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Strong Gamma Group

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Abstract

Objectives: The primary goal of this study is to present the concept of a strong $\Gamma-$ *group* as a generalization of $\Gamma-$ *group*. **Methods and Findings:** We have investigated some of the properties of the $\Gamma-$ *group* and extended it to introduce the idea of a strong $\Gamma-$ *group*. **Novelty:** Every strong $\Gamma-$ *group* is a $\Gamma-$ *group*, but not all $\Gamma-$ *groups* are strong $\Gamma-$ *groups*. Further if G is a non-empty $\Gamma-$ *semigroup* and for all $a, b\ \varepsilon\ G$, the equations $a\alpha x = b$ and $y\alpha a = b$ for all $x,\ y\ \varepsilon\ G$ and for all $\alpha\ \varepsilon\ \Gamma$ have unique solutions in G, then G is a strong $\Gamma-$ *group*. Also, we characterize that non-empty subset H of a strong $\Gamma-$ *group* G is a strong $\Gamma-$ *subgroup* if and only if for all $a, b\ \varepsilon H, a\alpha c\ \varepsilon\ H$ for all $\alpha\ \varepsilon\ \Gamma$ where c is strong inverse of b in G. Finally, we prove that the intersection of two strong $\Gamma-$ *subgroups* is again a strong $\Gamma-$ *subgroup* and the center of strong $\Gamma-$ *group* $C(G)$ is also a $\Gamma-$ *subgroup*.

Keywords: Semigroup; Strong Γ −group; Strong Γ −subgroup; Centre of Γ −group

1 Introduction

The notion of a ternary algebraic system was introduced by Lehmer in 1932. As a speculation of ring, the notion of a $\Gamma-$ *ring* was introduced by N Nobusawa in 1964. In 1981, M. K. Sen introduced the notion of a $\Gamma-$ *semigroup* as a generalization of semigroup. In 1995, Rao [1] introduced the notion of a $\Gamma-$ *semiring* as a generalization of Γ ring. The formal study of semi groups begins in the early 20th century. Rao studied ideals of $\Gamma-$ *semirings*, semirings, semigroups and $\Gamma-$ *semigroup*. In this paper, we study the concept of a strong $\Gamma-$ *group* as a generalization of $\Gamma-$ *group*. Further, we prove some basic results regarding strong $\Gamma-$ *subgroup*, centre of strong $\Gamma-$ *group* etc. and study some fundamental properties of a strong $\Gamma-$ *group*.

1.1 Preliminaries

We include some necessary preliminaries from [1–3] for the sake of completeness.

Definition 2.1. A semigroup is an algebraic system $(G, .)$ consisting of a non-empty set G together with an associative binary operation'.'

Definition 2.2. An algebraic system $(G, .)$ consisting of a non-empty set G together with an associative binary operation '.' is called a group if it satisfies:

(i) there exists $e \in G$ such that $x.e = e.x = x$ for all $x \in G$.

(ii) if for each $x \in G$, there exists $y \in G$ such that $x.y = y.x = e$.

Definition 2.3. Let G and Γ be non-empty sets. Then we call G a $\Gamma-$ *semigroup* if there exists a mapping $G \times \Gamma \times G \to G$, (images of (x, α, y) will be denoted by $x\alpha y$, $x, y \in G, \alpha \in \Gamma$)such that it satisfies $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in G$ and $\alpha, \beta \in \Gamma$.

Definition 2.4. A $\Gamma-$*semigroup* G is said to be commutative if $x\alpha y = y\alpha x$ *for all* $x, y \in G$ for all $\alpha \in \Gamma$.

Definition 2.5. Let G be a $\Gamma-$*semigroup*. An element $e \in G$ is said to be unity if for each $x \in G$, there exists $\alpha \in \Gamma$ such that $x\alpha e = e\alpha x = x$

Definition 2.6. In a $\Gamma-$*semigroup* G with unity e, an element $x \in G$ is said to be left invertible (right invertible) if there exists $y \in G, \alpha \in \Gamma$ such that $y\alpha x = e$ $(x\alpha y = e)$.

Definition 2.7. A $\Gamma-$*semigroup* G with unity e, an element $u \in G$ is said to be unit if there exists $v \in G$ and $\alpha \in \Gamma$ such that $u\alpha v = e = v\alpha u$.

Definition 2.8. A $\Gamma-$ *semigroup* G with zero element 0 is said to hold cancellation laws if $x \neq 0$, $x\alpha y = x\alpha z$, $y\alpha x = z\alpha x$, where $x, y, z \in G, \alpha \in \Gamma$ then $y = z$.

Definition 2.9. A $\Gamma-$ *semigroup* G is said to be $\Gamma-$ *group* if it satisfies:

(i) if there exists $e \in G$ and for each $x \in G$, there exists $\alpha \in \Gamma$ such that $x\alpha e = e\alpha x = x$.

(ii) if for each element $x \neq 0$, there exists $y \in G, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = e$.

Remark 2.10. Every group G is a $\Gamma-$ *group* if $\Gamma = G$ and ternary operation is $x\alpha y$ defined as the binary operation of the group. The unity of a $\Gamma-$*group* may not be unique.

Example 2.11. Let G and Γ be the set of all rational numbers and the set of all natural numbers respectively. Define the ternary operation $G \times \Gamma \times G \to G$ by $(x, \alpha, y) \to x\alpha y$ using the usual multiplication. Then G is a $\Gamma-$ *group*.

Definition 2.12. An element x of a $\Gamma-$ *semigroup* G is said to be a strong $\Gamma-$ *idempotent* if $x\gamma x = x$ for all $\gamma \in \Gamma$.

Definition 2.13. A $\Gamma-$ *semigroup* G is said to be strong $\Gamma-$ *idempotent* if every element of G is strong $\Gamma-$ *idempotent*.

Definition 2.14. Let G be a $\Gamma-$ *semigroup* An element $e \in G$ is said to be strong identity e if for each $x \in G$, we have $x\alpha e = e\alpha x = x$ for all $\alpha \in \Gamma$.

Definition 2.15. Let G be a $\Gamma-$ *semigroup* with strong identity $e \in G$. An element $x \in G$ is said to have strong inverse in G if there exists $y \in G$ such that $x\alpha y = y\alpha x = e$ for all $\alpha \in \Gamma$.

Definition 2.16. A $\Gamma-$ *semigroup* G is said to be a strong $\Gamma-$ *group* if it satisfies:

(i) if G has strong identity $e \in G$;

(ii) And every element of G has strong inverse in G.

Example 2.17. Let G be the set of all positive rational numbers and Γ be the set of all real numbers whose square is 1. Define the ternary operation $G\times\Gamma\times G \to G$ by $(x, \alpha, y) \to x.|\alpha|.y$, where $'.'$ is the usual multiplication. Then G is a strong $\Gamma-$*group*.

Example 2.18. Let G be a set of real solutions of the equation $x^2 = x$ and let Γ be the set of all non-positive integers. Define the ternary operation $G\times\Gamma\times G \to G$ *by* $(x, \alpha, y) \to Max\ (x.y, \ \alpha\}$, where $'.'$ is the usual multiplication. Then G is a strong $\Gamma-$ *group*.

Example 2.19. Let G be the set of all non-zero real numbers and let $\Gamma = \{2\pi ki : k\,\varepsilon\, N\}$.

Define the ternary operation $G \times \Gamma \times G \to G$ by $(x, \alpha, y) \to x.\,e^{\alpha}.\,y$, where $'.'$ is the usual multiplication. Then G is a strong $\Gamma-$ *group*.

2 Main Results

Theorem 3.1.Every strong $\Gamma-$ *group* is a $\Gamma-$ *group* .

Proof. Let G be a strong $\Gamma-$ *group*. Then G is a $\Gamma-$ *semigroup* with strong identity $e \in G$ and every element $a \in G$ has a strong inverse in G. This implies that for all $x \in G, \alpha \in \Gamma, x\alpha e = e\alpha x = x$ and for all $x \neq 0$ there exists $y \in G$ such that $x\alpha y = y\alpha x = e$.

Note: - Every $\Gamma-$ *group* need not be a strong $\Gamma-$ *group*.

Example 3.2.let G and Γ be the set of all rational numbers and the set of all natural numbers respectively. Define the ternary operation $G \times \Gamma \times G \to G$ by $(x, \alpha, y) \to x\alpha y$, where $'.'$ is the usual multiplication. Then G is a $\Gamma-$ *group*. Let e be the strong identity of G. Then for each $x \neq 0 \in G$, $x = x\alpha e = e\alpha x$ for all $\alpha \in \Gamma$. This implies that $e = 1/\alpha$ for all $\alpha \in \Gamma$ and hence e depends on α which is not possible. Thus, G does not have a strong identity. Therefore, G is not a strong $\Gamma-$ *group*.

Theorem 3.3. (**Cancellation Laws**) Let G be a strong $\Gamma-$ $group$. If $x\alpha y = x\alpha z$, $y\beta x = z\beta x$, where $x, y, z \in G$ and for all $\alpha, \beta \in \Gamma$ then $y = z$.

Proof. Let $x, y, z \in G$ and $\alpha, \gamma \in \Gamma$. Suppose e is the strong identity of G. Then $x\alpha(e\gamma y) = x\alpha(e\gamma z)$ implies that $(x\alpha e)\gamma y = (x\alpha e)\gamma z$. Now $x\alpha e \in G$, there exists $w \in G$ such that $(x\alpha e)\delta w = w\delta(x\alpha e) = e$ for all $\delta \in \Gamma$.Therefore $y = e\eta y = w\delta(x\alpha e)\eta y = w\delta(x\alpha(e\eta y)) = w\delta(x\alpha(e\eta z)) = w\delta(x\alpha e)\eta z = e\eta z = z$ for all $\eta \in \Gamma$ Similarly $y\beta x = z\beta x$ implies $y = z$.

Theorem 3.4. The strong identity of a strong $\Gamma-$ $group$ is unique.

Proof. If possible, let e_1, e_2 be two strong identities of a strong $\Gamma-$ $group$ G. Therefore, $e_1\alpha e_2 = e_1$ and $e_1\alpha e_2 = e_2$ for all $\alpha \in \Gamma$. Hence, $e_1 = e_2$.

Theorem 3.5. The strong inverse of each element of a strong $\Gamma-$ $group$ is unique.

Proof. Let e be the strong identity of a strong $\Gamma-$ $group$ G and $a \in G$ be an arbitrary element. If possible, let $b_1, b_2 \in G$ be two strong inverses of a. Therefore, $a\alpha b_1 = e = b_1\alpha a$ and $a\beta b_2 = e = b_2\beta a$ for all $\alpha, \beta \in \Gamma$. Now $b_1 = b_1\alpha e = b_1\alpha(a\beta b_2) = (b_1\alpha a)\beta b_2 = e\beta b_2 = b_2$.

Theorem 3.6. Let G be a strong $\Gamma-$ $group$. Then left strong identity and right strong identity are the same in G.

Proof. Let e_1 and e_2 be the left and right strong identities of G. Then $e_1\alpha x = x$ and $x\alpha e_2 = x$ for all $x \in G, \alpha \in \Gamma$. Now by taking $x = e_2$ and $x = e_1$ respectively in above relations, we have $e_1 = e_2$.

Theorem 3.7. Let G be a strong $\Gamma-$ $group$. Then left strong inverse and right strong inverse of every element in a strong $\Gamma-$ $group$ is same.

Proof. Let e be the strong identity of G and b, c be the left and right strong inverses of an element $\in G$. Then $b\alpha a = e$ and $a\beta c = e$ for all $\alpha, \beta \in \Gamma$. Hence, $b = b\alpha e = b\alpha(a\beta c) = (b\alpha a)\beta c = e\beta c = c$.

Theorem 3.8. Let G be a strong $\Gamma-$ $group$. Then the equations $a\alpha x = b$ and $y\alpha a = b$ have unique solutions in G for $a, b \in G, \alpha \in \Gamma$.

Proof. Let $a \in G$, therefore there exists $c \in G$ such that $a\alpha c = c\alpha a = e$ for all $\alpha \in \Gamma$ where e is the strong identity of G. Take $x = c\alpha b$, then $x \in G$. Now $a\alpha x = a\alpha(c\beta b) = (a\alpha c)\beta b = e\beta b = b$ for all $\alpha, \beta \in \Gamma$. Similarly, the solution of $y\alpha a = b$ exists. Further suppose that x_1 and x_2 are two solutions of the equation $a\alpha x = b$ in G. Therefore $a\alpha x_1 = b$ and $a\alpha x_2 = b$ for all $\alpha \in \Gamma$. This implies that $a\alpha x_1 = a\alpha x_2$. By left cancellation law $x_1 = x_2$. Hence the equation $a\alpha x = b$ has a unique solution in G. By similar arguments one can prove that the equation $y\alpha a = b$ also has a unique solution in G.

Theorem 3.9. Let G be a non-empty $\Gamma-$ $semigroup$. If for all $a, b \in G$, the equations $a\alpha x = b$ and $y\alpha a = b, \alpha \in \Gamma$ have solutions in G then G is a strong $\Gamma-$ $group$.

Proof. For any $a, b \in G$, let the equations $a\alpha x = b$ and $y\alpha a = b, \alpha \in \Gamma$ have a solution in G. Since G is non empty, so there exists $a_0 \in G$. Therefore, the equations $a_0\alpha x = a_0$ and $y\alpha a_0 = a_0, \alpha \in \Gamma$ have solutions in G. Let $x = g$ and $y = f$ be the respective solutions of these equations in G. Thus $g, f \in G$ and $a_0\alpha g = a_0$ and $f\alpha a_0 = a_0$ for all $\alpha \in \Gamma$. Now let $b \in G$ arbitrarily then there exist $x_0, y_0 \in G$ such that $a_0\alpha x_0 = b$ and $y_0\alpha a_0 = b$. By associativity of $\Gamma-$ $semigroup$ G, we have $b\beta g = (y_0\alpha a_0)\beta g = y_0\alpha(a_0\beta g) = y_0\alpha a_0 = b$. Also $f\beta b = f\beta(a_0\alpha x_0) = (f\beta a_0)\alpha x_0 = a_0\alpha x_0 = b$. Therefore $b\beta g = b$ and $f\beta b = b$ for all $b \in G$ and for all $\beta \in \Gamma$. Taking $b = f$ in $b\beta g = b$. This implies that $f\beta g = f$ and by taking $b = g$ in $f\beta b = b$, we have $f\beta g = g$. Thus $g = f$ for all $\beta \in \Gamma$. Putting $g = f$ in $b\beta g = b$ and $f\beta b = b$ for all $b \in G$ for all $\beta \in \Gamma$, we have $b\beta g = b$ and $g\beta b = b$ for all $b \in G, \beta \in \Gamma$. Thus g is the strong identity of G. Again, the equations $a\alpha x = g$ and $y\alpha a = g$ for all $\alpha \in \Gamma$ have solutions in G. Let $x = c$ and $y = d$ be their respective solutions in G. Therefore $a\alpha c = g$ and $d\alpha a = g$. Now $d = d\alpha g = d\alpha(a\alpha c) = (d\alpha a)\alpha c = g\alpha c = c$. Hence $a\alpha c = g$ and $c\alpha a = g$ for all $\alpha \in \Gamma$ implies that c is strong inverse of a for all $a \in G$. Thus, G is a strong $\Gamma-$ $group$.

Theorem 3.10. Let G be a strong $\Gamma-$ $group$. Then $x \in G$ is strong $\Gamma-$ $idempotent$ if and only if $x = e$, where e is strong identity of G.

Proof. Suppose $x \in G$ is strong $\Gamma-$ $idempotent$, then $x\alpha x = x$ for all $\alpha \in \Gamma$.Now $x\alpha x = e\alpha x$ for all $\alpha \in \Gamma$. By right cancellation law we have $x = e$. Conversely, assume that $x = e$, then $x\alpha x = e\alpha e = e = x$ for all $\alpha \in \Gamma$.

Theorem 3.11. (**Reversal Law**) Let G be a strong $\Gamma-$ $group$.Then for $x, y \in G$ and $\alpha \in \Gamma$, strong inverse of $x\alpha y$ is $z\beta w$ for all $\beta \in \Gamma$, where w and z are strong inverses of x and y respectively.

Proof. Let $x, y \in G$ and let w and z be strong inverses of x and y respectively.

Then for all $\alpha, \beta, \gamma, \delta \in \Gamma$, $(x\alpha y)\gamma(z\beta w) = x\alpha(y\gamma z)\beta w = x\alpha(e\beta w) = x\alpha w = e$. Also $(z\beta w)\delta(x\alpha y) = z\beta(w\delta x)\alpha y = z\beta(e\alpha y) = z\beta y = e$, where e is strong identity of G. This implies that strong inverse of $x\alpha y$ is $z\beta w$.

Definition 3.12. (Strong $\Gamma-$ $subgroup$) A non-empty subset H of a strong $\Gamma-$ $group$ G is said to be a strong $\Gamma-$ $subgroup$ of G if H itself is a strong $\Gamma-$ $group$.

Theorem 3.13. The strong identity of a strong $\Gamma-$ $group$ and strong $\Gamma-$ $subgroup$ are same.

Proof. Let G be a strong $\Gamma-$ *group* and H be its strong $\Gamma-$ *subgroup*. Let e and e' be the strong identities of G and H respectively. Suppose $a \in H$ is any element, then $a\alpha e' = e'\alpha a = a$ for all $\alpha \in \Gamma$. Since $a \in H$ and $H \subset G$, so $a \in G$. Thus $a\alpha e = e\alpha a = a$ for all $\alpha \in \Gamma$. Therefore $a\alpha e = a\alpha e'$ for all $\alpha \in \Gamma$. So, by left cancellation law $e = e'$.

Theorem 3.14. The strong inverse of any element of a strong $\Gamma-$ *subgroup* H is same as the strong inverse of the element regarded as the element of the strong $\Gamma-$ *group* G.

Proof. Let e be the strong identity of G and H. Since $H \subset G$, so for $a \in H$ we have $a \in G$.Let b be strong inverse of a in H and c be the strong inverse of a in G. This implies that $b\alpha a = e$ and $c\alpha a = e$ for all $\alpha \in \Gamma$. Thus $b\alpha a = c\alpha a$ for all $\alpha \in \Gamma$. So, by right cancellation law $b = c$.

Theorem 3.15. A non-empty subset H of a strong $\Gamma-$ *group* G is a strong $\Gamma-$ *subgroup* if and only if

(i) $a\alpha b \in H$ for all $a, b \in H$ and for all $\alpha \in \Gamma$.

(ii) for all $a \in H$ there exists $b \in H$ such that $a\alpha b = e$ for all $\alpha \in \Gamma$, where e is strong identity of G.

Proof. Suppose H is a strong $\Gamma-$ *subgroup*. Then H is a strong $\Gamma-$ *group* under the same ternary operation as that of G. Thus (i) and (ii) hold. Conversely, assume that (i) and (ii) hold in H. Since G is a strong $\Gamma-$*group* and $H \subset G$, so $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in H$ and for all $\alpha, \beta \in \Gamma$. Again $H \neq \varnothing$, by (ii) for $a \in H$ there exists $b \in H$ such that $a\alpha b = e$ for all $\alpha \in\Gamma$ Thus (i) implies, $e = a\alpha b \in H$.

Theorem 3.16. A non-empty subset H of a strong $\Gamma-$ *group* G is a strong $\Gamma-$ *subgroup* if and only if for all $a, b \in H$ and for all $\alpha \in \Gamma$ implies $a\alpha c \in H$ where c is strong inverse of b in G.

Proof. Suppose H is a non-empty strong $\Gamma-$ *subgroup* of a strong $\Gamma-$ *group* G. Then H is a strong $\Gamma-$ *group* under the same ternary operation as that of G. Therefore, for all $a, b \in H$ and for all $\alpha \in \Gamma$, $a\alpha c \in H$, where c is strong inverse of b in G. Conversely, since G is a strong $\Gamma-$*group* and $H \subset G$, $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in H$ and for all $\alpha, \beta \in \Gamma$. Let $a \in H$ be any arbitrary element. Then for $a \in H$ and for all $\alpha \in \Gamma$ implies $a\alpha b \in H$, where b is strong inverse of $(i.e\ a\alpha b = b\alpha a = e,\ where\ e\ is\ strong\ identity\ of\ G)$. Thus $e \in H$. Since $H \neq \varnothing$, So let $a \in H$. This implies that for $e, a \in H$, $e\alpha b \in H$ for all $\alpha \in \Gamma$ and $b = b\beta e = e\beta b \in H$ for all $\beta \in \Gamma$ where b is strong inverse of a. Hence H is a strong $\Gamma-$ *subgroup* .

Theorem 3.17. Intersection of two strong $\Gamma-$ *subgroups* is again a strong $\Gamma-$ *subgroup* of the strong $\Gamma-$ *group*.

Proof. Let H_1 and H_2 be two strong $\Gamma-$ *subgroups* of a strong $\Gamma-$ *group* G. Since $e \in H_1 \cap H_2$ so $H_1 \cap H_2 \neq \varnothing$, where e is the strong identity of strong $\Gamma-$ *group* G. Let $x, y \in H_1 \cap H_2$. This implies that $x, y \in H_1$ and $x, y \in H_2$ .Therefore $x\alpha z \in H_1$ and $x\alpha z \in H_2$ for all $\alpha \in \Gamma$, where z is strong inverse of y in G (since H_1 and H_2 are two strong $\Gamma-$ *subgroups*). Hence $x\alpha z \in H_1 \cap H_2$ for all $\alpha \in \Gamma$.

Definition 3.18. (Center of a strong $\Gamma-$ *group*) Let G be a strong $\Gamma-$ *group* then Center of a strong $\Gamma-$ *group* G is a subset of G consisting of all elements x of G such that $x\alpha y = y\alpha x$ for all $y \in G$ and for all $\alpha \in \Gamma$. It is denoted by $C(G)$.

Theorem 3.19. Let G be a strong $\Gamma-$ *group*. Then the Centre $C(G)$ of G is a strong $\Gamma-$ *subgroup* of G.

Proof. Let e be the strong identity of G, then $e\alpha x = x\alpha e$ for all $x \in G$ and for all $\alpha \in \Gamma$. Therefore $e \in C(G)$, so $C(G) \neq \varnothing$. Let $g_1, g_2 \in C(G)$ then $g_1\alpha y = y\alpha g_1$ and $g_2\alpha y = y\alpha g_2$ for all $y \in G$ and for all $\alpha \in \Gamma$. Since $g_2 \in G$, there exists strong inverse $g_3 \in G$ such that $g_2\alpha g_3 = g_3\alpha g_2 = e$ for all $\alpha \in \Gamma$. Now $g_3\alpha y = g_3\alpha(y\alpha e) = g_3\alpha(y\alpha(g_2\alpha g_3)) = g_3\alpha((y\alpha g_2)\alpha g_3) = g_3\alpha((g_2\alpha y)\alpha g_3) = ((g_3\alpha g_2)\alpha y)\alpha g_3 = (e\alpha y)\alpha g_3 = y\alpha g_3$ for all $y \in G$. Therefore, $g_3 \in C(G)$. Thus $y\alpha(g_1\alpha g_3) = (y\alpha g_1)\alpha g_3 = (g_1\alpha y)\alpha g_3 = g_1\alpha(y\alpha g_3) = g_1\alpha(g_3\alpha y) = (g_1\alpha g_3)\alpha y$ for all $y \in G$ and for all $\alpha \in \Gamma$. Therefore $g_1\alpha g_3 \in C(G)$ and hence $C(G)$ is a strong $\Gamma-$ *subgroup* G.

Theorem 3.20. Let G be a strong $\Gamma-$ *group*. Then G is a commutative strong $\Gamma-$ *group* if and only if $C(G) = G$.

Proof. Let G be commutative strong $\Gamma-$ *group*. Then for $x \in G$, $x\alpha y = y\alpha x$ for all $y \in G$ and for all $\alpha \in \Gamma$. Therefore, $x \in C(G)$ and thus $G \subset C(G)$. Clearly $C(G) \subset G$ being a strong $\Gamma-$ *subgroup* of G. Conversely, let $x, y \in G$. Since $C(G) = G$, then $x, y \in C(G)$ and hence $x\alpha y = y\alpha x$ for all $\alpha \in \Gamma$. This implies that G is commutative strong $\Gamma-$ *group*.

Definition 3.21. (Normalizer of an element of a strong $\Gamma-$ *group*) Let G be a strong $\Gamma-$ *group*, then Normalizer of an element a of G is a subset of G consisting of all elements x of G such that $x\alpha a = a\alpha x$ for all $\alpha \in \Gamma$. It is denoted by $N(a)$.

Theorem 3.22. Let G be a strong $\Gamma-$ *group*, then normalizer $N(a)$ is a strong $\Gamma-$ *subgroup* of G.

Proof. Let e be the strong identity of G, then $e\alpha x = x\alpha e$ for all $x \in G$ and for all $\alpha \in \Gamma$. In particular $e\alpha a = a\alpha e$ for all $\alpha \in \Gamma$. Therefore $e \in N(a)$, $so\ N(a) \neq \varnothing$. Let $g_1, g_2 \in N(a)$, then $g_1\alpha a = a\alpha g_1$ and $g_2\alpha a = a\alpha g_2$ for all $\alpha \in \Gamma$. Since $g_2 \in G$, there exists $g_3 \in G$ such that $g_2\alpha g_3 = g_3\alpha g_2 = e$ for all $\alpha \in \Gamma$. Now $g_3\alpha a = g_3\alpha(a\alpha e) = g_3\alpha(a\alpha(g_2\alpha g_3)) = g_3\alpha((a\alpha g_2)\alpha g_3) = g_3\alpha((g_2\alpha a)\alpha g_3) = ((g_3\alpha g_2)\alpha a)\alpha g_3 = (e\alpha a)\alpha g_3 = a\alpha g_3$.Therefore $g_3 \in N(a)$. Now $a\alpha(g_1\alpha g_3) = (a\alpha g_1)\alpha g_3 = (g_1\alpha a)\alpha g_3 = g_1\alpha(a\alpha g_3) = g_1\alpha(g_3\alpha a) = (g_1\alpha g_3)\alpha a$ for all $\alpha \in \Gamma$. Thus $g_1\alpha g_3 \in N(a)$. Hence, $N(a)$ is a strong $\Gamma-$ *subgroup* of G.

Theorem 3.23. Let G be a strong $\Gamma-$ *group*. Then G is commutative strong $\Gamma-$ *group*if and only if $N(a) = G$ for all $a \in G$.

Proof. Let G be commutative strong $\Gamma-$ *group*. Then for $x \in G$, we have $x\alpha y = y\alpha x$ for all $y \in G$ and for all $\alpha \in \Gamma$. In particular $x\alpha a = a\alpha x$ for all $\alpha \in \Gamma$. Therefore $x \in N(a)$ and thus $G \subset N(a)$. Clearly $N(a) \subset G$ being a strong $\Gamma-$ *subgroup* of G. Conversely, let $x,\ y \in G$. Since $N(a) = G$ for all $a \in G$, then in particular $N(x) = N(y)$. Therefore, $x\alpha y = y\alpha x$ for all $\alpha \in \Gamma$. This implies that G is commutative strong $\Gamma-$ *group*.

3 Conclusion

This article presents the idea of a strong $\Gamma-$ *group* and provides some key findings in this area. We pay particular attention to outcomes that hold true for strong $\Gamma-$ *groups* but not for $\Gamma-$ *groups*. For researchers, this article has a lot of potential.

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