u_m\| + M_2\,|\alpha_{m+1} - \alpha_m| + (1 - \alpha_{m+1}\tau)\,\|v_{m+1} - v_m\|
\end{aligned} \tag{9}$$

where $M_2 = \delta sup(||Vu_m||\} + \omega sup(||Gv_m||\} < \infty$.

Now,

$$\begin{aligned}
\|v_{m+1} - v_m\| &= \|\beta_{m+1}w_{m+1} + (1 - \beta_{m+1})\, T_{m+1}w_{m+1} - \beta_m w_m - (1 - \beta_m)\, T_m w_m\| \\
&= \|\beta_{m+1}w_{m+1} - \beta_{m+1}w_m + \beta_{m+1}w_m - \beta_m w_m \\
&\quad + (1 - \beta_{m+1})\, T_{m+1}w_{m+1} - (1 - \beta_m)\, T_m w_m\| \\
&= \|\beta_{m+1}\,(w_{m+1} - w_m) + (\beta_{m+1} - \beta_m)\, w_m \\
&\quad + (1 - \beta_{m+1})\,(T_{m+1}w_{m+1} - T_{m+1}w_m) + (1 - \beta_m)\,(T_{m+1}w_m - T_m w_m) \\
&\quad + (1 - \beta_{m+1})\, T_{m+1}w_m - (1 - \beta_m)\, T_{m+1}w_m\| \\
&= \|\beta_{m+1}\,(w_{m+1} - w_m) + (\beta_{m+1} - \beta_m)\, w_m \\
&\quad + (1 - \beta_{m+1})\,(T_{m+1}w_{m+1} - T_{m+1}w_m) + (1 - \beta_m)\,(T_{m+1}w_m - T_m w_m) \\
&\quad + (\beta_m - \beta_{m+1})\, T_{m+1}w_m\| \\
&= \|\beta_{m+1}\,(w_{m+1} - w_m) + (\beta_{m+1} - \beta_m)\,(T_{m+1}w_m - w_m) \\
&\quad + (1 - \beta_{m+1})\,(T_{m+1}w_{m+1} - T_{m+1}w_m) + (1 - \beta_m)\,(T_{m+1}w_m - T_m w_m)\,\| \\
&\leq \beta_{m+1}\,\|w_{m+1} - w_m\| + |\beta_{m+1} - \beta_m| \cdot \|T_{m+1}w_m - w_m\| \\
&\quad + (1 - \beta_{m+1})\,\|T_{m+1}w_{m+1} - T_{m+1}w_m\| + (1 - \beta_m)\,\|T_{m+1}w_m - T_m w_m\| \\
&\leq \beta_{m+1}\,\|w_{m+1} - w_m\| + K_m\,|\beta_{m+1} - \beta_m| \\
&\quad + (1 - \beta_{m+1})\,(\|w_{m+1} - w_m\| + a_{m+1}) + (1 - \beta_m)\, D_B\,(T_{m+1}, T_m) \\
&\leq \|w_{m+1} - w_m\| + K_m\,|\beta_{m+1} - \beta_m| + a_{m+1} + D_B\,(T_{m+1}, T_m)
\end{aligned} \tag{10}$$

where $K_m = sup\big(||T_{m+1}w_m||\} + sup\{||w_m||\} < \infty$.

Now,

$$\begin{aligned}
\|w_{m+1}-w_m\| &= \|\gamma_{m+1}u_{m+1}+(1-\gamma_{m+1})T_{m+1}u_{m+1}-\gamma_m u_m-(1-\gamma_m)T_m u_m\| \\
&= \|\gamma_{m+1}u_{m+1}-\gamma_{m+1}u_m+\gamma_{m+1}u_m-\gamma_m u_m \\
&+(1-\gamma_{m+1})T_{m+1}u_{m+1}-(1-\gamma_m)T_m u_m\| \\
&= \|\gamma_{m+1}(u_{m+1}-u_m)+(\gamma_{m+1}-\gamma_m)u_m \\
&+(1-\gamma_{m+1})(T_{m+1}u_{m+1}-T_{m+1}u_m)+(1-\gamma_m)(T_{m+1}u_m-T_m u_m) \\
&+(1-\gamma_{m+1})T_{m+1}u_m-(1-\gamma_m)T_{m+1}u_m\| \\
&= \|\gamma_{m+1}(u_{m+1}-u_m)+(\gamma_{m+1}-\gamma_m)u_m \\
&+(1-\gamma_{m+1})(T_{m+1}u_{m+1}-T_{m+1}u_m)+(1-\gamma_m)(T_{m+1}u_m-T_m u_m) \\
&+(\gamma_m-\gamma_{m+1})T_{m+1}u_m\| \\
&= \|\gamma_{m+1}(u_{m+1}-u_m)+(\gamma_{m+1}-\gamma_m)(T_{m+1}u_m-u_m) \\
&+(1-\gamma_{m+1})(T_{m+1}u_{m+1}-T_{m+1}u_m)+(1-\gamma_m)(T_{m+1}u_m-T_m u_m)\| \\
&\leq \gamma_{m+1}\|u_{m+1}-u_m\|+|\gamma_{m+1}-\gamma_m|\cdot\|T_{m+1}u_m-u_m\| \\
&+(1-\gamma_{m+1})\|T_{m+1}u_{m+1}-T_{m+1}u_m\|+(1-\gamma_m)\|T_{m+1}u_m-T_m u_m\| \\
&\leq \gamma_{m+1}\|u_{m+1}-u_m\|+L_m|\gamma_{m+1}-\gamma_m| \\
&+(1-\gamma_{m+1})(\|u_{m+1}-u_m\|+a_{m+1})+(1-\gamma_m)D_B(T_{m+1},T_m) \\
&\leq \|u_{m+1}-u_m\|+L_m|\gamma_{m+1}-\gamma_m|+D_B(T_{m+1},T_m)+a_{m+1}
\end{aligned} \tag{11}$$

where $L_m = sup(||T_{m+1}u_m||\} + sup\{||u_m||\} < \infty$.

From (10),

$$||v_{m+1}-v_m|| \leq ||u_{m+1}-u_m||+K_m(\beta_{m+1}-\beta_m|+L_m(\gamma_{m+1}-\gamma_m| \ +2a_{m+1}+2D_B(T_{m+1},T_m)$$

From (9),

$$\begin{aligned}
&||u_{m+2}-u_{m+1}|| \leq \alpha_{m+1}\delta L||u_{m+1}-u_m||+M_2(\alpha_{m+1}-\alpha_m| \\
&\quad +(1-\alpha_{m+1}\tau)[||u_{m+1}-u_m||+K_m(\beta_{m+1}-\beta_m|+L_m(\gamma_{m+1}-\gamma_m| \\
&\quad +2a_{m+1}+2D_B(T_{m+1},T_m)] \\
&\quad = [1-\alpha_{m+1}(\tau-\delta L)]||u_{m+1}-u_m||+M_2(\alpha_{m+1}-\alpha_m| \\
&\quad +(1-\alpha_{m+1}\tau)[K_m(\beta_{m+1}-\beta_m|+L_m(\gamma_{m+1}-\gamma_m| \\
&\quad +2a_{m+1}+2D_B(T_{m+1},T_m)]
\end{aligned}$$

Using lemma 2.6,

$$||u_{m+1}-u_m|| \to 0 \ as \ m \to \infty.$$

Step 3. $\lim_{m\to\infty}||u_m - Tu_m|| = 0$.

Now,

$$\begin{aligned}
\|u_{m+1}-T_{\mathrm{m}}u_m\|^2 &= \langle u_{m+1}-T_{\mathrm{m}}u_m, u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&= \langle P_E\left[\alpha_m\delta V u_m+(I-\alpha_m\omega G)v_m\right]-T_{\mathrm{m}}u_m, u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&\le \langle \alpha_m\delta V u_m+(I-\alpha_m\omega G)v_m-T_{\mathrm{m}}u_m, u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&= \langle \alpha_m\delta V u_m-\alpha_m\omega G(T_{\mathrm{m}}u_m)+\alpha_m\omega G(T_{\mathrm{m}}u_m) \\
&+(I-\alpha_m\omega G)v_m-\mathrm{I}(T_{\mathrm{m}}u_m), u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&= \langle \alpha_m(\delta V u_m-\omega G T_{\mathrm{m}}u_m)+(I-\alpha_m\omega G)v_m \\
&-(I-\alpha_m\omega G)T_{\mathrm{m}}u_m, u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&= \langle \alpha_m(\delta V u_m-\delta V T_{\mathrm{m}}u_m+\delta V T_{\mathrm{m}}u_m-\omega G T_{\mathrm{m}}u_m)+(I-\alpha_m\omega G)v_m \\
&-(I-\alpha_m\omega G)T_{\mathrm{m}}u_m, u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&= \langle \alpha_m\delta(V u_m-V T_{\mathrm{m}}u_m)+\alpha_m(\delta V-\omega G)T_{\mathrm{m}}u_m) \\
&+(I-\alpha_m\omega G)(v_m-T_{\mathrm{m}}u_m), u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&= \langle \alpha_m\delta(V u_m-V T_{\mathrm{m}}u_m), u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&+\langle \alpha_m(\delta V-\omega G)T_{\mathrm{m}}u_m, u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&+\langle (I-\alpha_m\omega G)(v_m-T_{\mathrm{m}}u_m), u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&\le \alpha_m\delta L\|u_m-T_{\mathrm{m}}u_m\|.\|u_{m+1}-T_{\mathrm{m}}u_m\| \\
&+\alpha_m\langle(\delta V-\omega G)T_{\mathrm{m}}u_m, u_{m+1}-T_{\mathrm{m}}u_m\rangle \\
&+(1-\alpha_m\tau)\|v_m-T_{\mathrm{m}}u_m\|.\|u_{m+1}-T_{\mathrm{m}}u_m\|
\end{aligned} \tag{12}$$

Now,

$$\begin{aligned}
\|v_m-T_{\mathrm{m}}u_m\| &= \|\beta_m w_m+(1-\beta_m)T_m w_m-T_{\mathrm{m}}u_m\| \\
&= \|\beta_m w_m-\beta_m T_{\mathrm{m}}u_m+\beta_m T_{\mathrm{m}}u_m+(1-\beta_m)T_m w_m-T_{\mathrm{m}}u_m\| \\
&= \|\beta_m(w_m-T_{\mathrm{m}}u_m)+(1-\beta_m)T_m w_m-(1-\beta_m)T_{\mathrm{m}}u_m\| \\
&= \|\beta_m(w_m-T_{\mathrm{m}}u_m)+(1-\beta_m)(T_m w_m-T_{\mathrm{m}}u_m)\| \\
&\le \beta_m\|w_m-T_{\mathrm{m}}u_m\|+(1-\beta_m)\|T_m w_m-T_{\mathrm{m}}u_m\| \\
&\le \beta_m\|w_m-T_{\mathrm{m}}u_m\|+(1-\beta_m)(\|w_m-u_m\|+a_m) \\
&\le \beta_m\|w_m-T_{\mathrm{m}}u_m\|+(1-\beta_m)\|w_m-u_m\|+a_m
\end{aligned} \tag{13}$$

Now,

$$\begin{aligned}
\|w_m-T_{\mathrm{m}}u_m\| &= \|\gamma_m u_m+(1-\gamma_m)T_m u_m-T_{\mathrm{m}}u_m\| \\
&= \|\gamma_m u_m-\gamma_m T_m u_m\| \\
&= \gamma_m\|u_m-T_m u_m\|
\end{aligned} \tag{14}$$

Thus,

$$||v_m-T_m u_m|| \le \beta_m\gamma_m||u_m-T_m u_m||+(1-\beta_m)||w_m-u_m||+a_m$$

Hence,

$$\begin{aligned}
||u_{m+1}-T_m u_m||^2 &\le \alpha_m\delta L||u_m-T_m u_m||.||u_{m+1}-T_m u_m|| \\
&+\alpha_m\langle(\delta V-\omega G)T_m u_m, u_{m+1}-T_m u_m\rangle \\
&+(1-\alpha_m\tau)[\beta_m\gamma_m||u_m-T_m u_m|| \\
&+(1-\beta_m)||w_m-u_m||+a_m].||u_{m+1}-T_m u_m|| \\
&= [\alpha_m\delta L+\beta_m\gamma_m(1-\alpha_m\tau)]||u_m-T_m u_m||.||u_{m+1}-T_m u_m|| \\
&+\alpha_m\langle(\delta V-\omega G)T_m u_m, u_{m+1}-T_m u_m\rangle \\
&+(1-\alpha_m\tau)[(1-\beta_m)||w_m-u_m||+a_m].||u_{m+1}-T_m u_m|| \\
&\le [\alpha_m\delta L+\beta_m\gamma_m(1-\alpha_m\tau)]\left(\frac{||u_m-T_m u_m||^2+||u_{m+1}-T_m u_m||^2}{2}\right] \\
&+\alpha_m\langle(\delta V-\omega G)T_m u_m, u_{m+1}-T_m u_m\rangle \\
&+(1-\alpha_m\tau)[(1-\beta_m)||w_m-u_m||+a_m].||u_{m+1}-T_m u_m||
\end{aligned}$$

$+\alpha_m\langle(\delta V-\omega G)\,T_m u_m,u_{m+1}-T_m u_m\rangle$
$+(1-\alpha_m\tau)\,[(1-\beta_m)\,||w_m-u_m||+a_m].||u_{m+1}-T_m u_m||$
$||u_{m+1}-T_m u_m||^2\le\left(\frac{\alpha_m\delta L+\beta_m\gamma_m(1-\alpha_m\tau)}{2-[\alpha_m\delta L+\beta_m\gamma_m(1-\alpha_m\tau)}\right].||u_m-T_m u_m||^2$
$+\frac{2\alpha_m}{2-[\alpha_m\delta L+\beta_m\gamma_m(1-\alpha_m\tau)}\langle(\delta V-\omega G)\,T_m u_m,u_{m+1}-T_m u_m\rangle$
$+\frac{2(1-\alpha_m\tau)}{2-[\alpha_m\delta L+\beta_m\gamma_m(1-\alpha_m\tau)}[(1-\beta_m)\,||w_m-u_m||+a_m].M_3$
where $M_3=sup\{||u_{m+1}-T_m u_m||\}$
$||u_{m+1}-T_m u_m||^2\le(\alpha_m\delta L+\beta_m\gamma_m\,(1-\alpha_m\tau)]\,.||u_m-T_m u_m||^2$
$+2\alpha_m\langle(\delta V-\omega G)\,T_m u_m,u_{m+1}-T_m u_m\rangle$
$+[(1-\beta_m)\,||w_m-u_m||+a_m].M_3$
By lemma 2.6,
$\underset{m\to\infty}{lim}||u_m-T_m u_m||^2=0$
or

$$\underset{m\to\infty}{lim}||u_m-T_m u_m||=0 \tag{15}$$

Now,
$||u_m-Tu_m||=||u_m-T_m u_m+T_m u_m-Tu_m||$
$\le|\,(u_m-T_m u_m|\,|+||T_m u_m-Tu_m||$
$\le|\,(u_m-T_m u_m|\,|+D_B(T_m,T)$
Taking limit as $m\to\infty$ and using (15),
$\therefore\ \underset{m\to\infty}{lim}||u_m-Tu_m||$ =0.
Step 4. We will prove that
—
where $u^*\in F$ is the unique solution of VI.
Since $Tu=\underset{m\to\infty}{lim}T_m u,\ \forall u\in E$ and each $T_m$ is a nonexpansive mapping, so T is a nonexpansive mapping.
Let $\left(u_{m_k}\right\}$ be a subsequence of $(u_m\}$ such that

$$\lim_{m\to\infty}\sup\langle(\delta V-\omega G)u^*,u_m-u^*\rangle=\lim_{k\to\infty}\left\langle(\delta V-\omega G)u^*,u_{m_k}-u^*\right\rangle \tag{16}$$

W.L.O.G, assume that $u_{m_k}\to u\ \in E$. By using lemma 2 4, we get $u\in Fi\,xT=F$ .
So, from (16) and VI,

$$\lim_{m\to\infty}\sup\langle(\delta V-\omega G)u^*,u_m-u^*\rangle=\langle(\delta V-\omega G)u^*,\mathrm{u}-u^*\rangle \tag{17}$$

Step 5. $u_m \to u^*$ *as* $m \to \infty$.

$$\begin{aligned}
\|u_{m+1}-u^*\|^2 &= \langle u_{m+1}-u^*, u_{m+1}-u^*\rangle \\
&= \langle P_E\left[\alpha_m \delta V u_m + (I-\alpha_m \omega G)\, v_m\right] - u^*, u_{m+1}-u^*\rangle \\
&\leq \langle \alpha_m \delta V u_m + (I-\alpha_m \omega G)\, v_m - u^*, u_{m+1}-u^*\rangle \\
&= \langle \alpha_m \delta V u_m - \alpha_m \omega G u^* + \alpha_m \omega G u^* \\
&\quad + (I-\alpha_m \omega G)\, v_m - I u^*, u_{m+1}-u^*\rangle \\
&= \langle \alpha_m (\delta V u_m - \omega G u^*) + (I-\alpha_m \omega G)\, v_m \\
&\quad - (I-\alpha_m \omega G)\, u^*, u_{m+1}-u^*\rangle \\
&= \langle \alpha_m (\delta V u_m - \delta V u^* + \delta V u^* - \omega G u^*) + (I-\alpha_m \omega G)\, v_m \\
&\quad - (I-\alpha_m \omega G)\, u^*, u_{m+1}-u^*\rangle \\
&= \langle \alpha_m \delta (V u_m - V u^*) + \alpha_m (\delta V - \omega G) u^* \\
&\quad + (I-\alpha_m \omega G)(v_m - u^*), u_{m+1}-u^*\rangle \\
&= \langle \alpha_m \delta (V u_m - V u^*), u_{m+1}-u^*\rangle \\
&\quad + \langle \alpha_m (\delta V - \omega G) u^*, u_{m+1}-u^*\rangle \\
&\quad + \langle (I-\alpha_m \omega G)(v_m - u^*), u_{m+1}-u^*\rangle \\
&\leq \alpha_m \delta L \|u_m - u^*\| \cdot \|u_{m+1}-u^*\| \\
&\quad + \alpha_m \langle (\delta V - \omega G) u^*, u_{m+1}-u^*\rangle \\
&\quad + (1-\alpha_m \tau) \|v_m - u^*\| \cdot \|u_{m+1}-u^*\|
\end{aligned} \tag{18}$$

Now,

$$\begin{aligned}
\|v_m - u^*\| &= \|\beta_m w_m + (1-\beta_m)\, T_m w_m - u^*\| \\
&= \|\beta_m w_m - \beta_m u^* + \beta_m u^* + (1-\beta_m)\, T_m w_m - T_m u^*\| \\
&= \|\beta_m (w_m - u^*) + (1-\beta_m)\, T_m w_m - (1-\beta_m)\, T_m u^*\| \\
&= \|\beta_m (w_m - u^*) + (1-\beta_m)(T_m w_m - T_m u^*)\| \\
&\leq \beta_m \|w_m - u^*\| + (1-\beta_m) \|T_m w_m - T_m u^*\| \\
&\leq \beta_m \|w_m - u^*\| + (1-\beta_m)(\|w_m - u^*\| + a_m) \\
&\leq \|w_m - u^*\| + a_m
\end{aligned} \tag{19}$$

Similarly,

$$\begin{aligned}
\|w_m - u^*\| &= \|\gamma_m u_m + (1-\gamma_m)\, T_m u_m - u^*\| \\
&\leq \|u_m - u^*\| + a_m \\
&\Rightarrow \|v_m - u^*\| \leq \|u_m - u^*\| + 2a_m
\end{aligned}$$

Hence,

$$\begin{aligned}
&||u_{m+1}-u^*||^2 \leq \alpha_m \delta L ||u_m - u^*||.||u_{m+1}-u^*|| \\
&\quad +\alpha_m \langle (\delta V - \omega G)\, u^*, u_{m+1}-u^*\rangle \\
&\quad +(1-\alpha_m \tau)(||u_m - u^*|| + 2a_m)\, ||u_{m+1}-u^*|| \\
&\quad = [1-\alpha_m (\tau - \delta L)] ||u_m - u^*||.||u_{m+1}-u^*|| \\
&\quad +\alpha_m \langle (\delta V - \omega G)\, u^*, u_{m+1}-u^*\rangle \\
&\quad +2(1-\alpha_m \tau)\, ||u_{m+1}-u^*|| a_m \\
&\quad \leq [1-\alpha_m (\tau - \delta L) \left( \frac{||u_m - u^*||^2 + ||u_{m+1}-u^*||^2}{2} \right] \\
&\quad +\alpha_m \langle (\delta V - \omega G)\, u^*, u_{m+1}-u^*\rangle \\
&\quad +2(1-\alpha_m \tau)\, ||u_{m+1}-u^*|| a_m \\
&\left( \frac{2-1+\alpha_m (\tau - \delta L)]}{2} \right] .||u_{m+1}-u^*||^2 \leq \frac{[1-\alpha_m (\tau - \delta L)]}{2} .||u_m - u^*||^2
\end{aligned}$$

$+\alpha_m\langle(\delta V-\omega G)\,u^*,u_{m+1}-u^*\rangle$
$+2\,(1-\alpha_m\tau)\,||u_{m+1}-u^*||a_m$
$\left(\frac{1+\alpha_m(\tau-\delta L)}{2}\right].||u_{m+1}-u^*||^2\leq\frac{[1-\alpha_m(\tau-\delta L)]}{2}.||u_m-u^*||^2$
$+\alpha_m\langle(\delta V-\omega G)\,u^*,u_{m+1}-u^*\rangle$
$+2\,(1-\alpha_m\tau)\,||u_{m+1}-u^*||a_m$
$||u_{m+1}-u^*||^2\leq\left(\frac{1-\alpha_m(\tau-\delta L)}{1+\alpha_m(\tau-\delta L)}\right].||u_m-u^*||^2$
$+\frac{2\alpha_m}{1+\alpha_m(\tau-\delta L)}\langle(\delta V-\omega G)\,u^*,u_{m+1}-u^*\rangle$
$+\frac{4(1-\alpha_m\tau)}{1+\alpha_m(\tau-\delta L)}||u_{m+1}-u^*||a_m$
$\leq[1-\alpha_m\,(\tau-\delta L)].||u_m-u^*||^2$
$+2\alpha_m\langle(\delta V-\omega G)\,u^*,u_{m+1}-u^*\rangle$
$+4a_m||u_{m+1}-u^*||$
$\leq[1-\alpha_m\,(\tau-\delta L)].||u_m-u^*||^2$
$+2\alpha_m\langle(\delta V-\omega G)\,u^*,u_{m+1}-u^*\rangle+4a_mM_4$

where $M_4=sup\{||u_{m+1}-u^*||\}$.

Assume that $s_m=\|u_m-u^*\|^2, t_m=\alpha_m(\tau-\delta L), b_m=\frac{2}{\tau-\delta L}\langle(\delta V-\omega G)u^*,u_{m+1}-u^*\rangle$
$c_m=4a_mM_4$. So,
$s_{m+1}\leq(1-t_m)\,s_m+t_mb_{m+}c_m$.

By Lemma 2.6,
$\lim\limits_{m\to\infty}||u_m-u^*||^2=0$
$\therefore||u_m-u^*||\to 0\ as\ m\to\infty.$
$\therefore\ u_m\to u^*\ as\ m\to\infty.$

This completes the proof.

**Theorem 3.2** Let V and G be as defined in theorem 3.1. Let $(T_m\}$ be a sequence of contractive mappings from E into itself. Let sequences $(\beta_m\}, (\gamma_m\}\ and\ (k_m\}$ in (0,1) also satisfy

(a) $\lim\limits_{m\to\infty}\beta_m=0,$

(b) $\lim\limits_{m\to\infty}\gamma_m=0,$

and the sequences $(x_m\}$ and $(u_m\}$ defined by algorithms (2) and (4). Then $(u_m\}$ converges faster than $(x_m\}$.

Proof. From (6),

$$\begin{aligned}&\|v_m-d\|\leq\beta_m\|w_m-\mathrm{d}\|+(1-\beta_m)\|T_mw_m-T_md\|\\&\leq\beta_m\|w_m-\mathrm{d}\|+(1-\beta_m)\,k_m\|w_m-d\|\\&=\beta_m\|w_m-d\|+k_m\|w_m-d\|-\beta_mk_m\|w_m-d\|\\&=\beta_m(1-k_m)\|w_m-d\|+k_m\|w_m-d\|\\&\therefore\lim_{m\to\infty}\|v_m-d\|\leq\lim_{m\to\infty}k_m\|w_m-d\|\end{aligned}\tag{21}$$

Similarly, from (7),

$$\lim_{m\to\infty}||w_m-d||\leq\lim_{m\to\infty}k_m||u_m-d||\tag{22}$$

From (5), using (21) and (22),
$\lim\limits_{m\to\infty}||u_{m+1}-d||\leq\lim\limits_{m\to\infty}[\alpha_m\delta L|\,(u_m-d|\,|+\alpha_m|\,((\delta V-\omega G)\,d|\,|+(1-\alpha_m\tau)k_m||w_m-d||]$
$\leq\lim\limits_{m\to\infty}k_m||w_m-d||$
$\leq\lim\limits_{m\to\infty}k_m{}^2||u_m-d||$
$\leq\lim\limits_{m\to\infty}\prod_{i=0}^{m}k_i{}^2||u_0-d||$

Taking $K=sup_mk_m$, then
$\lim\limits_{m\to\infty}||u_{m+1}-d||\leq\lim\limits_{m\to\infty}||u_0-d||K^{2(m+1)}$

In [14], from (3.3),

$\lim_{m\to\infty} ||x_{m+1} - d|| \leq \lim_{m\to\infty} ||x_0 - d|| K^{(m+1)}$
Define $a_m = ||u_0 - d|| K^{2(m+1)}$ and $b_m = ||x_0 - d|| K^{(m+1)}$, then
$\zeta_m = \frac{a_m}{b_m}$
$= \frac{||u_0 - d|| K^{2(m+1)}}{||x_0 - d|| K^{(m+1)}}$
$= K^{m+1}$
$\because \lim_{m\to\infty} \frac{\zeta_{m+1}}{\zeta_m} = \lim_{m\to\infty} \frac{K^{m+2}}{K^{m+1}} = K < 1,$
so by ratio test, $\sum \zeta_m < \infty$.
$\therefore \lim_{m\to\infty} \frac{||u_{m+1} - d||}{||x_{m+1} - d||} = \lim_{m\to\infty} \frac{a_m}{b_m} = \lim_{m\to\infty} \zeta_m = 0.$
$\implies (u_m\}$ converges faster than $(x_m\}$.

## Supportive Application

For the reliability on the present result, an application in support of the main result is as follows:

Let $M = R$ and $E = (0,1]$. Let the self-mapping T be defined by $Tu = 1 - u,\ \forall u \in E$, so T is a nonexpansive mapping. Let $G,\ V : E \to M$ be two mappings such that $Gu = 3u$ *and* $Vu = 3u,\ \forall u \in E$. Then G is a 3-Lipschitzian and 3-strongly monotone mapping and V is a 3-Lipschitzian mapping. Now, $0 < \omega < 2\theta/k^2$ and $0 < \delta L < \tau$, so we have $\omega = 1/3$ , $\tau = 1$ *and* $\delta = 1/4$. Taking $\alpha_m,\ \beta_m$ *and* $\gamma_m$ in (0,1) as $\alpha_m = \frac{1}{m},\ \beta_m = \frac{1}{m+1}$ *and* $\gamma_m = \frac{1}{m+2}$. Also taking $a_m = \frac{1}{m^2}$ . Now, $T_m : E \to E$ is defined as

$$T_m u = \begin{cases} 1 - u, \text{ if } u \in [0,1) \\ a_m, \text{ if } u = 1 \end{cases}$$

Then, the sequence $(T_m\}$ is of nearly nonexpansive mappings from E into itself. Also, $F = \cap Fix(T_m) = (1/2\}$ and $Tu = \lim_{m\to\infty} T_m u,\ \forall u \in E$.

So, all the conditions of Theorem 3.1 are satisfied and hence the sequence $(u_m\}$ obtained by algorithm (4) converges to the fixed point $(1/2\}$ and this fixed point is the solution of the corresponding variational inequality. The convergence of the sequence $(u_m\}$ for different initial values of $u_1$ is shown graphically in Figure 1.

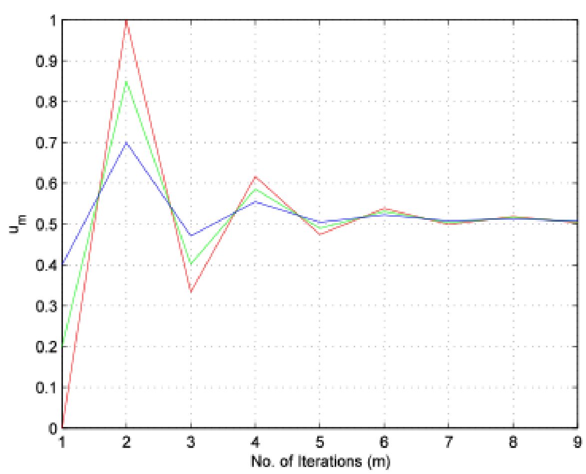


**Fig 1.** Graphical representation of m and $u_m$

Figure 2 is the graphical representation of convergence of the sequence $(u_m\}$ using algorithms (2), (3) and (4). It can be seen that convergence of the sequence $(u_m\}$ is the fastest by algorithm (4) among algorithms (2), (3) and (4).

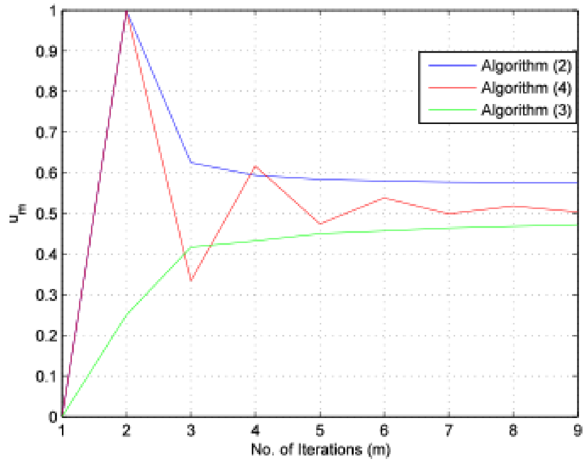


**Fig 2.** Gaphical representation of m and i for algorithms (2), (3) and (4)

## 4 Conclusion

In this paper, firstly, common fixed points of a sequence of nearly nonexpansive mappings are determined. After that, it is showed that these fixed points are the solutions of the corresponding VIP. Theoretically, algorithms (2) and (4) are compared for contractive mappings that establishes better convergence of algorithm (4) than algorithm (2). In addition, supportive application is given to validate the result by comparing algorithms (2), (3) and (4) with the help of Matlab programming. For nearly nonexpansive mappings, numerical comparison is done which shows that the improved algorithm (4) has the best convergence rate among algorithms (2), (3) and (4).

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