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Some Quadruple Fixed Points of Integral Type Contraction Mappings in Bipolar Metric Spaces with Application

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Abstract

Objectives: To establish a common quadruple fixed point theorem on complete bipolar metric spaces. **Methods:** By using integral type contraction of two covariant mappings. **Findings:** For covariant mappings of the complete metric space, which was able to obtain a unique common quadruple fixed point theorem and validate it with an appropriate example. **Novelty:** A unique common quadruple fixed point is produced by using integral contraction with covariant mappings and is a more reliable generalization of the previously published theorems.

Keywords: Bipolar metric space; Intergral type contraction; Completeness; ω - compatible mapping; Common Quadruple fixed point

1 Introduction

Non-linear analysis's section on fixed point theory is significant because of its potential applications. We use completeness, continuity, convergence, and numerous other topological properties to prove fixed point theorems. One of the most significant findings in non-linear analysis is the Banach Contraction Principle, sometimes known as the Banach fixed point theorem. By extending the underlying space or by considering it a common fixed-point theorem along with other self-maps, this theorem has been expanded in a broad range of ways. Many fixed-point theorems require the use of commutativity to be proven. The concept of compatibility was set-up as a generalization of commutativity due to the fact that it is a more permissive condition. Many extensions of metric spaces, including G-metric spaces, partial metric spaces, cone metric spaces, and bipolar-metric spaces, have appeared in a number of articles during the past few years. The study of fixed-point theory was expanded using these generalizations. One of the most recent generalizations is that of a bipolar metric space, introduced by Mutlu and G¨urdal [(1)] in 2016, and studied fixed point and coupled fixed point results on this space. Among these, Branciari [(2)] examined the use of Lebesgue integrals in the theory of metric fixed points in 2002 and established the existence and uniqueness of fixed points for integral contractions whenever the metric space (X; d) is complete.

Then, in [3–5] and the references therein, numerous authors studied various iterations of integral contractions and obtained fixed point solutions with respect to these contractions in various metric spaces.

E. Karapinar [6] demonstrated certain quadruple fixed results in partially ordered metric spaces very recently by using the idea of quadruple fixed point. Numerous researchers [7–12] then created quadruple fixed theorems in different metric spaces.

In this paper, we use Integral type contraction to present some frequent quadruple fixed point solutions in complete bipolar metric spaces. We also provide applications to integral equations as an example.

In what follows, we collect relevant definitions needed in our subsequent discussions.

1.1. Definition [1].

The mapping $d: X\times Y\to[0,\infty)$ is said to be Bipolar-metric on a pair of non-empty sets *(X,Y)*. If

$(B1) d(x,y)=0$ if and only if $x=y$;

$(B2) x=y$ implies that $d(x,y)=0$

(B3) if $(x,y)\in\ X\cap Y$ then $d(x,y)=d(y,x)$

(B4) $d(x_1,y_2)\leq d(x_1,y_1)+d(x_2,y_1)+d(x_2,y_2)$ for all $x,x_1,x_2\in X$ and $y,y_1,y_2\in Y$

Then the triple *(X,Y,d)* is called Bipolar-metric space.

1.2 Example$

Let $X=(1,\infty)$ and $Y=[-1,1]$. A mapping $d: X\times Y\to[0,+\infty)$ as

$d(x,y)=\left|x^2-y^2\right|$, for all $(x,y)\in(X,Y)$. Then the triple (X,Y,d) is called Bipolar-metric space.

1.3 Example [1]

A mapping $d: X\times Y\to[0,+\infty)$ as $d(\psi,a)=\psi(a)$ for all $(\psi,a)\in(X,Y)$

Where $X=\{\psi/\psi: R\to[1,3]\}$ be the set of all functions and $Y=R$. Then the triple *(X,Y,d)* is called a disjoint Bipolar-metric space.

1.4 Definition [1]

Let $f: X_1\cup Y_1\to X_2\cup Y_2$ be a function defined on two pairs of sets (X_1,Y_1) and (X_2,Y_2) is said to be

i) Covariant, if $f(X_1)\subseteq X_2$ and $f(Y_1)\subseteq Y_2$

ii) Contravariant, if $f(X_1)\subseteq Y_2$ and $f(Y_1)\subseteq X_2$.

1.5 Definition [1]

In a Bipolar metric space (X,Y,d) and $\xi\in(X\cup Y)$

i) Such ξ is a left point, if $\xi\in X$

ii) Such ξ is a right point, if $\xi\in Y$;

iii) Such ξ is a central point, if $\xi\in(X\cap Y)$

Also, $\{x_n\}$in X is a left sequence, $\{y_n\}$ in Y is a right sequence. In a Bipolar metric space, we call a sequence, a left or right one. A sequence $\{u_n\}$ is said to be convergent to u if and only if either $\{u_n\}$ is a left sequence, is a right point and $\lim_{n\to\infty} d(u_n,u)=0$ or $\{u_n\}$ is a right sequence, u is a left point and $\lim_{n\to\infty} d(u,u_n)=0$. The bisequence $(\{x_n\},\{y_n\})$ on $X\times Y$. If $\{x_n\}$ and $\{y_n\}$both are convergent, then the bisequence $(\{x_n\},\{y_n\})$ is convergent. And $(\{x_n\},\{y_n\})$is a Cauchy bisequence, if $\lim_{n\to\infty} d(u,u_n)=0$

Note that every convergent Cauchy bisequence is biconvergent. The bipolar metric space is complete, if each Cauchy bisequence is convergent (and so it is biconvergent).

For more properties of a bipolar metric space we refer the reader to [1,13].

2 Results and Discussion

2.1 Definition

Let *(X,Y,d)* is Bipolar-metric space, $f: P^4\cup Q^4\to P\cup Q$ be a covariant mapping. If $f(p,q,r,s)=p, f(q,r,s,p)=q, f(r,s,p,q)=r, f(s,p,q,r)=s$ for $p,q,r,s\in P\cup Q$ then *(p, q, r, s)* is called quadruple fixed point of f.

2.2. Definition

Let *(X,Y,d)* is Bipolar-metric space, $f : P^4 \cup Q^4 \to P \cup Q$ and $g : P \cup Q \to P \cup Q$ be two covariant mappings. An element (p, q, r, s) is said to be quadruple coincident point of f and g. If

$f(p,q,r,s) = gp, f(q,r,s,p) = gq, f(r,s,p,q) = gr, f(s,p,q,r) = gs.$

2.3 Definition

Let *(X,Y,d)* is Bipolar-metric space, $f : P^4 \cup Q^4 \to P \cup Q$ and $g : P \cup Q \to P \cup Q$ be two covariant mappings. An element (p, q, r, s) is said to be quadruple coincident point of f and g. If

$f(p,q,r,s) = gp = p, f(q,r,s,p) = gq = q, f(r,s,p,q) = gr = r, f(s,p,q,r) = gs = s.$

2.4 Definition

Let *(X,Y,d)* is Bipolar-metric space, $f : P^4 \cup Q^4 \to P \cup Q$ and $g : P \cup Q \to P \cup Q$ be two covariant mappings are called ω - compatible, if

$g(f(p,q,r,s)) = f(gp,gq,gr,gs), g(f(q,r,s,p)) = f(gq,gr,gs,gp),$

$g(f(r,s,p,q)) = f(gr,gs,gp,gq), g(f(s,p,q,r)) = f(gs,gp,gq,gr)$

Whenever $f(p,q,r,s) = gp, f(q,r,s,p) = gq, f(r,s,p,q) = gr, f(s,p,q,r) = gs.$

Let $\phi, \psi : [0,\infty) \to [0,\infty)$ be two functions. For convenience we consider the following properties of the functions:

i) ϕ is non-increasing on [0, ∞ 0), ii) $\phi$is Lebesgue integrable, iii) for any $t > 0, \int_0^t \phi(s)ds > 0, (iv)\phi$

and

i) ψ is non-decreasing on $[0,\infty)$, , ii) $\Psi(s) \le s$for all s > 0, iii) $\psi$is additive function, iv) $\sum_{i=0}^{\infty} 2^i \psi^i(s) < \infty$ for all s > 0.

2.5 Theorem

Let (P,Q,d) is Bipolar-metric space. Suppose that $T : P^4 \cup Q^4 \to P \cup Q$ and $f : P \cup Q \to P \cup Q$ be two covariant mappings satisfying

$$\int_0^{d(T(x,y,z,w),T(p,q,r,s)} \emptyset(s)ds \le \frac{\psi}{2}\left(\int_0^{d(fx,fp)+d(fy,fq)+d(fz,fr)+d(fw,fs)} \emptyset(s)ds\right) \tag{2.1}$$

for all $(x,y,z,w) \in P$ and $(p,q,r,s) \in Q$ and

$(a)T\left(P^4 \cup Q^4\right) \subseteq f(P \cup Q)$ and $f(P \cup Q)$ is complete,

(b)Pair (T,f) is ω - comptible.

Then there is a unique common quadruple fixed point of T,f in $P \cup Q$.

Proof. Let $x_0,y_0,z_0,w_0 \in P$ and $p_0,q_0,r_0,s_0 \in Q$. be arbitrary, and from (a) we construct the bisequences $(\{\alpha_n\},\{\zeta_n\}),(\{\beta_n\},\{\eta_n\}),(\{\gamma_n\},\{\chi_n\}),(\{\delta_n\},\{v_n\})$ in $P \cup Q$

as

$T(x_n,y_n,z_n,w_n) = fx_{n+1} = \alpha_n, \quad T(p_n,q_n,r_n,s_n) = fp_{n+1} = \zeta_n$

$T(y_n,z_n,w_n,x_n) = fy_{n+1} = \beta_n, T(q_n,r_n,s_n,p_n) = fq_{n+1} = \eta_n$

$T(z_n,w_n,x_n,y_n) = fz_{n+1} = \gamma_n, T(r_n,s_n,p_n,q_n) = fr_{n+1} = \chi_n$

$T(w_n,x_n,y_n,z_n) = fw_{n+1} = \delta_n, T(s_n,p_n,q_n,r_n) = fs_{n+1} = v_n$

Where $n = 0,1,2,3,\ldots\ldots.$

Then from (2.1), we can get

$$\begin{aligned}
&\int_0^{d(\alpha_n,\zeta_{n+1})} \phi(s)ds = \int_0^{d(T(x_n,y_n,z_n,w_n),T(p_{n+1},q_{n+1},r_{n+1},s_{n+1}))} \phi(s)^{ds}ds \\
&\le \frac{\psi}{2}\left(\begin{array}{c} d(fx_n,fp_{n+1}) + d(fy_n,fq_{n+1}) + d(fz_n,fr_{n+1}) + d(fw_n,fs_{n+1}) \\ \int_0 \phi(s)ds \end{array}\right) \\
&\le \frac{\psi}{2}\left(\begin{array}{c} d(\alpha_{n-1},\zeta_n) + d(\beta_{n-1},\eta_n) + d(\gamma_{n-1},\chi_n) + d(\delta_{n-1},v_n) \\ \int_0 \phi(s)ds \end{array}\right)
\end{aligned} \tag{2.2}$$

Similarly we can prove

$$\int_0^{d(\beta_n,\eta_{n+1})}\phi(s)ds \leq \frac{\psi}{2}\left(\begin{array}{c} d(\alpha_{n-1},\zeta_n)+d(\beta_{n-1},\eta_n)+d(\gamma_{n-1},\chi_n)+d(\delta_{n-1},v_n) \\ \int_0 \phi(s)ds \end{array}\right) \tag{2.3}$$

and

$$\int_0^{d(\gamma_n,\chi_{n+1})}\phi(s)ds \leq \frac{\psi}{2}\left(\begin{array}{c} d(\alpha_{n-1},\zeta_n)+d(\beta_{n-1},\eta_n)+d(\gamma_{n-1},\chi_n)+d(\delta_{n-1},v_n) \\ \int_0^{\phi(s)ds} \end{array}\right) \tag{2.4}$$

also

$$\int_0^{d(\delta_n,v_{n+1})}\phi(s)ds \leq \frac{\psi}{2}\left(\begin{array}{c} d(\alpha_{n-1},\zeta_n)+d(\beta_{n-1},\eta_n)+d(\gamma_{n-1},\chi_n)+d(\delta_{n-1},v_n) \\ \int_0 \phi(s)ds \end{array}\right) \tag{2.5}$$

For all $n \geq 0$ Since $\phi$is non-increasing function, we obtain

$$\int_0^{a+b+c+d}\phi(s)ds \leq \int_0^a \phi(s)ds + \int_0^b \phi(s)ds + \int_0^c \phi(s)ds + \int_0^d \phi(s)ds \tag{2.6}$$

For all $a,b,c,d \geq 0$ and $k_n = d(\alpha_n,\zeta_{n+1})+d(\beta_n,\eta_{n+1})+d(\gamma_n,\chi_{n+1})+d(\delta_n,v_{n+1})$

Now since ψ is linear and non-decreasing, it follows from (2.2) to (2.6), we conclude that

$$\begin{gathered} \leq 2\Psi \int_0^{d(\alpha_{n-1},\zeta_n)+d(\beta_{n-1},\eta_n)+d(\gamma_{n-1},\chi_n)+d(\delta_{n-1},v_n)} \varnothing(s)ds - - - - \leq \\ 2^n\Psi^n\left(\int_0^{d(\alpha_0,\zeta_1)+d(\beta_0,\eta_1)+d(\gamma_0,\chi_1)+d(\delta_0,v_1)} \emptyset(s)ds\right) \\ \leq 2^n\psi^n\left(\int_0^{k_0}\phi(s)ds\right) \end{gathered} \tag{2.7}$$

On the other hand

$$\begin{aligned} \int_0^{d(\alpha_{n+1},\zeta_n)}\phi(s)ds &= \int_0^{d(T(x_{n+1},y_{n+1},z_{n+1},w_{n+1}),T(p_n,q_n,r_n,s_n))}\phi(s)^d ds \\ \leq \frac{\psi}{2}&\left(\begin{array}{c} d(fx_{n+1},fp_n)+d(fy_{n+1},fq_n)+d(fz_{n+1},fr_n)+d(fw_{n+1},fs_n) \\ \int_0 \phi(s)ds \end{array}\right) \\ \leq \frac{\psi}{2}&\left(\begin{array}{c} d(\alpha_n,\zeta_{n-1})+d(\beta_n,\eta_{n-1})+d(\gamma_n,\chi_{n-1})+d(\delta_n,\nu_{n-1}) \\ \int_0 \phi(s)ds \end{array}\right) \end{aligned} \tag{2.8}$$

Similarly we can prove

$$\int_0^{d(\beta_{n+1},\eta_n)}\phi(s)ds \leq \frac{\psi}{2}\left(\begin{array}{c} d(\alpha_n,\zeta_{n-1})+d(\beta_n,\eta_{n-1})+d(\gamma_n,\chi_{n-1})+d(\delta_n,v_{n-1}) \\ \int_0 \phi(s)ds \end{array}\right) \tag{2.9}$$

and

$$\int_0^{d(\gamma_{n+1},\chi_n)}\phi(s)ds \leq \frac{\psi}{2}\left(\begin{array}{c} d(\alpha_n,\zeta_{n-1})+d(\beta_n,\eta_{n-1})+d(\gamma_n,\chi_{n-1})+d(\delta_n,v_{n-1}) \\ \int_0 \phi(s)ds \end{array}\right) \tag{2.10}$$

also

$$\int_0^{d(\delta_{n+1},v_n)}\phi(s)ds \leq \frac{\psi}{2}\left(\begin{array}{c} d(\alpha_n,\zeta_{n-1})+d(\beta_n,\eta_{n-1})+d(\gamma_n,\chi_{n-1})+d(\delta_n,v_{n-1}) \\ \int_0 \phi(s)ds \end{array}\right) \tag{2.11}$$

Let $k_n = d(\alpha_{n+1},\zeta_n)+d(\beta_{n+1},\eta_n)+d(\gamma_{n+1},\chi_n)+d(\delta_{n+1},v_n)$

Since ψ is linear and non-decreasing, it follows from (2.8) to (2.11), we conclude that

$$\int_0^{t_n}\phi(s)ds \le \int_0^{d(\alpha_{n+1},\zeta_n)}\phi(s)ds+\int_0^{d(\beta_{n+1},\eta_n)}\phi(s)ds+\int_0^{d(\gamma_{n+1},\chi_n)}\phi(s)ds+\int_0^{d(\delta_{n+1},v_n)}\phi(s)ds$$
$$\le 2^n\psi^n\begin{pmatrix} d(\alpha_1,\zeta_0)+d(\beta_1,\eta_0)+d(\gamma_1,\chi_0)+d(\delta_1,v_0) \\ \int_0\phi(s)ds \end{pmatrix}$$

$$\le 2^n\psi^n\left(\int_0^{t_0}\phi(s)ds\right) \tag{2.12}$$

Moreover,

$$\begin{aligned}\int_0^{d(\alpha_n,\zeta_n)}\phi(s)ds &= \int_0^{d(T(x_n,y_n,z_n,w_n),T(p_n,q_n,r_n,s_n))}\phi(s)^d ds\\ &\le \frac{\psi}{2}\left(\int_0^{d(fx_n,fp_n)+d(fy_n,fq_n)+d(fz_n,ff_n)+d(fw_n,fs_n)}\phi\right)\\ &\le \frac{\psi}{2}(s)ds\left(\int_0^{d(\alpha_{n-1},\zeta_{n-1})+d(\beta_{n-1},\eta_{n-1})+d(\gamma_{n-1},\chi_{n-1})+d(\delta_{n-1},v_{n-1})}\right)\end{aligned} \tag{2.13}$$

Similarly we can prove that

$$\int_0^{d(\beta_n,\eta_n)}\phi(s)ds \le \frac{\psi}{2}\begin{pmatrix} d(\alpha_{n-1},\zeta_{n-1})+d(\beta_{n-1},\eta_{n-1})+d(\gamma_{n-1},\chi_{n-1})+d(\delta_{n-1},v_{n-1}) \\ \int_0\phi(s)ds \end{pmatrix} \tag{2.14}$$

and

$$\int_0^{d(\gamma_n,\chi_n)}\phi(s)ds \le \frac{\psi}{2}\begin{pmatrix} d(\alpha_{n-1},\zeta_{n-1})+d(\beta_{n-1},\eta_{n-1})+d(\gamma_{n-1},\chi_{n-1})+d(\delta_{n-1},v_{n-1}) \\ \int_0\phi(s)ds \end{pmatrix} \tag{2.15}$$

also

$$\int_0^{d(\delta_n,v_n)}\phi(s)ds \le \frac{\psi}{2}\begin{pmatrix} d(\alpha_{n-1},\zeta_{n-1})+d(\beta_{n-1},\eta_{n-1})+d(\gamma_{n-1},\chi_{n-1})+d(\delta_{n-1},v_{n-1}) \\ \int_0\phi(s)ds \end{pmatrix} \tag{2.16}$$

Let $e_n = d(\alpha_n,\zeta_n)+d(\beta_n,\eta_n)+d(\gamma_n,\chi_n)+d(\delta_n,v_n)$

Since ψ is linear and non-decreasing, it follows from (2.13) to (2.16), we conclude that

$$\int_0^{e_n}\phi(s)ds \le \int_0^{d(\alpha_n,\zeta_n)}\phi(s)ds+\int_0^{d(\beta_n,\eta_n)}\phi(s)ds+\int_0^{d(\gamma_n,\chi_n)}\phi(s)ds+\int_0^{d(\delta_n,v_n)}\phi(s)ds$$
$$\le 2^n\psi^n\begin{pmatrix} d(\alpha_0,\zeta_0)+d(\beta_0,\eta_0)+d(\gamma_0,\chi_0)+d(\delta_0,v_0) \\ \int_0^{\phi(s)ds} \end{pmatrix}$$

$$\le 2^n\psi^n\left(\int_0^{e_0}\phi(s)ds\right) \tag{2.17}$$

Now, for each $m,n\in N$ with $m>n$. Then from (2.7), (2.12), (2.17) and using (B_4), we have

$$d(\alpha_n,\zeta_m)+d(\beta_n,\eta_m)+d(\gamma_n,\chi_m)+d(\delta_n,v_m)$$
$$\int_0\emptyset(s)ds$$

$$d(\alpha_n,\zeta_{n+1})+d(\beta_n,\eta_{n+1})+d(\gamma_n,\chi_{n+1})+d(\delta_n,v_{n+1})$$
$$\le\int_0\emptyset(s)ds$$

$$
\begin{aligned}
& d\left(\alpha_{n+1}, \zeta_{n+1}\right)+d\left(\beta_{n+1}, \eta_{n+1}\right)+d\left(\gamma_{n+1}, \chi_{n+1}\right)+d\left(\delta_{n+1}, v_{n+1}\right) \\
& +\int_0 \emptyset(s) ds+ \\
& d\left(\alpha_{m-1}, \zeta_{m-1}\right)+d\left(\beta_{m-1}, \eta_{m-1}\right)+d\left(\gamma_{m-1}, \chi_{m-1}\right)+d\left(\delta_{m-1}, v_{m-1}\right) \\
& +\int_0 \\
& \phi(s) ds \\
& d\left(\alpha_{m-1}, \zeta_m\right)+d\left(\beta_{m-1}, \eta_m\right)+d\left(\gamma_{m-1}, \chi_m\right)+d\left(\delta_{m-1}, v_m\right) \\
& +\int_0 \emptyset(s) ds
\end{aligned}
$$

$$
\begin{aligned}
& \leq \int_0^{k_n} \phi(s) ds+\int_0^{e_{n+1}} \phi(s) ds+\ldots\ldots\ldots+\int_0^{e_{m-1}} \phi(s) ds+\int_0^{k_{m-1}} \phi(s) ds \\
& \leq 2^n \psi^n\left(\int_0^{k_0} \phi(s) ds\right)+2^{n+1} \psi^{n+1}\left(\int_0^{e_0} \phi(s) ds\right)+\ldots\ldots\ldots+2^{m-1} \psi^{m-1}\left(\int_0^{e_0} \phi(s) ds\right)+2^{m-1} \psi^{m-1}\left(\int_0^{k_0} \phi(s) ds\right) \\
& \leq\left(2^n \psi^n+2^{n+1} \psi^{n+1}+\ldots\ldots\ldots+2^{m-1} \psi^{m-1}\right)\left(\int_0^{k_0} \phi(s) ds\right)+\left(2^{n+1} \psi^{n+1}+2^{n+2} \psi^{n+2} \ldots\ldots\ldots+2^{m-1} \psi^{m-1}\right) \int_0^{e_0} \phi(s) ds\Big) \\
& \leq \sum_{i=n}^{m-1} 2^i \psi^i\left(\int_0^{k_0} \phi(s) ds\right)+\sum_{i=n+1}^{m-1} 2^i \psi^i\left(\int_0^{e_0} \phi(s) ds\right) \\
& \qquad \leq \textstyle\sum_{i=0}^{\infty} 2^i \psi^i\left(\int_0^{k_0} \phi(s) ds+\int_0^{e_0} \phi(s) ds\right)
\end{aligned}
\tag{2.18}
$$

and

$$
\begin{aligned}
& \int_0^{d(\alpha_m,\zeta_n)+d(\beta_m,\eta_n)+d(\gamma_m,\chi_n)+d(\delta_m,\nu_n)} \emptyset(s) ds \\
\leq & \int_0^{d(\alpha_m,\zeta_{m-1})+d(\beta_m,\eta_{m-1})+d(\gamma_m,\chi_{m-1})+d(\delta_m,\nu_{m-1})} \emptyset(s) ds \\
& + \int_0^{d(\alpha_{m-1},\zeta_{m-1})+d(\beta_{m-1},\eta_{m-1})+d(\gamma_{m-1},\chi_{m-1})+d(\delta_{m-1},\nu_{m-1})} \emptyset(s) ds \\
& + \int_0^{d(\alpha_{n+1},\zeta_{m+1})+d(\beta_{n+1},\eta_{m+1})+d(\gamma_{n+1},\chi_{m+1})+d(\delta_{n+1},\nu_{m+1})} \emptyset(s) ds \\
& + \int_0^{d(\alpha_{n+1},\zeta_n)+d(\beta_{n+1},\eta_n)+d(\gamma_{n+1},\chi_n)+d(\delta_{n+1},\nu_n)} \emptyset(s) ds
\end{aligned}
$$

$$
\begin{aligned}
& \leq \int_0^{t_{m-1}} \phi(s) ds+\int_0^{e_{m-1}} \phi(s) ds+\ldots\ldots\ldots+\int_0^{e_{n+1}} \phi(s) ds+\int_0^{t_n} \phi(s) ds \\
& \leq 2^{m-1} \psi^{m-1}\left(\int_0^{t_0} \phi(s) ds\right)+2^{m-1} \psi^{m-1}\left(\int_0^{e_0} \phi(s) ds\right)+\ldots\ldots\ldots+2^{n+1} \psi^{n+1}\left(\int_0^{e_0} \phi(s) ds\right)+2^n \psi^n\left(\int_0^{t_0} \phi(s) ds\right) \\
& \leq\left(2^n \psi^n+2^{n+1} \psi^{n+1}+\ldots\ldots\ldots+2^{m-1} \psi^{m-1}\right)\left(\int_0^{t_0} \phi(s) ds\right)+\left(2^{n+1} \psi^{n+1}+2^{n+2} \psi^{n+2} \ldots\ldots\ldots+2^{m-1} \psi^{m-1}\right)\left(\int_0^{e_0} \phi(s) ds\right) \\
& \leq \sum_{i=n}^{m-1} 2^i \psi^i\left(\int_0^{t_0} \phi(s) ds\right)+\sum_{i=n+1}^{m-1} 2^i \psi^i\left(\int_0^{e_0} \phi(s) ds\right)
\end{aligned}
$$

$$\leq \sum_{i=0}^{\infty} 2^i \psi^i \left(\int_0^{t_0} \phi(s)ds + \int_0^{e_0} \phi(s)ds\right) \tag{2.19}$$

for $m > n$. Since $\sum 2^i \psi^i(s) < \infty, \forall s \in [0, +\infty)$, for an arbitrary $\varepsilon > 0$ such that

$\sum_{i=0}^{\infty} 2^i \psi^i \left(\int_0^{k_0} \phi(s)ds + \int_0^{e_0} \phi(s)ds\right) < \frac{\varepsilon}{3}$ and $\sum_{i=0}^{\infty} 2^i \psi^i \left(\int_0^{t_0} \phi(s)ds + \int_0^{e_0} \phi(s)ds\right) < \frac{\varepsilon}{3}$

From (2.18) and (2.19), we have

$d(\alpha_n, \zeta_m) + d(\beta_n, \eta_m) + d(\gamma_n, \chi_m) + d(\delta_n, \nu_m) < \frac{\varepsilon}{3}$

Then the bisequences $(\{\alpha_n\}, \{\zeta_n\}), (\{\beta_n\}, \{\eta_n\}), (\{\gamma_n\}, \{\chi_n\}), (\{\delta_n\}, \{\nu_n\}\}$ are Cauchy bisequences in (P, Q). Suppose $f(P \cup Q)$ is complete subspace of (P, Q, d) , then the sequences

$\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\zeta_n\}, \{\eta_n\}, \{\chi_n\}, \{\nu_n\} \subseteq f(P \cup Q)$ are converges in complete bipolar metric space $(f(P), f(Q), d)$. Therefore, there exist $a, b, c, d \in f(P)$ and $l, m, n, o \in f(Q)$ such that

$$\begin{aligned} &\lim_{n\to\infty} \alpha_n = l, \lim_{n\to\infty} \beta_n = m, \lim_{n\to\infty} \gamma_n = n, \lim_{n\to\infty} \delta_n = o \\ &\lim_{n\to\infty} \zeta_n = a, \lim_{n\to\infty} \eta_n = b, \lim_{n\to\infty} \chi_n = c, \lim_{n\to\infty} \nu_n = l \end{aligned} \tag{2.20}$$

Since $f : P \cup Q \to P \cup Q$ and $a, b, c, d \in f(P)$ and $l, m, n, o \in f(Q)$,

there exist $x, y, z, w \in P$ and $p, q, r, s \in Q$ such that $fx = a, fy = b, fz = c, fw = d$ and $fp = l, fq = m, fr = n, fs = o$.

$$\begin{aligned} &\lim_{n\to\infty} \alpha_n = l = fp, \lim_{n\to\infty} \beta_n = m = fq, \lim_{n\to\infty} \gamma_n = n = fr, \lim_{n\to\infty} \delta_n = o = fs \\ &\lim_{n\to\infty} \zeta_n = a = fx, \lim_{n\to\infty} \eta_n = b = fy, \lim_{n\to\infty} \chi_n = c = fz, \lim_{n\to\infty} \nu_n = l = fw_{-} \end{aligned} \tag{2.21}$$

Claim that $T(x, y, z, w) = l, T(y, z, w, x) = m, T(z, w, x, y) = n, T(w, x, y, z) = o$

and $T(p, q, r, s) = a, T(q, r, s, p) = b, T(r, s, p, q) = c, T(s, p, q, r) = d$

By using (2.1) we have

$$\begin{aligned} \int_0^{d(T(x,y,z,w),l)} \emptyset(s)ds &\leq \int_0^{d(T(x,y,z,w),\zeta_{n+1})} \emptyset(s)ds + \int_0^{d(\alpha_{n+1},\zeta_{n+1})} \emptyset(s)ds + \int_0^{d(\alpha_{n+1},l)} \emptyset(s)ds \\ &\leq \int_0^{d(T(x,y,z,w),T(p_{n+1},q_{n+1},r_{n+1},s_{n+1})} \emptyset(s)ds + \int_0^{d(\alpha_{n+1},\zeta_{n+1})} \emptyset(s)ds + \int_0^{d(\alpha_{n+1},l)} \emptyset(s)ds \\ &\leq \frac{\Psi}{2}\left(\int_0^{d(fx,fp_{n+1})+d(fy,fq_{n+1})+d((fz,fr_{n+1})+d(fw,fs_{n+1})} \emptyset(s)ds\right) + \int_0^{d(\alpha_{n+1},\zeta_{n+1})} \emptyset(s)ds \\ &\quad + \int_0^{d(\alpha_{n+1},l)} \emptyset(s)ds \\ &\leq \frac{\Psi}{2}\left(\int_0^{d(fx,\zeta_n)+d(fy,\eta_n)+d((fz,\chi_n)+d(fw,\nu_n)} \emptyset(s)ds\right) + \int_0^{d(\alpha_{n+1},\zeta_{n+1})} \emptyset(s)ds + \int_0^{d(\alpha_{n+1},l)} \emptyset(s)ds \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain $d(T(x, y, z, w), l) = 0$ which implies $T(x, y, z, w) = l$. Similarly we can prove that $T(y, z, w, x) = m, T(z, w, x, y) = n$ and $T(w, x, y, z) = o$ and $T(p, q, r, s) = a, T(q, r, s, p) = b, T(r, s, p, q) = c, T(s, p, q, r) = d$ Therefore it follows that $T(x, y, z, w) = l = fp, T(y, z, w, x) = m = fq, T(z, w, x, y) = n = fr, T(w, x, y, z) = o = fs$and $T(p, q, r, s) = a = fx, T(q, r, s, p) = b = fy, T(r, s, p, q) = c = fz, T(s, p, q, r) = d = fw$.

Since the pair (T, f) is ω compatible, we have $T(l, m, n, o) = fl, T(m, n, o, l) = fm, T(n, o, l, m) = fn$ and $T(o, l, m, n) = fo$ and $T(a, b, c, d) = fa, T(b, c, d, a) = fb, T(c, d, a, b) = fc$ and $T(d, a, b, c) = fd$

Now have to prove that $fl = l, fm = m, fn = n, fo = o$

and $fa = a, fb = b, fc = c, fd = d$

Now we have

$$\begin{aligned} \int_0^{d(fa,\zeta_{n+1})} \emptyset(s)ds &= \int_0^{d(T(a,b,c,d),\zeta_{n+1})} \emptyset(s)ds = \int_0^{d(T(a,b,c,d),T(p_{n+1},q_{n+1},r_{n+1},s_{n+1}))} \emptyset(s)ds \\ &\leq \frac{\Psi}{2}\left(\int_0^{d(fa,fp_{n+1})+d(fb,fq_{n+1})+d((fc,fr_{n+1})+d(fd,fs_{n+1})} \emptyset(s)ds\right) \\ &\leq \frac{\Psi}{2}\left(\int_0^{d(fa,\zeta_n)+d(fb,\eta_n)+d(fc,\chi_n)+d(fd,\nu_n)} \emptyset(s)ds\right) \end{aligned}$$

$$\int_0^{d(fa,\zeta_n)+d(fb,\eta_n)+d((fc,\chi_n)+d(T(fd,\nu_n)} \emptyset(s)ds \leq$$
$$2\Psi \int_0^{d(fa,\zeta_n)+d(fb,\eta_n)+d((fc,\chi_n)+d(T(fd,\nu_n)} \emptyset(s)ds).$$

Taking the limit as $n \to \infty$in the above inequality, and which possibility holds only $d(fa,a) = 0, d(fb,b) = 0, d(fc,c) = 0$ and $d(fd,d) = 0$ which implies that $fa = a, fb = b, fc = c$ and $fd = d$. Therefore

$T(a,b,c,d) = fa = a, T(b,c,d,a) = fb = b, T(c,d,a,b) = fc = c, T(d,a,b,c) = fd = d$

Similarly we can prove that

$T(l,m,n,o) = fl = l, T(m,n,o,l) = fm = m, T(n,o,l,m) = fn = n, T(o,l,m,n) = fo = o.$

Therefore

$T(p,q,r,s) = fx = a = fa = T(a,b,c,d)$; $T(x,y,z,w) = fp = l = fl = T(l,m,n,o)$
$T(q,r,s,p) = fy = b = fb = T(b,c,d,a)$; $T(y,z,w,x) = fq = m = fm = T(m,n,o,l)$
$T(r,s,p,q) = fz = c = fc = T(c,d,a,b)$; $T(z,w,x,y) = fr = n = fn = T(n,o,l,m)$
$T(s,p,q,r) = fw = d = fd = T(d,a,b,c)$; $T(w,x,y,z) = fs = o = fo = T(o,l,m,n)$

On the other hand from (2.20), we get

$$\int_0^{d(fx,fp)+d(fy,fq)+d(fz,fr)+d(fw,fs)} \emptyset(s)ds = \int_0^{d(a,l)+d(b,m)+d(c,n)+d(d,o)} \emptyset(s)ds$$
$$= \lim_{n\to\infty}\left(\int_0^{d(\zeta_n,\alpha_n)+d(\eta_n,\beta_n)+d(\chi_n,\gamma_n)+d(\nu_n,\delta_n)} \emptyset(s)ds\right) = 0$$

Thus, $a = l, b = m, c = n, d = o$ Therefore, $(a,b,c,d) \in P^4 \cap Q^4$is a common quadruple fixed point of covariant mappings T and f.

In the following we will show that, the uniqueness of common quadruple fixed point in $P^4 \cap Q^4$

For this purpose, assume that there is another quadruple fixed point (a', b', c', d') of T and f.

Then

$$\int_0^{d(a,a')} \emptyset(s)ds = \int_0^{d((T(a,b,c,d),T(a',b',c',d'))} \emptyset(s)ds$$

$$\leq \frac{\Psi}{2} \int_0^{d(fa,fa')+d(fb,fb')+d((fc,fc')+d(T(fd,fd')} \emptyset(s)ds$$

$$\int_0^{d(a,a')+d(b,b')+d((c,c')+d(d,d')} \emptyset(s)ds \leq 2\Psi \int_0^{d(a,a')+d(b,b')+d((c,c')+d(d,d')} \emptyset(s)ds$$

$$\leq \frac{\Psi}{2}\left(\int_0^{d(a,b)+d(b,c)+d(c,d)+d(d,a)} \emptyset(s)ds\right).$$

Therefore, we have

$$\int_0^{d(a,a')+d(b,b')+d((c,c')+d(d,d')} \emptyset(s)ds \leq 2\Psi \int_0^{d(a,a')+d(b,b')+d((c,c')+d(d,d')} \emptyset(s)ds$$

Hence, we get $a = a', b = b', c = c'$ and $d = d'$

Therefore, *(a,b,c,d)* is a unique common quadruple fixed point of covariant mappings T and f.

Finally we will prove that *a=b=c=d*.

$$\leq \frac{\Psi}{2}\left(\int_0^{d(a,b)+d(b,c)+d(c,d)+d(d,a)} \emptyset(s)ds\right).$$

Therefore, we have

$$\int_0^{d(a,b)+d(b,c)+d(c,d)+d(d,a)} \emptyset(s)ds \leq 2\Psi \int_0^{d(a,b)+d(b,c)+d(c,d)+d(d,a)} \emptyset(s)ds$$

.

Hence, we get *a*=*b*=*c*=*d*.

Which means that T and f have a unique common quadruple fixed point of the form *(a,a,a,a)*.

2.6 Theorem

Let (P,Q,d) is Bipolar-metric space. Suppose that $T:\left(P^4,Q^4\right)\to(P,Q)$ and $f:(P,Q)\to(P,Q)$ be two covariant mappings satisfying

$$\int_0^{d(T(a,b,c,d),T(x,y,z,w))}\phi(s)^2ds \leq \frac{\Psi}{2}\left(\int_0^{\max\{d(fa,fx)+d(fb,fy)+d(fc,fz)+d(fd,fw)\}}ds\right)ds\Bigg) \tag{2.22}$$

for all $(a,b,c,d)\in P$ and $(x,y,z,w)\in Q$ and

$(a)T\left(P^4\cup Q^4\right)\subseteq f(P\cup Q)$ and $f(P\cup Q)$ is complete, *(b)* pair *(T,f)* is ωcompatible.

Then there is a unique common quadruple fixed point of T,f in $P\cup Q$.

2.7 Corollary

Let (P,Q,d) is a complete Bipolar-metric space. Suppose that

$T:\left(P^4,Q^4\right)\to(P,Q)$ be a covariant mapping satisfying

$$\int_0^{d(T(a,b,c,d),T(x,y,z,w))}\emptyset(s)ds \leq \frac{\Psi}{2}\left(\int_0^{\max\{d(a,x),d(b,y),d(c,z),d(d,w)\}}\emptyset(s)ds\right) \tag{2.23}$$

for all $(a,b,c,d)\in P$ and $(x,y,z,w)\in Q$

Then there is a unique common quadruple fixed point of T in $P\cup Q$

2.8 Example

Let $P=U_m(R)$ and $Q=L_m(R)$ be the set of all *m x m* upper and lower triangular matrices over *R*. Defined $d:U_m(R)\times L_m(R)\to[0,\infty)$ as $d(P,Q)=\sum_{i,i=1}^{m}\left|p_{ij}-q_{ij}\right|$ for all

$P=(p_{ij})_{m\times m}\in U_m(R)$ and $Q=(q_{ij})_{m\times m}\in L_m(R)$. Then obviously $(U_m(R),L_m(R),d)$ is a bipolar metric space. And define $T:P^4\cup Q^4\to P\cup Q$ as $T(P,Q,R,S)=\left(\frac{p_{ij}+q_{ij}+r_{ij}+s_{ij}}{5}\right)_{m\times m}$

where $\left(P=(p_{ij})_{m\times m},Q=(q_{ij})_{m\times m},R=(r_{ij})_{m\times m},S=(s_{ij})_{m\times m}\right)\in U_m(R)^4\cup L_m(R)^4$

and defines $f:P\cup Q\to P\cup Q$ as $f(P)=2\,(p_{ij})_{m\times m}$ and $\psi(t)=\frac{t}{2}$ for all $t\in[0,\infty)$.

Then obviously, $T\left(P^4\cup Q^4\right)\subseteq f(P\cup Q)$ and the pair (T,f) is ω - compatible. In fact we have,

$$\int_0^{d(T(A,B,C,D),T(P,Q,R,S))}\emptyset(s)ds = \int_0^{d\left(\left(\frac{a_{ij}+b_{ij}+c_{ij}+d_{ij}}{5}\right)_{m\times m},\left(\frac{p_{ij}+q_{ij}+r_{ij}+s_{ij}}{5}\right)_{m\times m}\right)}\emptyset(s)ds$$

$$\leq \frac{1}{10}\int_0^{\sum_{i,j}^{m}|2a_{i,j}-2p_{i,j}|+\sum_{i,j}^{m}|2b_{i,j}-2q_{i,j}|+\sum_{i,j}^{m}|2c_{i,j}-2r_{i,j}|+\sum_{i,j}^{m}|2d_{i,j}-2s_{i,j}|}\emptyset(s)ds$$

$$\leq \frac{\Psi}{2}\left(\int_0^{d(fA,fP)+d(fB,fQ)+d(fC,fR)+d(fD,fS)}\emptyset(s)ds\right)$$

Thus all the conditions of the theorem (2.5) are satisfied and $(O_{m\times m},O_{m\times m},O_{m\times m},O_{m\times m})$ is unique quadruple fixed point.

3 Application part to integral equations

As an application to the corollary (2.7), we examine the existence of a unique solution to an initial value problem in this section.

3.1 Theorem

Let the initial value problem

$$x^1(t) = T(t,(x,y,z,w)(t)), t \in I = [0,1], (x,y,z,w)(0) = (x_0, y_0, z_0, w_0) \tag{3.1}$$

Where $T : I \times \left(E_1^4 \cup E_2{}^4\right) \to R$ and $x_0, y_0, z_0, w_0 \in E_1 \cup E_2$, where $E_1 \cup E_2$is a Lebesgue measurable set with $m(E_1 \cup E_2) < \infty$. Then there exists unique solution in $C(I, L^\infty(E_1) \cup L^\infty(E_2))$for the initial value problem (3.1).

Proof. Initial value problem's (3.1), equivalent integral equation is

$x(t) = x_0 + 4\int_{E_1 \cup E_2} T(s,(x,y,z,w)(s))ds$

Let $P = C(I, L^\infty(E_1)), Q = C(I, L^\infty(E_2))$ and $d(f,g) = \|f - g\|$ for all $f, g \in P \cup Q$ and $\psi(t) = t$for all $t \in [0,\infty)$ Define $R : P^4 \cup Q^4 \to P \cup Q$ by

$$R(x,y,z,w)(t) = \frac{x_0}{4} + \int_{E_1 \cup E_2} T(s,(x,y,z,w)(s))ds \tag{3.2}$$

Now

$$\int_0^{d(R(x,y,z,w)(t),R(a,b,c,d)(t))} \phi(s)\,ds = \int_0^{\|R(x,y,z,w)(t)-R(a,b,c,d)(t)\|} \phi(s)\,ds = \int_0^{\left\|\left(\frac{x_0}{4}+\int_{E_1\cup E_2} T(s,(x,y,z,w)(s))ds\right)-\left(\frac{a_0}{4}+\int_{E_1\cup E_2} T(s,(a,b,c,d)(s))ds\right)\right\|} \phi(s)\,ds$$

$$= \int_0^{\frac{1}{4}\|x(t)-a(t)\|} \phi(s)ds$$

$$\leq \frac{1}{4}\int_0^{d(x,a)} \phi(s)ds \leq \frac{1}{4}\int_0^{\max\{d(x,a),d(y,b)d(z,c)d(w,d)\}} \phi(s)\,ds \leq \frac{\psi}{2}\left(\int_0^{\max\{d(x,a),d(y,b)d(z,c)d(w,d)\}} \phi(s)\,ds\right)$$

It follows from corollary (2.7), we conclude that R has unique fixed point in $P \cup Q$.

4 Application part to homotopy

Now we study the existence of a unique solution to Homotopy theory.

4.1 Theorem

Let (P,Q,d)be a complete bipolar metric space, *(X,Y)* be an open subset of (P,Q) and $(\bar{X},\bar{Y})$ closed subset of (P,Q) such that $(X,Y) \subseteq (\bar{X},\bar{Y})$.

Suppose $H_p : \left(\bar{X}^4 \cup \bar{Y}^4\right) \times [0,1] \to P \cup Q$ be an operator with the following conditions are satisfying.

$$\int_0^{d\left(H_p(a,b,c,d,\lambda),H_p(x,y,z,w,\lambda)\right)} \emptyset(s)ds \leq k\int_0^{\max\{d(a,x),d(b,y),d(c,z),d(d,w)\}} \emptyset(s)ds \tag{4.1.1}$$

For all $a,b,c,d \in \bar{X}, x,y,z,w \in \bar{Y}, \lambda \in [0,1]$ and $\rho \in (0,1)$

$$a \neq H_p(a,b,c,d,\lambda), b \neq H_p(b,c,d,a,\lambda), c \neq H_p(c,d,a,b,\lambda) \text{ and } d \neq H_p(d,a,b,c,\lambda) \tag{4.1.2}$$

for each $a,b,c,d \in \partial X \cup \partial Y$ and $\lambda \in [0,1]$ where $\partial X \cup \partial Y$ is the boundary of $X \cup Y$ in $P \cup Q$.

$$\exists M \geq 0 \ni \int_0^{d\left(H_p(a,b,c,d,\mu),H_p(x,y,z,w,\tau)\right)} \emptyset(s)ds \leq \int_0^{M|\mu-\tau|} \emptyset(s)ds \tag{4.1.3}$$

for every $a,b,c,d \in \bar{X}, x,y,z,w \in \bar{Y}$ and $\mu, \tau \in [0,1]$.

Then $H_p(.,0)$ has a quadruple fixed point $\Leftrightarrow H_p(.,1)$ has a quadruple fixed point.

Proof. Consider the sets

$$A = \left\{ \begin{array}{c} \lambda \in [0,1] : H_p(a,b,c,d,\lambda) = a, H_p(b,c,d,a,\lambda) = b, H_p(c,d,a,b,\lambda) = c \\ H_p(d,a,b,c,\lambda) = d, \text{ for some } a,b,c,d \in X \end{array} \right\}$$

$$B = \left\{ \begin{array}{c} \xi \in [0,1] : H_p(x,y,z,w,\xi) = x, H_p(y,z,w,x,\xi) = y, H_p(z,w,x,y,\xi) = z \\ H_p(w,x,y,z,\xi) = w, \text{ for some } x,y,z,w \in Y \end{array} \right\}$$

Since $H_p(.,0)$ has a quadruple fixed point in $X^4 \cup Y^4$, so $(0,0,0,0) \in A^4 \cap B^4$.
Then $A \cap B \neq \phi$ Now we will show that $A \cap B$is both closed and open in [0,1] and hence by connectedness $A = B = [0,1]$.
Let $(\{\lambda_n\}_{n=1}^{\infty}, \{\xi\}_{n=1}^{\infty}) \subseteq (X,Y)$ with $(\lambda_n, \xi_n) \to (\lambda, \lambda) \in [0,1]$ as $n \to \infty$
We must show that $\lambda \in A \cap B$.
Since $(\lambda_n, \xi_n) \in (A,B)$ for $n = 0,1,2,3,\ldots\ldots$ there exist bisequences
$(a_n, x_n), (b_n, y_n), (c_n, z_n), (d_n, w_n)$ with
$a_{n+1} = H_p(a_n, b_n, c_n, d_n, \lambda_n), b_{n+1} = H_p(b_n, c_n, d_n, a_n, \lambda_n)$,
$c_{n+1} = H_p(c_n, d_n, a_n, b_n, \lambda_n), d_{n+1} = H_p(d_n, a_n, b_n, c_n, \lambda_n)$
and
$x_{n+1} = H_p(x_n, y_n, z_n, w_n, \xi_n), y_{n+1} = H_p(y_n, z_n, w_n, x_n, \xi_n)$
$z_{n+1} = H_p(z_n, w_n, x_n, y_n, \xi_n), w_{n+1} = H_p(w_n, x_n, y_n, z_n, \xi_n)$
Consider

$$\int_0^{d(a_n, x_{n+1})} \phi(s)ds = \int_0^{d(H_p(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}, \lambda_{n-1}), H_p(x_n, y_n, z_n, w_n, \xi_n))} \phi(s)\, ds$$

$$\leq \int_0^{d(H_p(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}, \lambda_{n-1}), H_p(x_n, y_n, z_n, w_n, \xi_{n-1})) + d(H_p(a_n, b_n, c_n, d_n, \lambda_{n-2}), H_p(x_n, y_n, z_n, w_n, \xi_{n-1})) + d(H_p(a_n, b_n, c_n, d_n, \lambda_{n-2}), H_p(x_n, y_n, z_n, w_n, \xi_n))} \phi(s)ds$$

$$\leq \int_0^{d(H_p(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}, \lambda_{n-1}), H_p(x_n, y_n, z_n, w_n, \xi_{n-1}))} \phi(s)\, ds + \int_0^{M|\lambda_{n-2} - \xi_{n-1}|} \phi(s)ds + \int_0^{M|\lambda_{n-2} - \xi_n|} \phi(s)ds$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \int_0^{d(a_n, x_{n+1})} \phi(s)ds \leq \lim_{n\to\infty} \int_0^{d(H_p(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}, \lambda_{n-1}), H_p(x_n, y_n, z_n, w_n, \xi_{n-1}))} \phi(s)\, ds$$

$$\leq \lim_{n\to\infty} \rho \int_0^{\max\{d(a_{n-1}, x_n), d(b_{n-1}, y_n), d(c_{n-1}, z_n), d(d_{n-1}, w_n)\}} \phi(s)\, ds$$

Therefore, we have

$$\lim_{n\to\infty} \int_0^{\max\{d(a_n, x_{n+1}), d(b_n, y_{n+1}), d(c_n, z_{n+1}), d(d_n, w_{n+1})\}} \phi(s)\, ds$$

$$\leq \lim_{n\to\infty} \rho \int_0^{\max\{d(a_{n-1}, x_n), d(b_{n-1}, y_n), d(c_{n-1}, z_n), d(d_{n-1}, w_n)\}} \phi(s)\, ds$$

$$\leq \lim_{n\to\infty} \rho^2 \int_0^{\max\{d(a_{n-2}, x_{n-1}), d(b_{n-2}, y_{n-1}), d(c_{n-2}, z_{n-1}), d(d_{n-2}, w_{n-1})\}} \phi(s)\, ds$$

$$\vdots$$

$$\leq \lim_{n\to\infty} \rho^n \int_0^{\max\{d(a_0, x_1), d(b_0, y_1), d(c_0, z_1), d(d_0, w_1)\}} \phi(s)\, ds = 0$$

It follows $d(a_n, x_{n+1}) \to 0, d(b_n, y_{n+1}) \to 0, d(c_n, z_{n+1}) \to 0, d(d_n, w_{n+1}) \to 0$, as $n \to \infty$. Similarly, we can show that
$d(a_{n+1}, x_n) \to 0, d(b_{n+1}, y_n) \to 0, d(c_{n+1}, z_n) \to 0, d(d_{n+1}, w_n) \to 0$, as $n \to \infty$ and
$d(a_n, x_n) \to 0, d(b_n, y_n) \to 0, d(c_n, z_n) \to 0, d(d_n, w_n) \to 0$, as $n \to \infty$.
For each $n, k \in N, n < k$
Using the property *(B4)*, we have

$$\int_0^{d(a_n, x_k) + d(b_n, y_k) + d(c_n, z_k) + d(d_n, w_k)} \phi(s)\, ds \leq \int_0^{d(a_n, x_{n+1}) + d(b_n, y_{n+1}) + d(c_n, z_{n+1}) + d(d_n, w_{n+1})} \phi(s) + \int_0^{d(a_{n+1}, x_{n+1}) + d(b_{n+1}, y_{n+1}) + d(c_{n+1}, z_{n+1}) + d(d_{n+1}, w_{n+1})} \phi(s) \ldots$$

$$+ \int_0^{d(a_{k-1}, x_{k-1}) + d(b_{k-1}, y_{k-1}) + d(c_{k-1}, z_{k-1}) + d(d_{k-1}, w_{k-1})} \phi(s) + \int_0^{d(a_{k-1}, x_k) + d(b_{k-1}, y_k) + d(c_{k-1}, z_k) + d(d_{k-1}, w_k)} \phi(s)$$

$$\leq 4(\rho^n + \rho^{n+1} + \rho^{n+2} + \ldots\ldots + \rho^{k-1}) \int_0^{\max\{d(a_0, x_1), d(b_0, y_1), d(c_0, z_1), d(d_0, w_1)\}} \phi(s)\, ds +$$

$$4(\rho^{n+1} + \rho^{n+2} + \ldots\ldots + \rho^{k-1}) \int_0^{\max\{d(a_0, x_0), d(b_0, y_0), d(c_0, z_0), d(d_0, w_0)\}} \phi(s)\, ds$$

$$\leq 4\frac{\rho^n}{1-\rho} \int_0^{\max\{d(a_0, x_1), d(b_0, y_1), d(c_0, z_1), d(d_0, w_1)\}} \phi(s)\, ds + 4\frac{\rho^{n+1}}{1-\rho} \int_0^{\max\{d(a_0, x_0), d(b_0, y_0), d(c_0, z_0), d(d_0, w_0)\}} \phi(s)\, ds \to 0 \; as \; n,k \to \infty$$

It follows that
$\lim_{n,k\to\infty}(d(a_n, x_k) + d(b_n, y_k) + d(c_n, z_k) + d(d_n, w_k)) = 0$ similarly, we can show that $\lim_{n,k\to\infty}(d(a_k, x_n) + d(b_k, y_n) + d(c_k, z_n) + d(d_k,$
0
Therefore, $(a_n, x_n), (b_n, y_n), (c_n, z_n)$ and (d_n, w_n) are Cauchy bisequence in *(X,Y)*.

By completeness, there exist $a, b, c, d \in X$ and $x, y, z, w \in Y$ with

$$\begin{aligned}&\lim_{n\to\infty} a_n = x, \lim_{n\to\infty} b_n = y, \lim_{n\to\infty} c_n = z, \lim_{n\to\infty} d_n = w\\&\lim_{n\to\infty} x_n = a, \lim_{n\to\infty} y_n = b, \lim_{n\to\infty} z_n = c, \lim_{n\to\infty} w_n = d\end{aligned} \tag{4.1.4}$$

Now consider

$$\begin{aligned}\int_0^{d(H_p(a,b,c,d,\lambda),x)} \phi(s)ds &\leq \int_0^{d(H_p(a,b,c,d,\lambda),x_{n+1})} ds + \int_0^{d(a_{n+1},x_{n+1})} \phi(s)ds + \int_0^{d(a_{n+1},x)} \phi(s)ds\\&\leq \lim_{n\to\infty} d\left(H_p(a,b,c,d,\lambda), H_p\left(x_n,y_n,z_n,w_n,\lambda\right)\right)\\&\qquad \int_0 \phi(s)ds\\&\leq \lim_{n\to\infty} \rho \quad \int_0^{\max\{d(a,x_n),d(b,y_n),d(c,z_n),d(d,w_n)\}} \phi(s)ds \quad = 0\end{aligned}$$

It follows $d\left(H_p(a,b,c,d,\lambda),x\right) = 0$ implies that $H_p(a,b,c,d,\lambda) = x$. Similarly, we get

$H_p(b,c,d,a,\lambda) = y, H_p(c,d,a,b,\lambda) = z, H_p(d,a,b,c,\lambda) = w$ and

$H_p(x,y,z,w,\xi) = a, H_p(y,z,w,x,\xi) = b, H_p(z,w,x,y,\xi) = c, H_p(w,x,y,z,\xi) = d$

On the other hand from (4.1.3), we get

$$\int_0^{d(a,x)} \phi(s)ds = \int_0^{d(\lim_{n\to\infty} x_n, \lim_{n\to\infty} a_n)} \phi(s)ds = \int_0^{\lim_{n\to\infty} d(a_n,x_n)} \phi(s)ds = 0$$

Therefore a=x. Similarly we can prove that b=y, c=z, d=w.

Thus, $\lambda \in A \cap B$. Clearly $A \cap B$ is closed in [0.1].

Let, $(\lambda_0, \xi_0) \in (A,B)$, then there exist bisequences $(a_0,x_0), (b_0,y_0), (c_0,z_0), (d_0,w_0)$ with

$a_0 = H_p\left(a_0,b_0,c_0,d_0,\lambda_0\right), b_0 = H_p\left(b_0,c_0,d_0,a_0,\lambda_0\right), c_0 = H_p\left(c_0,d_0,a_0,b_0,\lambda_0\right), d_0 = H_p\left(d_0,a_0,b_0,c_0,\lambda_0\right)$

and

$x_0 = H_p\left(x_0,y_0,z_0,w_0,\xi_0\right), y_0 = H_p\left(y_0,z_0,w_0,x_0,\xi_0\right), z_0 = H_p\left(z_0,w_0,x_0,y_0,\xi_0\right), w_0 = H_p\left(w_0,x_0,y_0,z_0,\xi_0\right)$

Since $A \cup B$ is open, then there exist $\theta > 0$ such that $B_d\left(a_0,\theta\right) \subseteq A \cup B$ and

$B_d\left(b_0,\theta\right) \subseteq A \cup B, B_d\left(c_0,\theta\right) \subseteq A \cup B, B_d\left(d_0,\theta\right) \subseteq A \cup B$

$B_d\left(x_0,\theta\right) \subseteq A \cup B, B_d\left(y_0,\theta\right) \subseteq A \cup B, B_d\left(z_0,\theta\right) \subseteq A \cup B, B_d\left(w_0,\theta\right) \subseteq A \cup B$

choose $\lambda \in \left(\xi_0 - \varepsilon, \xi_0 + \varepsilon\right), \xi \in \left(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon\right)$ such that

$|\lambda - \xi_0| \leq \frac{1}{M^n} < \frac{\varepsilon}{2}, |\xi - \lambda_0| \leq \frac{1}{M^n} < \frac{\varepsilon}{2}, |\lambda_0 - \xi_0| \leq \frac{1}{M^n} < \frac{\varepsilon}{2}$

Then for

$x \in \bar{B}_{A\cup B}\left(a_0,\theta\right) = \left\{x, x_0 \in Q/d\left(a_0,x\right) \leq \theta + d\left(a_0,x_0\right)\right\},$

$y \in \bar{B}_{A\cup B}\left(b_0,\theta\right) = \left\{y, y_0 \in Q/d\left(b_0,y\right) \leq \theta + d\left(b_0,y_0\right)\right\},$

$z \in \bar{B}_{A\cup B}\left(c_0,\theta\right) = \left\{z, z_0 \in Q/d\left(c_0,z\right) \leq \theta + d\left(c_0,z_0\right)\right\},$

$w \in \bar{B}_{A\cup B}\left(d_0,\theta\right) = \left\{w, w_0 \in Q/d\left(d_0,w\right) \leq \theta + d\left(d_0,w_0\right)\right\},$

$a \in \bar{B}_{A\cup B}\left(\theta,x_0\right) = \left\{a, a_0 \in P/d\left(a,x_0\right) \leq \theta + d\left(a_0,x_0\right)\right\},$

$b \in \bar{B}_{A\cup B}\left(\theta,y_0\right) = \left\{b, b_0 \in P/d\left(b,y_0\right) \leq \theta + d\left(b_0,y_0\right)\right\},$

$c \in \bar{B}_{A\cup B}\left(\theta,z_0\right) = \left\{c, c_0 \in P/d\left(c,z_0\right) \leq \theta + d\left(c_0,z_0\right)\right\},$

$d \in \bar{B}_{A\cup B}\left(\theta,w_0\right) = \left\{d, d_0 \in P/d\left(d,w_0\right) \leq \theta + d\left(d_0,w_0\right)\right\},$

$$\int_0^{d(H_p(a,b,c,d,\lambda),x_0)} \phi(s)\,ds = \int_0^{d(H_p(a,b,c,d,\lambda),H_p(x_0,y_0,z_0,w_0,\xi_0))} \phi(s)\,ds$$
$$\leq \int_0^{d(H_p(a,b,c,d,\lambda),H_p(x,y,z,w,\xi_0))} \phi(s)\,ds + \int_0^{d(H_p(a_0,b_0,c_0,d_0,\lambda),H_p(x,y,z,w,\xi_0))} \phi(s)\,ds + \int_0^{d(H_p(a_0,b_0,c_0,d_0,\lambda),H_p(x_0,y_0,z_0,w_0,\xi_0))} \phi(s)\,ds$$
$$\leq 2\int_0^{M|\lambda-\xi_0|} \phi(s)ds + \rho \int_0^{\max\{d(a_0,x),d(b_0,y),d(c_0,z),d(d_0,w)\}} \phi(s)\,ds$$
$$\leq \frac{2}{M^{n-1}}\int_0^{1} \phi(s)ds + \rho \int_0^{\max\{d(a_0,x),d(b_0,y),d(c_0,z),d(d_0,w)\}} \phi(s)\,ds$$

Taking limit as $n \to \infty$

$$\int_0^{d(H_p(a,b,c,d,\lambda),x_0)} \phi(s)\,ds = \rho \int_0^{\max\{d(a_0,x),d(b_0,y),d(c_0,z),d(d_0,w)\}} \phi(s)\,ds$$

Therefore, we deduce that

$$\int_0^{\max\{d(H_p(a,b,c,d,\lambda),x_0),d(H_p(b,c,d,a,\lambda),y_0),d(H_p(c,d,a,b,\lambda),z_0),d(H_p(d,a,b,c,\lambda),w_0)\}} \phi(s)\,ds = \rho \int_0^{\max\{d(a_0,x),d(b_0,y),d(c_0,z),d(d_0,w)\}} \phi(s)\,ds$$
$$< \int_0^{\max\{d(a_0,x),d(b_0,y),d(c_0,z),d(d_0,w)\}} \phi(s)\,ds$$
$$\leq \int_0^{\max\left\{\begin{array}{l}\theta+d(a_0,x_0),\\ \theta+d(b_0,y_0),\\ \theta+d(c_0,z_0),\\ \theta+d(d_0,w_0)\end{array}\right\}} \phi(s)\,ds$$

Similarly, we can prove that

$$\int_0^{\max\{d(a_0,H_p(x,y,z,w,\xi)),d(b_0,H_p(y,z,w,x,\xi)),d(c_0,H_p(z,w,x,y,\xi)),d(d_0,H_p(w,x,y,z,\xi))\}} \phi(s)\,ds \leq \int_0^{\max\left\{\begin{array}{l}\theta+d(a_0,x_0),\\ \theta+d(b_0,y_0),\\ \theta+d(c_0,z_0),\\ \theta+d(d_0,w_0)\end{array}\right\}} \phi(s)\,ds$$

On the other hand

$$\int_0^{d(a_0,x_0)} \phi(s)ds = \int_0^{d(H_p(a_0,b_0,c_0,d_0,\lambda_0),H_p(x_0,y_0,z_0,w_0,\xi_0))} \phi(s)\,ds \leq \int_0^{M|\lambda_0-\xi_0|} \phi(s)ds \leq \frac{1}{M^{n-1}}\int_0^{1} \phi(s)ds \to 0 \text{ as } n \to \infty$$

So $a_0 = x_0$. Similarly we get $b_0 = y_0, c_0 = z_0, d_0 = w_0$ and hence $\lambda = \xi$.
Thus each fixed
$\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$,
$H_p(.,\lambda) : \bar{B}_{A\cup B}(a_0,\theta) \to \bar{B}_{A\cup B}(a_0,\theta)$,
$H_p(.,\lambda) : \bar{B}_{A\cup B}(b_0,\theta) \to \bar{B}_{A\cup B}(b_0,\theta)$,
$H_p(.,\lambda) : \bar{B}_{A\cup B}(c_0,\theta) \to \bar{B}_{A\cup B}(c_0,\theta)$,
$H_p(.,\lambda) : \bar{B}_{A\cup B}(d_0,\theta) \to \bar{B}_{A\cup B}(d_0,\theta)$.
Then all conditions of theorem (4.1) are satisfied.
Thus we conclude that $H_p(.,\lambda)$ has a quadruple fixed point in $\bar{X} \cap \bar{Y}$. But this must be in $X \cap Y$
Since (4.1.2) holds, Therefore, $\lambda \in A \cap B$ for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$.
Hence $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subseteq A \cap B$. Then clearly $A \cap B$ is open in [0,1]
To prove the reverse part, we can use the similar process.

5 Conclusion

We proved the existence and uniqueness of a common quadruple fixed point for two mappings in the class of bipolar metric spaces via Integral type contraction with an example. And also an illustrated application towards integral equations and Homotopy theory has been provided.

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