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RESEARCH ARTICLE

Vertex *k*-Prime Labeling of Theta Graphs

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Abstract

Objectives: To analyse some theta related graphs that admit vertex k-prime labeling for each positive integer k. **Methods:** In this study, vertices of the graphs are assigned with k, k+1,...,k+|V|-1 such that each pair of labels of adjacent vertices are relatively prime. Justifications for the proof are given. **Findings:** We examine the theta related graphs such as generalised theta graphs, uniform theta graphs, centralised uniform theta graphs for m = 1 are vertex k-prime. In addition, we introduce another structure of theta graph known as centralised generalised theta graph and show that vertex k-prime labeling exists for the graph. **Novelty:** Vertex k-prime labeling is a new variant of prime labeling and theta families of graphs exhibiting the labeling is a new finding. Another structure of theta graph known as centralised generalised theta graph is introduced and proved that vertex k-prime labeling exists for the graph.

Keywords: Vertex k-Prime Labeling; Generalised Theta Graphs; Uniform Theta Graphs; Centralised Uniform Theta Graphs; Centralised Generalised Theta Graph

1 Introduction

In the discipline of mathematics known as graph theory, a graph labeling is a map that transforms graph elements including vertices, edges or both to numerals (positive integers) based on particular constraints. There has been substantial literature dealing with different kinds of graph labeling over the last three decades, and we refer to Gallian J. A [(1)] for a summary of diverse graph labeling discoveries.

The theory of prime labeling was first developed by Roger Entringer in an article through Tout, Dabboucy and Howalla [(2)]. A prime labeling is an assignment of the integers 1 to n as labels of the vertices such that each pair of labels of adjacent vertices are relatively prime. Teresa Arockiamary S and Vijayalakshmi G [(3)] established the idea of vertex k-prime labeling. For each positive integer k, they proved that triangular snake T_n, pentagonal snake mC_5, nanogonal snake mC_9, cyclic snakes mC_n, corona $T_n \odot K_1$, $mC_5 \odot K_1$, $mC_9 \odot K_1$, $mC_n \odot K_1$ for m > 1 and $n \geq 3$, class of planar graphs Pl_n are vertex k-prime [(3,4)].

A generalized theta graph $\theta(S_1, S_2, ..., S_n)$ was defined by Rajan et al. [(5)] as having two end vertices joined by n internally disjoint paths, each of which has at least two internal vertices $S_1, S_2, \ldots, S_n$. A generalised theta graph is called a uniform theta graph when all paths connecting the two end vertices have the same number of internal vertices. The uniform theta graph with n $\geq$ 3 paths connecting the end vertices and m $\geq$ 1 internal vertex in each path is denoted as θ(n,m). By merging one of their end vertices of r disjoint copies of uniform theta graphs, Putra and Susanti [6] constructed the centralized uniform theta graph. The centralized uniform theta graphs are represented by the notation θ^*(n,m,r), which is created from r $\geq$ 3 uniform theta graphs θ(n,m).

Numerous writers have studied various types of graph labeling for theta-related graphs, but labeling the graph's vertices from some k $\geq$ 1 such that the gcd of each labeled pair of adjacent vertices is 1 is novel research, and justification is provided for all results. Compared to the prime labeling where the labels of the vertices of the graph begin with 1 wherein for vertex k-prime labeling, the labels of the vertices of the graphs begin with k.

Definition 1.1 A vertex k-prime labeling of a graph G is a bijective function $f: V \rightarrow (k, k+1, k+2, \ldots, k+(V|-1\}$ for some positive integer k such that $gcd(f(u), f(v)) = 1 \; \forall \, e = uv \; \in E(G)$. A graph G that admits vertex k-prime labeling is called a vertex k-prime graph.

Main Results

Theorem 2.1. The generalized theta graph $\theta(S_1, S_2, \ldots, S_n)$ is vertex k-prime for k, m $\geq$ 1, n $\geq$ 3.

Proof of Theorem 2.1. Let $\theta(S_1, S_2, \ldots, S_n)$ be the generalized theta graph of order $S_1 + S_2 + \ldots + S_n + 2$ and size $S_1 + S_2 + \ldots + S_n$ + n. The edge set and vertex set of $\theta(S_1, S_2, \ldots, S_n)$ consists of:

V($\theta(S_1, S_2, \ldots, S_n)$) = {$u_1, u_2, v_{i,j}$ where $1 \leq j \leq S_a$, $1 \leq i \leq n$, $1 \leq a \leq n$}

E($\theta(S_1, S_2, \ldots, S_n)$) = {$u_1v_{i,1}$: $1 \leq i \leq n$} $\bigcup$ {$u_2v_{i,j}$: $j \in S_a$, $1 \leq i \leq n$, $1 \leq a \leq n$} $\bigcup$ {$v_{i,j}\, v_{i,j+1}$: $1 \leq j \leq S_a - 1$, $1 \leq i \leq n$, $1 \leq a \leq n$}

Define a bijective function f : V ($\theta(S_1, S_2, \ldots, S_n)$) $\rightarrow$ {k, k + 1, k + 2,…, k + $S_1 + S_2 + \ldots + S_n$ + 1} as follows:

Case 1. k and k + $S_1 + S_2 + \ldots + S_n$ + 1 are prime

f(u_1) = k

f(u_2) = k + $S_1 + S_2 + \ldots + S_n$ + 1

f($v_{i,j}$) = k + $\sum_{a=1}^{i-1} S_a$ + j, $1 \leq j \leq S_a$, $1 \leq i \leq n$, $1 \leq a \leq n$

For any edge $u_1v_{i,1}$ E(($\theta(S_1, S_2, \ldots, S_n)$)), gcd(f($u_1$), f($v_{i,1}$)) = gcd(k, k + $\sum_{a=1}^{i-1} S_a$ + 1) = 1 since k is prime and k + $\sum_{a=1}^{i-1} S_a$ + 1 will not be a multiple of k. For any $u_2v_{i,j}$ E(($\theta(S_1, S_2, \ldots, S_n)$)), gcd(f($u_2$), f($v_{i,j}$)) = gcd(k + $S_1 + S_2 + \ldots + S_n$ + 1, k + $\sum_{a=1}^{i-1} S_a$ + j) = 1 since k + $S_1 + S_2 + \ldots + S_n$ + 1 is prime. For any edge $v_{i,j}\, v_{i,j+1}$ E((($(S_1, S_2, \ldots, S_n)$)), gcd(f($v_{i,j}$), f($v_{i,j+1}$)) = gcd(k + $\sum_{a=1}^{i-1} S_a$ + j, k + $\sum_{a=1}^{i-1} S_a$ + j + 1) = 1 since k + $\sum_{a=1}^{i-1} S_a$ + j and k + $\sum_{a=1}^{i-1} S_a$ + j + 1 are consecutive positive integers.

Case 2. k is not prime and k + $S_1 + S_2 + \ldots + S_n$ + 1 is prime

f(u_1) = k

f(u_2) = k + $S_1 + S_2 + \ldots + S_n$ + 1

f($v_{i,j}$) = k + $\sum_{a=1}^{i-1} S_a$ + j, $1 \leq j \leq S_i$, $1 \leq i \leq n$, $1 \leq a \leq n$

For any edge $u_1v_{i,1} \in$ E(($\theta(S_1, S_2, \ldots, S_n)$), gcd(f($u_1$), f($v_{i,1}$)) = gcd(k, k + $\sum_{a=1}^{i-1} S_a$ + 1) = 1 since k + $\sum_{a=1}^{i-1} S_a$ + 1 is not a multiple of k. For any $u_2v_{i,j} \in$ E(($\theta(S_1, S_2, \ldots, S_n)$), gcd(f($u_2$), f($v_{i,j}$)) = gcd(k + $S_1 + S_2 + \ldots + S_n$ + 1, k + $\sum_{a=1}^{i-1} S_a$ + j) = 1 since k + $S_1 + S_2 + \ldots + S_n$ + 1 is prime. For any edge $v_{i,j}\, v_{i,j+1} \in$ E(($\theta(S_1, S_2, \ldots, S_n)$), gcd(f($v_{i,j}$), f($v_{i,j+1}$)) = gcd(k + $\sum_{a=1}^{i-1} S_a$ + j, k + $\sum_{a=1}^{i-1} S_a$ + j + 1) = 1 since k + $\sum_{a=1}^{i-1} S_a$ + j and k + $\sum_{a=1}^{i-1} S_a$ + j + 1 are consecutive positive integers.

Case 3. k is prime and k + $S_1 + S_2 + \ldots + S_n$ + 1 is not prime

f(u_1) = k

f(u_2) = k + $S_1 + S_2 + \ldots + S_n$ + 1

f($v_{i,j}$) = k + $\sum_{a=1}^{i-1} S_a$ + j, $1 \leq j \leq S_i$, $1 \leq i \leq n$, $1 \leq a \leq n$

For any edge $u_1v_{i,1} \in$ E(($\theta(S_1, S_2, \ldots, S_n)$), gcd(f($u_1$), f($v_{i,1}$)) = gcd(k, k + $\sum_{a=1}^{i-1} S_a$ + 1) = 1 since k is prime and k + $\sum_{a=1}^{i-1} S_a$ + 1 is not a multiple of k. For any $u_2v_{i,j} \in$ E(($\theta(S_1, S_2, \ldots, S_n)$), gcd(f($u_2$), f($v_{i,j}$)) = gcd(k + $S_1 + S_2 + \ldots + S_n$ + 1, k + $\sum_{a=1}^{i-1} S_a$ + j) = 1 since k + $\sum_{a=1}^{i-1} S_a$ + j is not a multiple of any factor of k + $S_1 + S_2 + \ldots + S_n$ + 1. For any edge $v_{i,j}v_{i,j+1} \in$ E(($\theta(S_1, S_2, \ldots, S_n)$), gcd(f($v_{i,j}$), f($v_{i,j+1}$)) = gcd(k + $\sum_{a=1}^{i-1} S_a$ + j, k + $\sum_{a=1}^{i-1} S_a$ + j + 1) = 1 since k + $\sum_{a=1}^{i-1} S_a$ + j and k + $\sum_{a=1}^{i-1} S_a$ + j + 1 are consecutive positive integers.

Case 4. k and k + $S_1 + S_2 + \ldots + S_n$ + 1 are not prime

f(u_1) = k

f(u_2) = k + $S_1 + S_2 + \ldots + S_n$ + 1

f($v_{i,j}$) = k + $\sum_{a=1}^{i-1} S_a$ + j, $1 \leq j \leq S_i$, $1 \leq i \leq n$, $1 \leq a \leq n$

For any edge $u_1v_{i,1} \in$ E(($\theta(S_1, S_2, \ldots, S_n)$), gcd(f($u_1$), f($v_{i,1}$)) = gcd(k, k + $\sum_{a=1}^{i-1} S_a$ + 1) = 1 since k + $\sum_{a=1}^{i-1} S_a$ + 1 is not a multiple of k. For any $u_2v_{i,j} \in$ E(($\theta(S_1, S_2, \ldots, S_n)$), gcd(f($u_2$), f($v_{i,j}$)) = gcd(k + $S_1 + S_2 + \ldots + S_n$ + 1, k + $\sum_{a=1}^{i-1} S_a$ + j) = 1 since k + $\sum_{a=1}^{i-1} S_a$ + j is not a multiple of any factor of k + $S_1 + S_2 + \ldots + S_n$ + 1. For any edge $v_{i,j}v_{i,j+1} \in$ E(($\theta(S_1, S_2, \ldots, S_n)$), gcd(f($v_{i,j}$), f($v_{i,j+1}$)) = gcd(k + $\sum_{a=1}^{i-1} S_a$ + j, k + $\sum_{a=1}^{i-1} S_a$ + j + 1) = 1 since k + $\sum_{a=1}^{i-1} S_a$ + j and k + $\sum_{a=1}^{i-1} S_a$ + j + 1 are consecutive positive integers.

Hence Generalized theta graph $\theta(S_1,\ S_2,\ \ldots,\ S_n))$ is vertex k-prime for k $\geq$ 1.

An illustration for case 1 and case 3 are shown in Figure 1 .

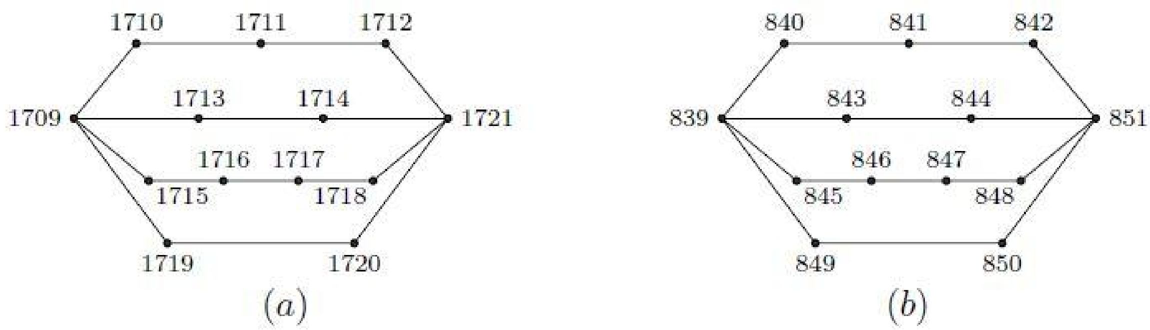

Fig 1. (a) vertex k-prime of $\theta(S_1,\ S_2,\ \ldots,\ S_n))$ for k = 1709 (b) vertex k-prime of $\theta(S_1,\ S_2,\ \ldots,\ S_n))$ for k = 839

Theorem 2.2. Uniform theta graph $\theta\,(n,m)$ is vertex k-prime for $k,\ m \geq 1$, n $\geq$ 3.

Proof of Theorem 2.2. Let $\theta\,(n,m)$ be the uniform theta graph of order nm + 2 and size n(m + 1). The edge set and vertex set of $\theta(n,\ m)$ are represented as:

V(($\theta\,(n,m)$)) = {a_0, a_1, $x_{i,j}$: 1 $\leq$ j $\leq$ m, 1 $\leq$ i $\leq$ n}

E(($\theta\,(n,m)$)) = {$a_0 x_{i,1}$, $a_1 x_{i,m}$, $x_{i,j-1} x_{i,j}$: 2 $\leq$ j $\leq$ m, 1 $\leq$ i $\leq$ n}

Define a bijective function f : V ($\theta\,(n,m)$) $\rightarrow$ {k, k + 1, k + 2,…, k + nm + 1} as follows:

Case 1. k and k + nm + 1 are prime

f(a_0) = k

f(a_1) = k + nm+ 1

f($x_{i,j}$) = k + (i – 1)m + j, 1 $\leq$ j $\leq$ m, 1 $\leq$ i $\leq$ n

For any edge $a_0 x_{i,1} \in$ E($\theta\,(n,m)$), gcd(f(a_0), f($x_{i,1}$)) = gcd(k, k + $(i1)m$ + 1) = 1 since k is prime and k is not a multiple factor of mt + 1, 1 $\leq$ t $\leq$ n – 1. For any $a_1 x_{i,m} \in$ E($\theta\,(n,m)$), gcd(f(a_1), f($x_{i,m}$)) = gcd(k + nm+ 1, k + im) = 1 since k + nm + 1 is prime. For any edge $x_{i,j-1} x_{i,j} \in$ E($\theta\,(n,m)$), gcd(f($x_{i,j-1}$), f($x_{i,j}$)) = gcd(k + (i – 1)m + j – 1, k + (i – 1)m + j) = 1 since k + (i – 1)m + j – 1 and k + (i – 1)m + j are consecutive positive integers.

Case 2. k is prime and k + nm + 1 is not prime

f(a_0) = k

f(a_1) = k + nm+ 1

f($x_{i,j}$) = k + (i – 1)m + j, 1 $\leq$ j $\leq$ m, 1 $\leq$ i $\leq$ n

For any edge $a_0 x_{i,1} \in$ E($\theta\,(n,m)$), gcd(f(a_0), f($x_{i,1}$)) = gcd(k, k + $(i1)m$ + 1) = 1 since k is prime and k is not a multiple factor of mt + 1, 1 $\leq$ t $\leq$ n – 1. For any $a_1 x_{i,m} \in$ E($\theta\,(n,m)$), gcd(f(a_1), f($x_{i,m}$)) = gcd(k + nm+ 1, k + im) = 1 since k + nm + 1 is not a multiple factor of mt + 1, 1 $\leq$ t $\leq$ n – 1. For any edge $x_{i,j-1} x_{i,j} \in$ E($\theta\,(n,m)$), gcd(f($x_{i,j-1}$), f($x_{i,j}$)) = gcd(k + (i – 1)m + j – 1, k + (i – 1)m + j) = 1 since k + (i – 1)m + j – 1 and k + (i – 1)m + j are consecutive positive integers.

Case 3. k is not prime and k + nm + 1 is prime

f(a_0) = k

f(a_1) = k + nm+ 1

f($x_{i,j}$) = k + (i – 1)m + j, 1 $\leq$ j $\leq$ m, 1 $\leq$ i $\leq$ n

For any edge $a_0 x_{i,1} \in$ E($\theta\,(n,m)$), gcd(f(a_0), f($x_{i,1}$)) = gcd(k, k + $(i1)m$ + 1) = 1 since k is not a multiple factor of mt + 1, 1 $\leq$ t $\leq$ n – 1 . For any $a_1 x_{i,m} \in$ E($\theta\,(n,m)$), gcd(f(a_1), f($x_{i,m}$)) = gcd(k + nm+ 1, k + im) = 1 since k + nm + 1 is prime. For any edge $x_{i,j-1} x_{i,j} \in$ E($\theta\,(n,m)$), gcd(f($x_{i,j-1}$), f($x_{i,j}$)) = gcd(k + (i – 1)m + j – 1, k + (i – 1)m + j) = 1 since k + (i – 1)m + j – 1 and k + (i – 1)m + j are consecutive positive integers.

Case 4. k and k + nm+ 1 are not prime

f(a_0) = k

$f(a_1) = k + nm + 1$

$f(x_{i,j}) = k + (i-1)m + j,\ 1 \leq j \leq m,\ 1 \leq i \leq n$

For any edge $a_0x_{i,1} \in E(\theta(n,m))$, $gcd(f(a_0), f(x_{i,1})) = gcd(k, k + (i1)m + 1) = 1$ since k is not a multiple factor of mt + 1, $1 \leq t \leq n-1$. For any $a_1x_{i,m} \in E(\theta(n,m))$, $gcd(f(a_1), f(x_{i,m})) = gcd(k + nm + 1, k + im) = 1$ since k + nm + 1 is not a multiple factor of mt + 1, $1 \leq t \leq n-1$. For any edge $x_{i,j-1}x_{i,j} \in E(\theta(n,m))$, $gcd(f(x_{i,j-1}), f(x_{i,j})) = gcd(k + (i-1)m + j - 1, k + (i-1)m + j) = 1$ since $k + (i-1)m + j - 1$ and $k + (i-1)m + j$ are consecutive positive integers.

Hence Uniform theta graph $\theta(n, m)$ is vertex k-prime for $k \geq 1$.

An illustration for case 3 and case 4 are shown in Figure 2 .

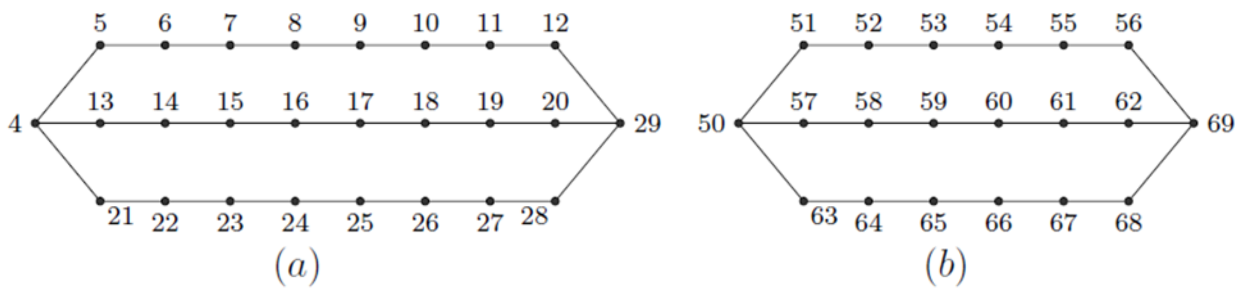

Fig 2. (a) Vertex k-prime of $\theta(3, 8)$ for k = 4 (b) Vertex k-prime of $\theta(3, 6)$ for k = 50

Theorem 2.3. Centralized uniform theta graph $\theta * (n,1,r)$ is vertex k-prime for $k \geq 1$, n, $r \geq 3$ when k + p – 1 is prime.

Proof of Theorem 2.3. Let $G(p,q) = \theta * (n,1,r)$ be the centralized uniform theta graph $\theta * (n,m,r)$ for m = 1 of order (n + 1)r + 1 and size 2n + r. The edge set and vertex set of $\theta * (n,1,r)$ are represented as shown below:

$V(G) = \{q_0\} \bigcup \{q_s : 1 \leq s \leq r\} \bigcup \{u_{i,1,s} : 1 \leq i \leq n,\ 1 \leq s \leq r\}$

$E(G) = \{q_0u_{i,1,s} : 1 \leq i \leq n,\ 1 \leq s \leq r\} \bigcup \{u_{i,1,s}q_s : 1 \leq i \leq n,\ 1 \leq s \leq r\}$

Observe Figure 3 (a). Define a bijective function $f : V(G) \rightarrow \{k, k+1, k+2, \ldots, k + (n+1)r + 1\}$ as follows:

Let l_s be the largest prime number for $1 \leq s \leq r$.

Case 1. Odd n and k

$f(q_0) = k + (n+1)r$

Subcase 1: k is the largest prime from k to k + (n + 1)r + 1

$f(q_s) = k + sn + (s-1),\ 1 \leq s \leq r$

$f(u_{i,1,s}) = k + (s-1)n + i + (s-2),\ 1 \leq i \leq n$

Subcase 2: k + (n + 1)r + 1 is prime

$f(q_s) = k + (s-1)\,n + (s-1),\ 1 \leq s \leq r$

$f(u_{i,1,s}) = k + (s-1)n + i + (s-1),\ 1 \leq i \leq n$

Subcase 3: $k + sn + (s-1) = l_s + t,\ t \geq 1,\ 1 \leq s \leq r$

$f(q_s) = l_s$

$$f(u_{i,1,s}) = \begin{cases} k+(s-1)n+i+(s-2) : 1 \leq i \leq n-t \\ k+(s-1)n+i+(s-1) : n-t+1 \leq i \leq n-1 \\ l_s+t : i = n \end{cases}$$

Case 2. n, k even and r odd

$f(q_0) = k + (n+1)r$

Subcase 1: k is the largest prime from k to k + (n + 1)r + 1

$f(q_s) = k + sn + (s-1),\ 1 \leq s \leq r$

$f(u_{i,1,s}) = k + (s-1)n + i + (s-2),\ 1 \leq i \leq n$

Subcase 2: k + (n + 1)r + 1 is prime

$f(q_s) = k + (s-1)\,n + (s-1),\ 1 \leq s \leq r$

$f(u_{i,1,s}) = k + (s-1)n + i + (s-1),\ 1 \leq i \leq n$

Subcase 3: $k + sn + (s-1) = l_s + t,\ t \geq 1,\ 1 \leq s \leq r$

$f(q_s) = l_s$

$$f(u_{i,1,s}) = \begin{cases} k+(s-1)n+i+(s-2) : 1 \le i \le n-t \\ k+(s-1)n+i+(s-1) : n-t+1 \le i \le n-1 \\ l_s + t : i = n \end{cases}$$

Case 3. n, r even and k odd

$f(q_0) = k + (n + 1)r$

Subcase 1: k is the largest prime from k to k + (n + 1)r + 1

$f(q_s) = k + sn + (s - 1), 1 \le s \le r$

$f(u_{i,1,s}) = k + (s - 1)n + i + (s - 2), 1 \le i \le n$

Subcase 2: k + (n + 1)r + 1 is prime

$f(q_s) = k + (s - 1) n + (s - 1), 1 \le s \le r$

$f(u_{i,1,s}) = k + (s - 1)n + i + (s - 1), 1 \le i \le n$

Subcase 3: $k + sn + (s - 1) = l_s + t, t \ge 1, 1 \le s \le r$

$f(q_s) = l_s$

$$f(u_{i,1,s}) = \begin{cases} k+(s-1)n+i+(s-2) : 1 \le i \le n-t \\ k+(s-1)n+i+(s-1) : n-t+1 \le i \le n-1 \\ l_s + t : i = n \end{cases}$$

From the labeling pattern defined in all the above cases, it is easy to observe that the greatest common divisor for every adjacent vertex is 1.

Hence centralized uniform theta graph $\theta * (n,1,r)$ is vertex k – prime for $k \ge 1, n \ge 3, r \ge 3$.

An illustration is given for case 2 in Figure 3 (b).

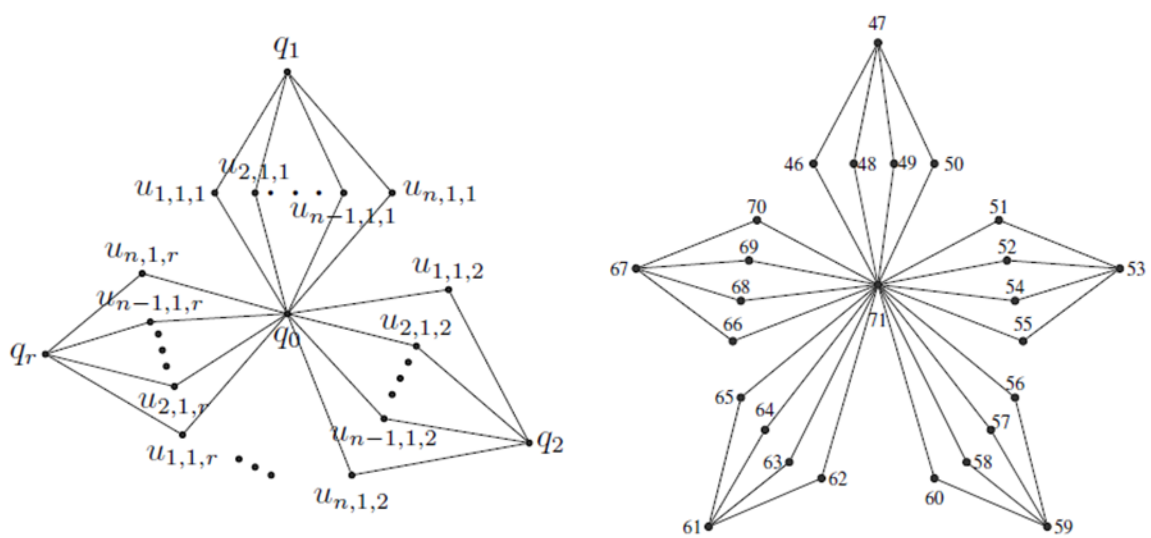

Fig 3. (a) Centralized uniform theta graph $\theta * (n,1,r)$ (b) Vertex k-prime labeling $\theta * (4,1,5)$ for $k = 46$

Theorem 2.4. Centralized uniform theta graph $\theta * (n,m,r)$ is not vertex k – prime for $k \ge 1$, n, r ≥ 3 and m ≥ 2.

Proof of Theorem 2.4. Let $\theta * (n,m,r)$ be the centralized uniform theta graph of order (nm + 1)r + 1 and size n(m + 1) r. Let q_0 denote the merged pole and q_r denote the unmerged poles of r$^{\text{th}}$ uniform theta graph $\theta(n,m)$. The edge set and vertex set of $\theta * (n,1,r)$ are represented as shown below:

$V(\theta * (n,m,r)) = \{q_0\} \bigcup \{q_s : 1 \le s \le r\} \bigcup \{u_{i,j,s} : 1 \le i \le n, 1 \le j \le m, 1 \le s \le r\}$

$E(\theta * (n,m,r)) = \{q_0 u_{i,j,s} : 1 \le i \le n, 1 \le j \le m, 1 \le s \le r\} \bigcup \{u_{i,j,s} q_s : 1 \le i \le n, 1 \le j \le m, 1 \le s \le r\}$

The graph $\theta * (n,m,r)$ is not vertex k – prime for the following two cases.

Case 1. k + p – 1 is not prime

As a contrary, let us assume G is vertex k – prime when k + p – 1 is not a prime number.

Define a bijective function $f : V(\theta * (n,m,r)) \to \{k, k + 1, k + 2, \ldots, k + (nm + 1)r\}$ as follows:

$f(q_0) = k + (nm + 1)r$

For any edge $q_0 u_{i,j,s} \in E(\theta * (n,m,r))$, $\gcd(f(q_0), f(u_{i,j,s})) = \gcd(k + (nm + 1)r, f(u_{i,j,s})) > 1$ since k + (nm + 1)r is not a prime number. This is a contradiction to our assumption.

Case 2. q_s not prime for $1 \leq s \leq r$

As a contrary, let us assume G is vertex k – prime when q_s not prime for $1 \leq s \leq r$.

Define a bijective function $f : V(\theta * (n,m,r)) \rightarrow \{k, k+1, k+2, \ldots, k+(nm+1)r\}$ as follows:

$f(q_0) = k + (nm + 1)r$

Assuming $k + (nm + 1)r$ is prime, the vertices labeled q_s and $u_{i,j,s}$ are adjacent. Hence for any edge $q_s u_{i,j,s} \in E(\theta * (n,m,r))$, $\gcd(f(q_s), f(u_{i,j,s})) > 1$ since $f(q_s)$ is not a prime number which is a contradiction.

Hence centralized uniform theta graph $\theta * (n,m,r)$ is not vertex k – prime for $k \geq 1$, $n \geq 3$, $r \geq 3$ and $m \geq 2$.

2 Centralized generalized theta graphs

Construction: Motivated by the concept of Generalized theta graphs and Centralized uniform theta graphs, we construct a graph called as Centralized Generalized Theta graph by collecting some generalized theta graphs $\theta(S_1,\ S_2,\ \ldots,\ S_{n_r})$ and merging to one of their poles. Centralized generalized theta graph is denoted by $\theta * (n_r, S_{i,r}, t)$, $r \geq 1$, $i \geq 1$, $t \geq 3$.

We label the vertices of Centralized generalized theta graph $\theta * (n_r, S_{i,r}, t)$ in the following way: We denote the merged pole by P_0, the jth internal vertex of ith path from rth generalized theta graphs $\theta(S_1,\ S_2,\ \ldots,\ S_{n_r})$ by $v_{i,j,r}$ and the unmerged poles of rth generalized theta graphs $\theta(S_1,\ S_2,\ \ldots,\ S_{n_r})$ by P_r respectively. Figure 4 shows the centralized generalized theta graphs $\theta * (n_r, S_{i,r}, t)$ with the above notation.

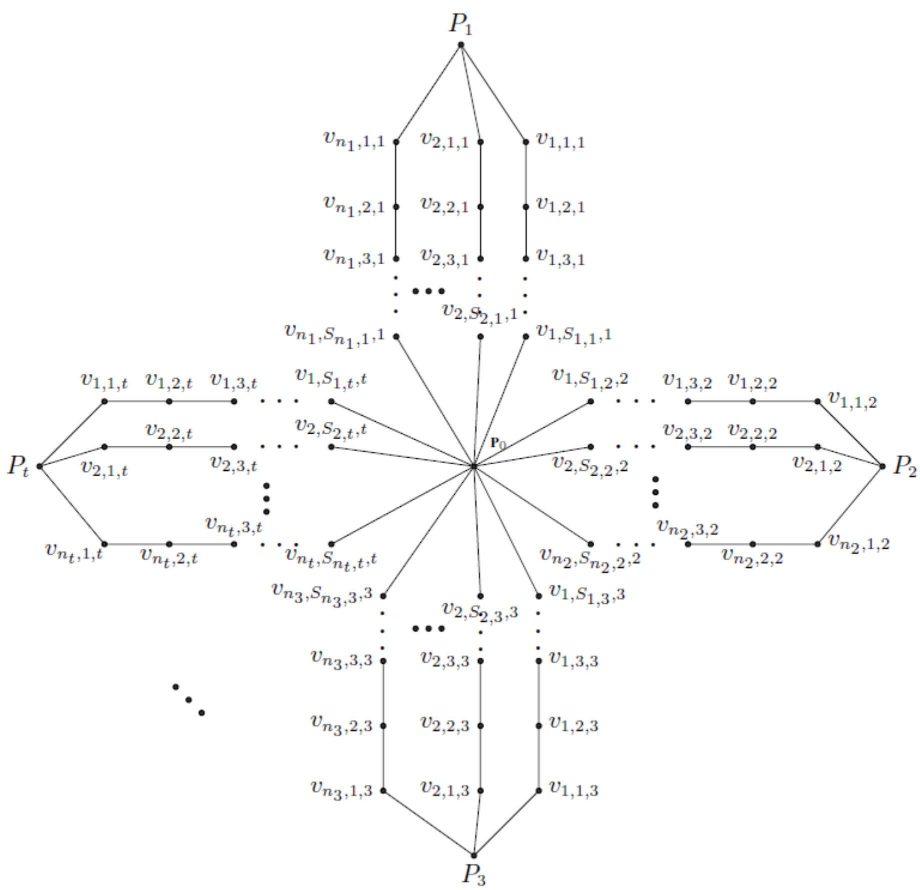

Fig 4. Centralized generalized theta graph $\theta * (n_r, S_{i,r}, t)$

Theorem 3.1. Centralized generalized theta graph $\theta * (n_r, S_{i,r}, t)$ is vertex k – prime for $k,\ i,\ r \geq 1$, $t \geq 3$ when $k + p - 1$ is prime.

Proof of Theorem 3.1. Let $\theta * (n_r, S_{i,r}, t)$ be the centralized generalized theta graph $\theta * (n_r, S_{i,r}, t)$ of order $\sum_{r=1}^{t} \sum_{i=1}^{n_r} S_{i,r} + t + 1$ and size $\sum_{r=1}^{t} \sum_{i=1}^{n_r} S_{i,r} + \sum_{r=1}^{t} n_r$. The edge set and vertex set of $\theta * (n_r, S_{i,r}, t)$ are represented as shown below:

V$(\theta*(n_r,S_{i,r},t))$ = {P$_0$}$\bigcup$ {P$_r$: 1 $\leq$ r $\leq$ t} $\bigcup$ {u$_{i,j,r}$: 1 $\leq$ i $\leq$ n_r, 1 $\leq$ j $\leq$ $S_{i,r}$, 1 $\leq$ r $\leq$ t}

E$(\theta*(n_r,S_{i,r},t))$ = {P$_0$v$_{i,Si,r,r}$: 1 $\leq$ i $\leq$ n_r, 1 $\leq$ r $\leq$ t} $\bigcup$ {P$_r$v$_{i,1,r}$: 1 $\leq$ i $\leq$ n_r, 1 $\leq$ r $\leq$ t} $\bigcup$ {v$_{i,j,r}$v$_{i,j+1,r}$: 1 $\leq$ i $\leq$ n_r, 1 $\leq$ j $\leq$ $S_{i,r}$, 1 $\leq$ r $\leq$ t}

Define a bijective function f : V$(\theta*(n_r,S_{i,r},t)))$ $\to$ {k, k + 1, k + 2,..., k + $\sum_{r=1}^{t}\sum_{i=1}^{n_r}S_{i,\,r}+t$} as follows:

Case 1. When P$_r$ is a prime number for v$_{i,1,r}$ is not a multiple factor of f(P$_r$).

f(P$_0$) =k + $\sum_{r=1}^{t}\sum_{i=1}^{n_r}S_{i,\,r}+t$, 1 $\leq$ i $\leq$ n_r, 1 $\leq$ r $\leq$ t

f(P$_1$) = k

f(P$_r$) = k + $\sum_{r=2}^{t}\sum_{i=1}^{n_{r-1}}S_{i,\,r-1}+(r-1)$, 2 $\leq$ r $\leq$ t

f(v$_{1,j,r}$) = k + $\sum_{r=1}^{t}\sum_{j=1}^{n_{r-1}}S_{i,\,r-1}+\ j+(r-1)$, 1 $\leq$ j $\leq$ $S_{i,r}$, 1 $\leq$ r $\leq$ t

f(v$_{i,j,r}$) = k + $\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,\,r-1}+\sum_{i=2}^{n_r}(S_{i-1,\,r}-1)+i+\ j+(r-2)$, 1 $\leq$ j $\leq$ $S_{i,r}$

For any edge

$P_0v_{i,Si,r,r}\in E\left(\theta*(n_r,S_{i,r},t)\right),\gcd\left(f\left(P_0\right),f\left(v_{i,Si,r,r}\right)\right)=\gcd\left(k+\sum_{r=1}^{t}\sum_{i=1}^{n_r}S_{i,r}+t,k+\right.$
$\left.\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+S_{i,r}+(r-2)\right)=$
1 since $k+\sum_{r=1}^{t}\sum_{i=1}^{n_r}S_{i,r}+t$ is prime.

For any edge $P_rv_{i,1,r}\in E\left(\theta*(n_r,S_{i,r},t)\right),\gcd\left(f\left(P_r\right),f\left(v_{i,1,r}\right)\right)$
$=\gcd\left(k+\sum_{r=2}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+(r-1),k+\right.$
$\left.\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+1+(r-2)\right)$
$=1$ since $k+\sum_{r=2}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+(r-1)$ is
prime and $v_{i,1,r}$ is not a multiple factor of $f\left(P_r\right)$.

For any edge $v_{i,j,r}v_{i,j+1,r}\in E\left(\theta*(n_r,S_{i,r},t)\right),\gcd\left(f\left(v_{i,j,r}\right),f\left(v_{i,j+1,r}\right)\right)=$
$\gcd\left(k+\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+j+(r-2),\right.$
$\left.k+\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+j+1+(r-2)\right)=1$
since $k+\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+j+(r-2)$ and
$k+\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+j+1+(r-2)$
are consecutive positive integers.

Case 2. When P$_r$ is not a prime number for v$_{i,1,r}$ is not a multiple factor of f(P$_r$).

f(P$_0$) =k + $\sum_{r=1}^{t}\sum_{i=1}^{n_r}S_{i,\,r}+t$, 1 $\leq$ i $\leq$ n_r, 1 $\leq$ r $\leq$ t

f(P$_1$) = k

f(P$_r$) = k + $\sum_{r=2}^{t}\sum_{i=1}^{n_{r-1}}S_{i,\,r-1}+(r-1)$, 2 $\leq$ r $\leq$ t

f(v$_{1,j,r}$) = k + $\sum_{r=1}^{t}\sum_{j=1}^{n_{r-1}}S_{i,\,r-1}+\ j+(r-1)$, 1 $\leq$ j $\leq$ $S_{i,r}$, 1 $\leq$ r $\leq$ t

f(v$_{i,j,r}$) = k + $\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,\,r-1}+\sum_{i=2}^{n_r}(S_{i-1,\,r}-1)+i+\ j+(r-2)$, 1 $\leq$ j $\leq$ $S_{i,r}$

For any edge P$_0$v$_{i,Si,r,r}\in$ E$(\theta*(n_r,S_{i,r},t))$, gcd(f(P$_0$), f(v$_{i,Si,r,r}$)) = gcd(k + $\sum_{r=1}^{t}\sum_{i=1}^{n_r}S_{i,\,r}+t$, k + $\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,\,r-1}+\sum_{i=2}^{n_r}(S_{i-1,\,r}-1)+i+S_{i,r}+(r-2)$) = 1 since k + $\sum_{r=1}^{t}\sum_{i=1}^{n_r}S_{i,\,r}+t$ is prime. For any edge P$_r$v$_{i,1,r}\in$ E$(\theta*(n_r,S_{i,r},t))$, gcd(f(P$_r$), f(v$_{i,1,r}$)) = gcd(k + $\sum_{r=2}^{t}\sum_{i=1}^{n_{r-1}}S_{i,\,r-1}+(r-1)$, k + $\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,\,r-1}+\sum_{i=2}^{n_r}(S_{i-1,\,r}-1)+i+1+(r-2)$) = 1 since k + $\sum_{r=2}^{t}\sum_{i=1}^{n_{r-1}}S_{i,\,r-1}+(r-1)$ is prime and v$_{i,1,r}$ is not a multiple factor of f(P$_r$).

For any edge

$v_{ij,r}v_{i,j+1,r}\in E\left(\theta*(n_r,S_{i,r},t)\right),\gcd\left(f\left(v_{ij,r}\right),f\left(v_{i,j+1,r}\right)\right)=$

$$\gcd\left(k+\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+j+(r-2)\right.$$

$$\left.k+\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+j+1+(r-2)\right)=1$$

$$\text{since } k+\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+j+(r-2)\text{ and}$$

$$k+\sum_{r=1}^{t}\sum_{i=1}^{n_{r-1}}S_{i,r-1}+\sum_{i=2}^{n_r}\left(S_{i-1,r}-1\right)+i+j+1+(r-2)$$

Hence centralized generalized theta graph $\theta*(n_r,S_{i,r},t)$ is vertex k – prime for k, r, i $\geq$ 1, t $\geq$ 3.

An illustration is given for case 1 in Figure 5 .

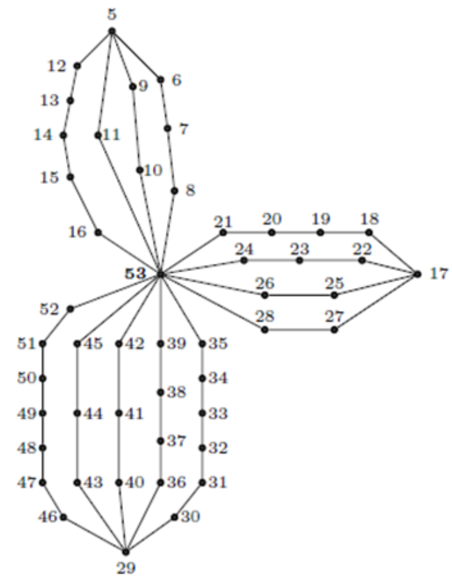

Fig 5. Vertexk-prime labeling of θ^* (4, 4, 5, $S_{i,r}$(3, 2, 1, 5, 4, 3, 2, 2, 6, 4, 3, 3, 7); 3) for k = 5

3 Conclusion

In this study, theta-related graphs including generalised theta graphs, uniform theta graphs, and centralised uniform theta graphs have been shown to exhibit vertex k-prime properties. We introduce the centralised generalised theta graph, a different type of theta network, and show that it has vertex k-prime labelling. One can find out that more families of graphs admit vertex k-prime labeling for some k. Our future work and analysis will focus on comparing prime labeling and vertex k-prime labeling to determine which method is best for time complexity.

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