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Common Fixed-Point Theorems and Applications in Complex Valued b-Metric Spaces

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Abstract

Objective: The purpose of this paper is to prove some common fixed-point theorems satisfying the rational type of contractive condition in the frame work of complex valued b metric spaces. **Methodology:** Some well-known common fixed-point result we studied in the complex valued b –metric spaces and replaced the constant of contractive condition by some control function to obtain the new common fixed point theorems and also proving the uniqueness of fixed point for a pair of mappings. **Findings:** Results based on common fixed point theorems are derived in the context of complex valued –b metric spaces for improved rational inequality, which was the version of the result of Aiman A Mukheimer(1), with an application which shows the applicability and validity of our main result. **Novelty:** This study revealed the Common fixed-point theorems by replacing the constant of contraction through control function, with suitable application.

Keywords: Common fixed point; Partial order; Cauchy sequence; ComplexValued b metric space; weakly compatible mappings

1 Introduction

Banach contraction principle(2) played important role for many researchers to the study of nonlinear functional analysis. The notion of b-metric space given by Bakhtin(3). concept of complex-valued metric space is given by A. Azam, B. Fisher, M. Khan(4) in 2011. R. Rao,P. Swamy , J. Prasad(5) found the concept of Complex-valued b-metric spaces. Azam et al(4) proved the common fixed theorem for two self-mappings satisfying rational inequality in complex valued metric space with constant of contraction. S. Bhatt, Chaukiyal, Dimri R.C(6) also proved some results related to common fixed-point theorem for other rational type inequality in complex valued metric space with constant of contraction for different condition. Further Amian A Mukheimer(1) generalized the work of A. Azam, B. Fisher, M. Khan(4) and S. Bhatt, Chaukiyal, Dimri R.C(6) and proved the common fixed point theorems for two pair of self-mappings satisfying a rational inequality on complex valued b –metric spaces instead of complex valued metric spaces. The main purpose of this paper is to present common fixed point theorems satisfying rational inequality on complex valued b metric spaces. The result presented in this paper is generalization of the work done by Amian A Mukheimer[1] by replacing the constant of contraction to control function in complex valued b metric space. We also solve an integral equation with the help of main result which shows the applicability and validity of our main result. Subsequently at present many researchers (see[7–11]) have contributed to different concepts and proved common fixed-point theorem in complex valued b metric spaces. Consistent with A. Azam, B. Fisher, M. Khan[4], the Subsequent consequences will be required for the main result.

Definition 1.1.[4]

Suppose that and X is a non-empty set, and let $d : X \times X \to \mathbb{C}$ be a mapping fulfils entirely the conditions of metric space $\forall x, y, z$ in X and $\preccurlyeq$ is partial order defined on $\mathbb{C}$ by A. Azam, B. Fisher, M. Khan[4]

Thus, (X, d) is termed as the "complex valued metric spaces".

Definition 1.2.[5] A non-empty set X with $s \geq 1$, and where $d : X \times X \to \mathbb{C}$ satisfying subsequent

(i) $0 \preccurlyeq d(x,y) \forall x, y$ in X with $d(x,y) = 0 \Leftrightarrow x = y$.

(ii) $d(x,y) = d(y,x) \forall x, y$ in X

(iii) $d(x,y) \preccurlyeq s[d(x,z) + d(z,y)], \forall x, y, z$ in X

Thus, $(X, d)$is termed as the complex valued b- metric space.

Definition1.3.[5] The sequence $\{x_n\}$, where $x \in X$

(i) Is called convergent if for every complex no $z, d(x_p, x) \prec c$, with $0 \prec z$ there exist p_0 be a natural no.

(ii) is called Cauchy sequence if $d(x_p, x_{p+q})$ for every z be a complex no. with $0 \prec z$ There exist a natural number $p_0 \in N$ such that $p > p_0$.

Definition1.4.[5] Let S and A be two self-mappings on X will be weakly compatible if both the mappings hold commutative law for their coincidence points.

Lemma1.5.T be a self-mapping on X . Then the mapping whose domain is $E$and co-domain is X is called one to one if $T(E) = T(X)$ where E is the sub set of X .

2 Main Results

Theorem2.1. Suppose that in a complex valued b-metric space (X, d) where the number's is greater and equal than 1 with two self-mappings U and V on X if $\exists$ a function $\delta : X \to [0, 1)$so as to $\forall x, y$ ò X

satisfying the following

(i)$\delta(Ux) \leq \delta(x)$

(ii) $\delta(Vx) \leq \delta(x)$

(iii)$\delta(x) < 1$

iv)

$$d(Ux, Vy) \preccurlyeq \frac{\delta(x)[d(x,Ux)d(x,Vy) + d(y,Vy)d(y,Ux)]}{d(x,Vy) + d(y,Ux)} \tag{2.1}$$

Then there exists a common fixed point of U and V.

Proof. Construct the sequence $\{x_n\}$ for arbitrary point x_0 and since $U(X)$ and $V(X)$ is the subset of X so as to $x_{2n+1} = Ux_{2n}, x_{2n+2} = Vx_{2n+1}, \forall n$ greater and equal than 0.

$d(x_{2n+1}, x_{2n+2}) = d(Ux_{2n}, Vx_{2n+1}) \preccurlyeq \frac{\delta(x_{2n})[d(x_{2n},Ux_{2n})d(x_{2n},Vx_{2n+1}) + d(x_{2n+1},Vx_{2n+1})d(x_{2n+1},Ux_{2n})}{d(x_{2n},Vx_{2n+1}) + d(x_{2n+1},Ux_{2n})}$

$\preccurlyeq \delta(x_{2n-2}) d(x_{2n}, x_{2n+1})$

Thus,

$$d(x_{2n+1}, x_{2n+2}) \preccurlyeq \delta(x_0) d(x_{2n}, x_{2n+1}) \tag{2.2}$$

Now, we set $\alpha = \delta(x_0)$

Thus,

$$"|d(x_{2n+1}, x_{2n+2})| \preccurlyeq \alpha^{2n+1} |d(x_0, x_1)|" \tag{2.3}$$

For any m greater than n and m, n be a natural no., by triangle inequality

$|d(x_n, x_m)| \leq$

$$s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3 \mid d(x_{n+2}, x_m \mid + .. + s^{m-n-2} \left| d(x_{m-3}, x_{m-2}) \right| + s^{m-n-1} \left| d(x_{m-2}, x_{m-1} \right| + s^{m-n} |d(x_{m-1}, x_m)| \tag{2.4}$$

From (2.3) we have

$$\text{"}\left|d\left(x_n,x_m\right)\right| \leq \frac{(s\alpha)^n d\left(x_0,x_1\right)}{1-s\alpha}\text{"} \to 0 \text{ as } m,n \to \infty \text{ as } \alpha \in [0,1) \text{ and } s \geq 1 \tag{2.5}$$

Thus $\{x_n\}$ is a Cauchy sequence, since X is complete thus $\exists u$ in X so as to x_n is converge to u as n to ∞. Further we prove $Su = u$.If not, there exist z so as to,

$|d(u,su)| = |z| > 0$

$Z = d(u,U\ u) \preccurlyeq sd\left(u,x_{2n+2}\right) + sd\left(Uu,Vx_{2n+1}\right)$

This concludes that

$|d(u,U\ u)| \leq s\left|d\left(u,x_{2n+2}\right)\right| + \underline{s}\left[\frac{\delta(u)[|d(u,U\ u)||d(u,x_{2n+2})|+|d(x_{2n+1},x_{2n+2})||d(x_{2n+1},U\ u)|]}{|d(u,x_{2n+2})|+|d(x_{2n+1},U\ u)|}\right]$

As n tends to ∞,

$|z| \leq 0$, logically this is not possible. Then $|z|$ is equal to zero.

Hence Uu is equal to u .

Similarly, Vu is equal to u

Uniqueness of u.

Assume that w is one more fixed point of both the mappings. Then by (2.1)

$d(u,w) = d(Uu,Vw) \preccurlyeq \frac{\delta(u)[d(u,U\ u)d(u,Vw)+d(w,Vw)d(w,U\ u)]}{d(u,Vw)+d(w,U\ u)}$

Which implies that

$d(u,w) \preccurlyeq 0$ and hence $u = w$.

Therefore "common fixed point" of both the mappings is w.

Corollary 2.2. In a complex valued b-metric space (X,d) where the number's is greater and equal than 1 with a self-mapping V on X if $\exists$ a function $\delta : X \to [0,1)$, so as to $\forall x,y \in X$ so as to

(i) $\delta(Vx) \leq \delta(x)$

(ii)$\delta(x) < 1$;

(iii) $d(Vx,Vy) \preccurlyeq \frac{\delta(x)[d(x,Vx)d(x,Vy)+d(y,Vy)d(y,Vx)]}{d(x,Vy)+d(y,Vx)}$.

Thus, the mapping V hold a fixed point of uniqueness.

Proof. Proof of the theorem is obvious by taking V is equal to U in our main consequence.

Corollary 2.3. In "complex valued b-metric space"(X,d) where the numbers is greater and equal than 1 with a self-mapping T on X if $\exists$ a function $\delta : X \to [0,1)$, so as to $\forall x$ and y in X so as to

(i)$\delta\left(\mathrm{V^n x}\right) \leq \delta\quad(\mathrm{x})$

(ii)$\delta(\mathrm{x}) < 1$

(iii) $d\left(V^n x,V^n y\right) \preccurlyeq \frac{\delta(x)[d(x,V^n x)d(x,V^n y)+d(y,V^n y)d(y,V^n x)]}{d(x,V^n y)+d(y,V^n x)}$

Wherever $\delta(\mathrm{X})$ in $[0,1]$Then there exists a unique fixed point of T.

Proof. Using (2.2), we have unique fixed point of V^n which is u

$V^n(Vu) = V\left(V^n u\right) = V(u)$

Then Vu is the fixed point of V^n. Therefore $Vu = u$

Thus V hold a fixed point of uniqueness.

Theorem2.4. In a complex valued b-metric space (X,d) with two self-mappings U and V on X such that $V(X)$ is the subset of $U(X)$ When $U(X)$

is complete, if $\exists$ a function $\delta : X \to [0,1)$

satisfying the following conditions

(i) $\delta(Vx) \leq \delta(Ux)$

(ii) $\delta(Ux) < 1$

(iii) $d(Vx,Vy) \preccurlyeq \frac{\delta(Ux)[d(Ux,Vx)d(Ux,Vy)+d(Uy,Vy)d(Uy,Vx)]}{d(Ux,Vy)+d(Uy,Vx)}$

Thus, both the mappings contain a unique coincidence point in X Furthermore, if both the mappings are weakly compatible, thus there exist a "common fixed point "uniquely in X.

Proof. Using (1.5) Since $V(E)$ is the subset of U(E) and $V(X)$ is the subset of $U(X)$

which is equal to $U(E)$.

Let define a self-mapping Φ on $U(E)$ such that

$$\Phi(Ux) = Vx \tag{2.6}$$

we have

$$\delta(\Phi(Ux)) \leq \delta(\mathrm{U}x) \tag{2.7}$$

From (iii) and (2.6), we get ,

$$d(\Phi(Ux),\Phi(Uy)) \preccurlyeq \frac{\delta(Ux)[d(Ux,\Phi(Ux))d(Ux,\Phi(Uy))+d(Uy,\Phi(Uy))d(Uy,\Phi(Ux))]}{d(Ux,\Phi(Uy))+d(Uy,\Phi(Ux))} \tag{2.8}$$

by the completeness of $U(E)$ which is equal to $U(X)$and by use of corollary (2.2) with a mapping Φ in that case there will be a fixed point u in $U(X)$ uniquely, such that $\Phi(u) = u$.

Then we have $u = Uv$ for some u in X.

So $\Phi(Uv) = Uv,$ to is $Vv = Uv$

Thus, both the mappings U and V have point of coincidence uniquely.

By the weakly compatibility of U and $V \& u = Vv = Uv$

$Uu = UVv = VU\ v = Vu$ Hence $U\ u = Vu$ coincidence point of both the mappings U and V.

Since u is the only point of coincidence of U and V , we have $u = Uu = Vu$

Uniqueness of u. Let w be one more common-fixed point of both the mappings, by(iii)

$d(u,w) = d(Vu,Vw) \preccurlyeq \frac{\delta(Uu)[d(Uu,Vu)d(Uu,Vw)+d(Uw,Vw)d(Uw,Vu)]}{d(Uu,Vw)+d(Uw,Vu)} \leq 0$

$d(u,w) \preccurlyeq 0$

Therefore $u = w$

3 Applications

In this segment, we solve system of Uryshon integral equations through applying our main result

Theorem3.1. Let $X = (C[c,d],R^n)$, where $[\mathrm{c,d}] \subseteq R^+$ and $d : X \times X \to \mathbb{C}$, such that

$d(x,y) = \max_{v\in[a,b]} \|x(v) - y(v)\|_\infty \sqrt{1} + a^2 e^{i\tan^{-1} a}$

with

$$\text{"}\mathrm{x(v)} = \textstyle\int_c^d L_1(v,u,x(u))du + m(t)\text{"} \tag{3.1}$$

$$\mathrm{x(v)} = \textstyle\int_c^d L_2(v,u,x(u))du + n(t) \tag{3.2}$$

When t in $[a,b]$

Let $\mathrm{L_1,\ L_2 : [c,d]x[c,d] \times R^n \to R^n}$ are such that $\mathrm{Px, Qx}$ in X for all $x \in X$where

$\mathrm{P_x(t)} = \int_c^d L_1(v,u,x(u))du$

and

$\mathrm{Q_x(t)} = \int_c^d L_2(v,u,x(u))du$

$\forall$t in $[c,d]$

If $\exists$ a function $\delta : X \to [0,1)$ so as to $\forall x,y$ in X,

(i) $\delta\,(\mathrm{P_x + g}) \leq \delta(X)$

(ii) $\delta\,(\mathrm{Q_x + h}) \leq \delta(X)$

(iii) $\delta(\mathrm{X}) < 1$

$$\preccurlyeq \delta(x)[C\,(x,y)\,(v) + D(x,y)(v)$$

i) $\left\|P_x(v) - Q_y(v) + m(v) - n(v)\right\|_\infty \sqrt{1} + a^2 e^{i\tan -1a} \preccurlyeq \delta(x)\,[C(x,y)(v) + D(x,y)(v)]$

Where

"$C(x,y)(v) = \frac{\|x(v)-y(v)\|_\infty \sqrt{1}+a^2 e^{i\tan^{-1} a}}{d(x,Vy)+d(y,U\ x)}$"and

$$x(t) = \textstyle\int_c^d L_2(v,u,x(u))du + n(t) \tag{3.4}$$

$$\text{"}D(x,y)(v) = \frac{\max_{v\in[a,b]} \left\|Q_y(v)+n(v)-y(v)\right\|_\infty \max_{v\in[a,b]} \|P_x(v)+m(v)-y(v)\|_\infty \sqrt{1}+a^2 e^{i\tan^{-1} a}}{d(x,Vy)+d(y,U\ x)}$$

Thus there exist a unique common solution of equation (3.1) and (3.2).

Proof: Let U and V be the two self-mappings define on (X,d) such that $Ux = P_x + g$ and $Vx = Q_x + h$.

$d(UI\,x, Vy) = \max_{t\in[a,b]} \left\| P_x(t) - Q_y(t) + m(t) - n(t) \right\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}$,

$d(x, U\,x) = \max_{t\in[a,b]} \| P_x(t) + m(t) - x(t) \|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}$

$d(x, Vy) = \max_{t\in[a,b]} \left\| Q_y(t) + n(t) - x(t) \right\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}$

$d(y, Vy) = \max_{t\in[a,b]} \left\| Q_y(t) + n(t) - y(t) \right\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}$

$d(y, U\,x) = \max_{t\in[a,b]} \| P_x(t) + m(t) - y(t) \|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}$

Since all the condition of theorem (2.1) is hold,

Thus, there exists a common fixed point for U and V such that

$x = Ux = Vx$

$x = Ux = P_x + g$ and

$x = Vx = Q_x + h$ that is

$$x(t) = \int_c^d L_1(v,u,x(u))du + m(t) \tag{3.5}$$

and

$$x(t) = \int_c^d L_2(v,u,x(u))du + n(t) \tag{3.6}$$

Therefore, the Urysohn integral (3.1), (3.2) have a common fixed point uniquely.

4 Conclusion

In present research paper, we proved some results related to common fixed-point theorems for rational type contraction in the context of complex-valued b-metric space which is the version of Amian A Mukheimer[(1)]. In the present work, we replace the constant of contraction through control function and also solve system of Uryshon integral equations through applying our main result**.** Derived results may be the encouragement to other authors in their findings, to be an appropriate tool for their outcomes to study more generalized contractive mappings in the view of more generalized spaces and diverse applications in analysis.

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