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k-Irreducible Ideals, Common Right Divisors and Euclidean Norms in Gamma Semirings

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Abstract

Objectives: The main objective of this paper is to derive some of the results of $k-$irreducible ideals, common right divisors and Euclidean $\Gamma-$ $s$emiring. **Methods:** To establish the main results in $\Gamma-s$emirings, we use some conditions like commutativity, simple, semi subtractive, centreless, multiplicative $\Gamma-$idempotent, strong multiplicative $\Gamma-$idempotent, additively cancellation and the concept of common right divisor and Euclidean norm. **Findings:** First we study some results regarding $k-$irreducible ideals and define a $\alpha-$ generated ideal by any element of R. In connection with different conditions, we characterize some results of irreducibility in ideals, primary ideals, common right divisors and Euclidean norms. **Novelty:** An $\alpha-$ generated ideal by any element of R, say a, denoted by $< a\alpha >$ is $k-$ideal if R is simple, semi subtractive and additive cancellative. Again, the conditions mentioned above are used to prove that every $k-$irreducible ideal of a $\Gamma-s$emiring R is primary ideal of R. Furthermore, the concept of Euclidean $\Gamma-s$emiring developed to establish various results in the theory of $\Gamma-s$emirings.

Keywords: k-Irreducible Ideal; Common Right Divisor; Euclidean Norm; Primary Ideals; Noetherian Γ-Semirings

1 Introduction

The first mathematical structure we encounter is the set of non-negative integers N with usual addition and multiplication provides a natural example of a semiring. For a given positive integer n, the set of all $n \times n$ matrices over a semiring R forms a semiring with usual matrix addition and multiplication over R. But the situation for the set of all non-positive integers and for the set of all $m \times n$ matrices over a semiring R are different. They do not form semirings with the above operation, because multiplication in the above sense are no longer binary compositions. This notion provides a new kind of algebraic structure known as a $\Gamma-$ semiring. In 1995, an author[1] as a generalization of a semiring as well as $\Gamma-$ring introduced the concept of $\Gamma-s$emiring. Sharma and Ranote[2], Sharma and Gupta[3] introduced the concept of a commutative, simple, additive idempotent and multiplicative $\Gamma-$idempotent $\Gamma-$semiring and studied the consequences of imposing these conditions on a $\Gamma-$semiring R.

It is worthy to be note here that many authors in different aspects also focused the study of $\Gamma-$semirings with irreducible, prime and primary ideals. The main aim of this study is to generalize some fundamental results of $k-$ irreducible ideals, common right divisors and Euclidean norm of $\Gamma-$semirings proved in [4–6] and investigate the conditions for which $k-$ irreducible ideals of R becomes primary ideals of R and for commutative $\Gamma-$semirings, the notions of left and right Euclidean norms coincides. Finally, we introduce the concept of Principal left $k-$ ideal $\Gamma-$semiring ((PLKI) $\Gamma-$semiring.

1.1 Preliminaries

For the definition of $\Gamma-$semiring and their identity elements 0 and 1, centreless, simple, semi subtractive, noetherian $\Gamma-$semiring, $k-$ideal, maximal ideal, smallest $k-$ideal, multiplicative $\Gamma-$ idempotent, strong multiplicative $\Gamma-$ idempotent and additive cancellable, one can refer to [2]. Now we include some necessary preliminaries for the sake of completeness. An ideal I of R is said to be prime ideal if for ideals A and B of R with $A\Gamma B \subseteq I$ then either $A \subseteq I$ or $B \subseteq I$. For an ideal I of a commutative $\Gamma-$semiring R, the prime radical of I, $r(I)$ is defined as $r(I) = \left(x \in R \mid (x\alpha)^{n-1}x \in I\right\}$ for some positive integer n and for all $\alpha \in \Gamma$. An ideal P of R is said be a primary ideal if $x\alpha y \in P$, for $\alpha \in \Gamma$ and $x, y \in R$, then either $x \in P$ or $(y\beta)^{n-1}y \in P$ for all $\beta \in \Gamma$ and some positive integer n. or An ideal P of a $\Gamma-$ semiring R is said be a primary ideal if for any two ideals A and B of $R, A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq r(P)$. A proper $k-$ideal Q of R is said to be $k-$irreducible ideal of R if for any two $k-$ideals, I and J of $R, Q = I \cap J$ implies that either $Q = I$ or $Q = J$ and strongly $k-$irreducible ideal of R if for any two $k-$ ideals, I and J of $R, I \cap J \subseteq Q$ implies that either $Q = I$ or $Q = J$.

Remark: Let R will denote a commutative $\Gamma-$ semiring with 0 and identity 1. We will use this notation throughout this paper unless otherwise stated.

2 Methodology

Following [4–6], we will establish some results of irreducible ideals and principal ideals by using some conditions on $\Gamma-$semirings. Further, we propose an algorithm on Euclidean norm that based on elements of $\Gamma-$semirings with conditions of strong identity and multiplicative $\Gamma-$idempotent and it will provide other class of Euclidean norms.

3 Results and Discussion

3.1 k-irreducible ideals in a $\Gamma-$semiring

In this section, we characterize some of the results of $k-$irreducible ideals by using some conditions such as simple, multiplicative $\Gamma-$ idempotent, strong identity, additive cancellative and $\alpha-$ generated set by an element of R in $\Gamma-$ semirings.

Lemma 4.1.1. Let R be a simple and multiplicative $\Gamma-$ idempotent $\Gamma-$ semiring with strong identity then any ideal I of R is a $k-$ideal.

Proof. Let $x \in I$ and $x + y \in I$ for any $y \in R$. Since R is multiplicatively $\Gamma-$ idempotent, so there exists $\alpha \in \Gamma$ and $y \in R$ such that $y\alpha y = y$. Now, y $= 1\alpha y = (x+1)\alpha y = x\alpha y + y = x\alpha y + y\alpha y = (x+y)\alpha y$. But $x + y \in I$ and I is an ideal so $y \in I$. Hence, I is a $k-$ ideal.

Lemma 4.1.2. Let Q be a $k-$ ideal of a $\Gamma-$ semiring R, then the set $(Q : a) = \{r \in R \mid a\Gamma r \subseteq Q\}$ is a $k-$ ideal.

Proof. Let $y, z \in (Q : a)$ such that $x + y = z$. Then $a\Gamma y \subseteq Q$ and $a\Gamma z \subseteq Q$. This implies that for all $\alpha \in \Gamma$, we have $a\alpha y, a\alpha z \in Q$. Now as $x + y = z$, so $a\alpha x + a\alpha y = a\alpha z$. But Q is a $k-$ideal so $a\alpha x \in Q$, for all $\alpha \in \Gamma$. This implies that $a\Gamma x \subseteq Q$ so $x \in (Q : a)$. Hence, $(Q : a)$ is a $k-$ideal.

The proof of the following Theorems are simple and straightforward, so we only give the statements.

Theorem 4.1.3. Let R be a $\Gamma-$ semiring. If Q is a maximal $k-$ideal of R, then it is a $k-$irreducible ideal.

Theorem 4.1.4. Every prime $k-$ideal of R is strongly $k-$irreducible ideal of R.

Theorem 4.1.5. If Q is strongly $k-$irreducible ideal of R then it is $k-$irreducible.

Corollary 4.1.6. If Q is a prime $k-$ideal then it is $k-$irreducible.

Theorem 4.1.7. Let R be a noetherian $\Gamma-$ semiring. If I is a $k-$ideal of R and $r \in R \setminus \{0\}$ such that $r \notin I$, then there will be a $k-$irreducible ideal M of R such that $I \subseteq M$ and $r \notin M$.

Theorem 4.1.8. Let R be a Noetherian $\Gamma-$ semiring. If I is a proper $k-$ideal of R then it is equal to the intersection of $k-$irreducible ideals of R which contains I.

Proof. Let for any indexed set $\Delta, (Q_i \mid i \in \Delta\}$ be the family of $k-$irreducible ideals of R such that $I \subseteq Q_i$ for each $i \in \Delta$. Now, it is clear that $I \subseteq \bigcap_{i\in\Delta} Q_i$. Rest to prove $\bigcap_{i\in\Delta} Q_i \subseteq I$. Let $\bigcap_{i\in\Delta} Q_i \nsubseteq I$, then there exists an element say r such that $r \in \bigcap_{i\in\Delta} Q_i$ and $r \notin I$. So by Theorem 4.1.7, there will be a $k-$irreducible ideal, say Q, such that $I \subseteq Q$ and $r \notin Q$. But then Q will be one of

Q_i and $r \notin Q$ implies that $r \notin \bigcap_{i \in \Delta} Q_i$, which is a contradiction. Hence, we can't find such an element so that $r \in \bigcap_{i \in \Delta} Q_i$ and $r \notin I$. Therefore, $\bigcap_{i \in \Delta} Q_i \subseteq I$. Hence, $I = \bigcap_{i \in \Delta} Q_i$.

Definition 4.1.9. Let R be a $\Gamma-$ semiring. If $a \in R$ and $\alpha \in \Gamma$ then the set defined by $< a\alpha >_r = \{a\alpha r \mid$ for all $r \in R\}$, will be termed as a right $\alpha-$ generated ideal by a. Similarly, $< \alpha a >_l = \{r\alpha a$ for all $r \in R\}$ is a left ideal of R. In case if R is commutative then both (left and right) ideals will be the same. Therefore, in case of commutativity, we will denote this ideal by $< a\alpha >$.

Theorem 4.1.10. Let R be centreless, semi subtractive and additive cancellative $\Gamma-$ semiring then for any $a \in R, Q =< a\alpha >$ is a $k-$ideal of R.

Proof. Let $y, z \in Q$ such that $x + y = z$ for any $x \in R$. Then $y = a\alpha s$ and $z = a\alpha t$ for some $s, t \in R$. Since R is semi subtractive, so there will be an element, say $u \in R$ such that either $u + s = t$ or $u + t = s$. If $u + s = t$ then $x + y = z = a\alpha t = a\alpha(u + s) = a\alpha u + a\alpha s = a\alpha u + y$. This implies that $x = a\alpha u$, since R is additively cancellative. Therefore, $x \in Q$. Further, if $u + t = s$, then $a\alpha t = z = x + y = x + a\alpha s = x + a\alpha(u + t) = x + a\alpha u + a\alpha t$. This implies that $x + a\alpha u = 0$. Therefore, $x = a\alpha u = 0 \in Q$, since R is centreless. Hence, Q is a $k-$ideal.

Theorem 4.1.11. Let K and L be two ideals of a $\Gamma-$ semiring R. Then $r(Sl(K \cap L)) = r(Sl(K) \cap Sl(L)) = r(Sl(K)) \cap r(Sl(L))$

The following theorem is proved in [6] for semirings.

Theorem 4.1.12. Let R be a centreless, semi subtractive, additively cancellative and commutative noetherian $\Gamma-$ semiring, then every $k-$irreducible ideal of R is primary ideal of R.

Proof. Let Q be a $k-$ irreducible ideal of R and let $a\beta b \in Q$ such that $b \notin Q, a, b \in R$. Now, we construct two ideals I and J of R such that $I =< \left((a\beta)^{m-1}a\right)\alpha > +Q$ and $J =< b\alpha > +Q$, for $\alpha, \beta \in \Gamma$. Now, by Theorem 4.1.10, $< \left((a\beta)^{m-1}a\right)\alpha >$ and $< b\alpha >$ are $k-$ideals of R, therefore I and J, being sum of $k-$ideals are also $k-$ ideals. Now, it is clear that $Q \subseteq I \cap J$. Let $y \in I \cap J$. Then $y = \left((a\beta)^{n-1}a\right)\alpha z + q$ for some $z \in R$ and $q \in Q$. Again, $a\beta J \subseteq Q$, since $a\beta b \in Q$. So $a\beta y \in Q$, since $y \in J$. Therefore, $a\beta y = ((a\beta)^n a)\alpha z + a\beta q \in Q$. Also, $a\beta q \in Q$, since Q is an ideal. Thus, it follows that $((a\beta)^n a)\alpha z \in Q$, since Q is a $k-$ideal. Further, let us construct a new set $A_n = \left(x \in R \mid \left((a\beta)^{n-1}a\right)\alpha x \in Q\right\}$. Now, it is clear that A_n is an ideal of R and $A_1 \subseteq A_2 \ldots$ is an ascending chain of ideals in R, since R is noetherian, therefore $A_n = A_{n+1} = \ldots$ for some $n \in Z^+$. Again, $((a\beta)^n a)\alpha z \in Q$. This implies that $z \in A_{n+1} = A_n$. Therefore, $\left((a\beta)^{n-1}a\right)\alpha z \in Q$, which implies that $y \in Q$. Thus, $I \cap J = Q$. Now by Theorem 4.1.11, we have $r(Sl(I \cap J) = r(Sl(I) \cap Sl(J)) = r(Sl(I)) \cap r(Sl(J))$. Therefore, $r(Sl(I \cap J)) = r((Sl(I)) \cap r((Sl(J))$. Now as $Q = I \cap J$, so $Q = Sl(Q) = Sl(I \cap J)$, since Q is a $k-$ideal so $Q = Sl(Q)$. Thus, $r(Q) = r(Sl(I \cap J)) = r(Sl(I)) \cap r(Sl(J))$. As Q is $k-$ irreducible so $r(Q)$ is also $k-$ irreducible. Again, $r(Q) = Q$, since $r(Q)$ is $k-$ irreducible. So $Q = r(Sl(I)) \cap r(Sl(J))$. Now $b \in J$ implies that $b \in r(Sl(J))$. But $b \notin Q$ therefore $Q \neq r(Sl(J))$. Therefore, $Q = r(Sl(I))$, since Q is k-irreducible. So $(a\beta)^{n-1}a \in I$ implies that $(a\beta)^{n-1}a \in r(Sl(I)) = Q$. Hence Q is primary.

Common Right Divisor of a Γ – semiring

In this section we characterize the results regarding right divisors $RD(x)$, common right divisors $CRD(x)$ and greatest common right divisors $GCRD(x)$ in a $\Gamma-$semiring R. An element x of R is unit if and only if there exists an element y of R and $\alpha \in \Gamma$ satisfying $x\alpha y = 1 = y\alpha x$. The element y of R is called the inverse of x in R. Let us denote the set of all elements of R having inverse in R by $U(\Gamma R)$. This set is non-empty since $1 \in U(\Gamma R)$ and not all of R.

Let x be an element of a $\Gamma-$semiring R. Then the set of all right divisors of x in R is $RD(x) = \{y \in R \mid x \in R\Gamma y\} = \{y \in R \mid R\Gamma x \subseteq R\Gamma y\}$. Since $y \in RD(y)$, for all $y \in R$, so it is clear that $y \in RD(x)$ if and only if $RD(y) \subseteq RD(x)$. Note that if R is a $\Gamma-$semiring and $y \in RD(x)$ then there exists an element $r \in R$, such that $x = r\alpha y$, for all $\alpha \in \Gamma$.

Lemma 4.2.1. Let R be a simple $\Gamma-$ semiring with a strong identity. If $y \in RD(x)$ then $x + y = y$.

Proof. Let R be a simple $\Gamma-$ semiring and $y \in RD(x)$, then there exists an element r of R and $\alpha \in \Gamma$ such that $x = r\alpha y$. Thus we have $x + y = r\alpha y + y = (r + 1)\alpha y = 1\alpha y = y$.

If x is an element of a $\Gamma-$ semiring R then $U(\Gamma R) \subseteq RD(1) \subseteq RD(x)$. If $x \notin U(\Gamma R)$ and $RD(x) = U(\Gamma R) \cup \{x\}$ then x is said to be irreducible from the right. Irreducibility from the left is defined similarly.

Definition 4.2.2. Let X be a nonempty subset of a $\Gamma - s$emiring R. Then the set of common right divisors of X is $CRD(X) = \cap\{RD(x) \mid x \in X\} = \{y \in R \mid R\Gamma X \subseteq R\Gamma y\}$. An element $y \in CRD(X)$ is the greatest common right divisor (GCRD) of X if and only if $CRD(X) = RD(y)$

Example 4.2.3. Let R be a $\Gamma-$semiring. If $R = M_2(N)$, where N is a set of all nonnegative integers and $\Gamma = \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\}$ then the only elements of R having determinant 1 which are irreducible from the right are $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$.

Theorem 4.2.4. Let R be a $\Gamma-$ semiring and X be a nonempty subset of R then an element $y \in R$ is a GCRD of X if and only if $R\Gamma X \subseteq R\Gamma y$ and if $z \in R$ satisfies $R\Gamma X \subseteq R\Gamma z$ then $R\Gamma y \subseteq R\Gamma z$

Proof. Let y be a $GCRD$ of X. Then $y \in CRD(X)$ and so $y \in RD(x)$ for all $x \in X$. Thus, $R\Gamma x \subseteq R\Gamma y$ for all $x \in X$. This implies that $R\Gamma X \subseteq R\Gamma y$. Further, if $R\Gamma X \subseteq R\Gamma z$ for some $z \in R$ then $z \in CRD(X) = RD(y)$ and so $R\Gamma y \subseteq R\Gamma z$. Conversely, assume that

$R\Gamma X \subseteq R\Gamma y$ and $z \in R$ satisfies $R\Gamma X \subseteq R\Gamma z$. Now let $y \in CRD(X)$. This implies that $RD(y) \subseteq CRD(X)$. Again, if $z \in CRD(X)$ then $R\Gamma X \subseteq R\Gamma z$ and so $R\Gamma y \subseteq R\Gamma z$. Therefore, $z \in RD(y)$, gives that $CRD(X) \subseteq RD(y)$. Hence, $CRD(X) = RD(y)$. That is, y is $GCRD$ of X.

Corollary 4.2.5. Let R be $\Gamma-$semiring. If every left ideal of R is principal ideal then every non-empty subset of R has a GCRD.

Lemma 4.2.6. Let R be a $\Gamma-$semiring and x, y and z are the elements of R. If m is a $GCRD$ of $\{x, y\}$ and n is a $GCRD$ of $\{z, m\}$ then n is a GCRD of $\{x, y, z\}$.

If x and y are the elements of a $\Gamma-$semiring R then clearly $CRD(\{x, y\}) \subseteq CRD(\{x+y, y\}$. We now investigate the conditions for having equality.

Theorem 4.2.7. Let R be a Γ-semiring, then $CRD(\{x, y\}) = CRD(\{x+y, y\})$ for all $x, y \in R$ if and only if every principal left ideal of R is $k-ideal$.

3.2 Euclidean Norm of a $\Gamma-$ semiring

A left Euclidean norm f defined on a $\Gamma-$semiring R is a function $f : R \setminus \{0\} \longrightarrow N($ set of all non-negative integers) satisfying the condition that if $x, y \in R$ with $y \neq 0$ then there exist elements q and r of R and $\alpha \in \Gamma$ satisfying $x = q\alpha y + r$ with $r = 0$ or $f(r) < f(y)$. A right Euclidean norm f defined on a $\Gamma-$semiring R is a function $f : R \setminus \{0\} \longrightarrow N$ satisfying the condition that if $x, y \in R$ with $y \neq 0$ then there exist elements q and r of R and $\alpha \in \Gamma$ such that $x = y\alpha q + r$ with $r = 0$ or $f(r) < f(y)$. A $\Gamma-$semiring R is left (right) Euclidean if and only if there exists a left (right) Euclidean norm defined on R. For commutative $\Gamma-$semiring, the notions of left and right Euclidean norm coincides.

Theorem 4.3.1. Let R be $\Gamma-$semiring. If f is a left Euclidean norm defined on R then there exists another left Euclidean norm g defined on R such that $g(x) \leq f(x)$ for all $x \in R \setminus \{0\}$ and $g(y) \leq f($ r$\alpha y)$ for all $r, y \in R$ and $\alpha \in \Gamma$ satisfying r $\alpha y \neq 0$.

Proof. Let $g(x) = min\{f(r\alpha x) \mid r\alpha x \neq 0\}$, for all $0 \neq x \in R$ and $\alpha \in \Gamma$. Then clearly g satisfies both conditions. Now to show that g is a left Euclidean norm on R. Let x and y be non-zero elements of R satisfying $g(x) \geq g(y)$. Then there exists an element $r_1 \in R$ and $\alpha \in \Gamma$ such that $g(y) = f(r_1\alpha y)$. This implies that $f(x) \geq f(r_1\alpha y)$. Therefore, there exist elements $q, r \in R$ and $\alpha, \beta \in \Gamma$ such that $x = q\beta(r_1\alpha y) + r_2$ where either $r_2 = 0$ or $f(r_2) < f(r_1\alpha y) = g(y)$. Hence, g is a left Euclidean norm on R.

If (R, f) is a left Euclidean $\Gamma-$ semiring we can without loss of generality, assume that f satisfies the condition that $f(y) \leq f(r_2\alpha y)$ for all $0 \neq y \in R, r_2 \in R$ and $\alpha \in \Gamma$ such that $r_2\alpha y \neq 0$. A left Euclidean norm satisfying this condition said to be $\Gamma-$ submultiplicative. A left Euclidean norm f defined on a $\Gamma-$ semiring R is $\Gamma-$ multiplicative if and only if $f(x\alpha y) = f(x)f(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$ such that $x\alpha y \neq 0$. That is, f is $\Gamma-$multiplicative if and only if it is a Γ-semigroup homomorphism from $R\setminus\{0\}$ to N(set of non-negative integers).

Theorem 4.3.2. Let R be a $\Gamma-$semiring with a strong identity. Let $\Gamma-$ submultiplicative Euclidean norm be defined by $f : R \setminus \{0\} \longrightarrow N$ and if $M_f = \{r \in R \mid f(r)\}$ is a minimal element of image f then $1 \in M_f$ and if $x \in M_f$ then there exists an element $q \in R$ and $\alpha \in \Gamma$ satisfying $q\alpha x = 1$.

Proof. Let $0 \neq x \in R$ then by definition of $\Gamma-$ sub multiplicative $f(1) \leq f(x\alpha 1) = f(x)$ so $1 \in M_f$. Again if $x \in M_f$ then there exist elements q and r of R and $\alpha \in \Gamma$ satisfying $1 = q\alpha x + r$ with either $r = 0$ or $f(r) < f(x)$. But $f(r) < f(x)$ is not possible by minimality. Hence $q\alpha x = 1$

Theorem 4.3.3. Let R be a strong multiplicative $\Gamma-$idempotent $\Gamma-$semiring with a strong identity. Let $\Gamma-$ submultiplicative Euclidean norm be defined by $f : R \setminus \{0\} \longrightarrow N$ and if $M_f = \{r \in R \mid f(r)\}$ is minimal elements of image (f) then $M_f \cap I_s^{\times}(\Gamma R) = \{1\}$ and $U(\Gamma R) \subseteq M_f$, equality holds if R is commutative.

Proof. Let $z \in M_f \cap I_s^{\times}(\Gamma R)$ then by Theorem 4.3.2, there exists an element $q \in R$ and $\alpha \in \Gamma$ satisfying $1 = q\alpha z$. Now, since $z \in I_s^{\times}(\Gamma R)$ so $z = 1\beta z = (q\alpha z)\beta z = q\alpha(z\beta z) = q\alpha z = 1$, for all $\beta \in \Gamma$. Thus, $M_f \cap I_s^{\times}(\Gamma R) = \{1\}$. Further, if $x \in U(\Gamma R)$ then there exists an element $y \in R$ and $\alpha \in \Gamma$ satisfying $1 = y\alpha x$, so $f(x) \leq f(y\alpha x) = f(1)$. Since $1 \in M_f$, thus equality holds and $x \in M_f$. Hence, $U(\Gamma R) \subseteq M_f$. Finally, if R is commutative, then by Theorem 4.3.2 equality holds.

Remark 4.3.4. Let x be an element of a Γ- semiring R. An element y of R is an additive inverse of x if and only if $x + y = 0 = y + x$. We will denote the additive inverse of an element x, if it exists, by $-x$. Let us denote the set of all elements of R having additive inverse by $A(\Gamma R)$

Theorem 4.3.5. Let R be a commutative cancellative $\Gamma-$ semiring with strong identity and f be a $\Gamma-$ sub multiplicative Euclidean norm defined on R. Then $f(x) = f(-x)$ for all $x \in A(\Gamma R)$ (set of all elements having additive inverse).

Proof. Let us assume that the result is not true and X be a non-empty set of all non-zero elements x' of $A(\Gamma R)$ satisfying $f\left(x'\right) > f\left(-x'\right)$. Let $x \in X$ be such that $f(-x)$ is minimal. Then there exist elements q and r of R and $\alpha \in \Gamma$ satisfying $x = q\alpha(-x) + r$ where either $r = 0$ or $f(r) < f(-x)$. Let $r \neq 0$ then $q\alpha(-x) + (-x) + r = 0$. Therefore, $z = q\alpha(-x) + (-x) \in V(\Gamma R)$ and $-[q\alpha(-x) + (-x)] = r$. Further, $z \notin X$, otherwise there is a contradiction to the choice of x. Thus, $f(z) = f(r)$, which is not possible otherwise by $\Gamma-$ sub multiplicity $f(r) = f(z) = f([q+1]\alpha(-x)) \geq f(-x)$. Thus, $r = 0$. Hence, $x = q\alpha(-x)$. Then

$q\alpha x + x = q\alpha(x + (-x)) = q\alpha 0 = 0$ and so $-x = q\alpha x + x + (-x) = q\alpha x$. This implies that $f(-x) \geq f(x)$, contradicting the assumption that $x \in X$. Hence X must be empty.

Definition 4.3.6. A function $f : R \longrightarrow S$, where R and S are $\Gamma-$semiring is said to be a $\Gamma-$morphism of $\Gamma-$ semiring if

(i) $f(x+y) = f(x) + f(y)$

(ii) $f(x\alpha y) = f(x)\alpha f(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Theorem 4.3.7. Let R be a $\Gamma - s$emiring. If $h : R \longrightarrow S$ be a surjective $\Gamma-$ morphism of R and f is a left Euclidean norm on R then there exists a left Euclidean norm g on S defined by $g(z) = min\left(f(x) \mid x \in h^{-1}(z)\right\}$ for all $0 \neq x \in S$.

Proof. Let m and n be elements of S with $n \neq 0$. Then there exist elements $x, y \neq 0$ of R such that $h(x) = m$ and $h(y) = n$. Further, choose y so that $g(n) = f(y)$. Since f is a left Euclidean norm on R, so there exist elements q and r of R and $\alpha \in \Gamma$ satisfying $x = q\alpha y + r$ where either $r = 0$ or $f(r) < f(y) = g(n)$. Thus, $m = h(x) = h(q)\alpha n + h(r)$, where either $h(r) = 0$ or $g((h(r)) \leq f(r) < f(y) = g(n)$. Hence, g is left Euclidean norm on S.

Theorem 4.3.8. Let R be a $\Gamma-$semiring and f be a left Euclidean norm defined on R then every left $k-$ideal of R is the principal ideal.

Proof. Let f be a left Euclidean norm on R and let J be a left $k-$ideal of R. Then $\{f(x) \mid x \in J\}$ has minimal element say $f(y)$. Let $x \in J \setminus R\Gamma y$ then there exists an element $r \in R \setminus \{0\}$ and $\alpha \in \Gamma$ such that $x = q\alpha y + r$ and $f(r) < f(y)$. But $r \in J$ since J is $k-$ideal, contradicting the minimality of $f(y)$. Hence, $J = R\Gamma y$, so is the principal.

Definition 4.3.9. A $\Gamma-$semiring for which Theorem 4.2.7 holds will be called Principal left $k-$ideal (PLKI) $\Gamma-$semiring.

Theorem 4.3.10. Let R be a left euclidean $\Gamma-$ semiring then R is (PLKI) $\Gamma-$ semiring if and only if there exists a left Euclidean norm f defined on R satisfying the condition that $x = q\alpha y + r$ for $r \in R \setminus \{0\}, \alpha \in \Gamma$ and $f(r) < f(y)$ then $x \notin R\Gamma y$.

Proof. By Theorem 4.3.1, we know that there exists a left Euclidean norm f defined on R satisfying the condition $f(s) \leq f(r\alpha s)$ for all $r, s \in R \setminus \{0\}$. Let $x = q\alpha y + r$ for $r \in R \setminus \{0\}, \alpha \in \Gamma$ and $f(r) < f(y)$. If $x \in R\Gamma y$ then as R is (PLKI) $\Gamma-$semiring, therefore we must have $r = z\alpha y$ for some $z \in R$ and $\alpha \in \Gamma$. Therefore, $f(r) \geq f(y)$, which is a contradiction. Hence, $x \notin R\Gamma y$. Conversely, let $x, y \in R$ and $t \in CRD(\{x+y, y\})$. Then we can write $x + y = m\alpha t$ and $y = n\alpha t$ for $m, n \in R$ and $\alpha \in R$. By the choice of f, we know that $f(x) \geq f(t)$. Therefore, either $x = q\alpha t$ or $x = q\alpha t + r$ for some $0 \neq r \in R$ satisfying $f(r) \leq f(t)$. If $x = q\alpha t + r$ then $m\alpha t = (n+q)\alpha t + r$, which again contradicts the stated condition. Thus, we must have $x = q\alpha t$ so $t \in RD(x)$. Since $t \in RD(y)$, so by the choice of t we have $t \in CRD(\{x, y\})$. Hence, R is $PLKI - \Gamma-$ semiring.

Theorem 4.3.11. Let R be a left euclidean (PLKI) $\Gamma-$ semiring with strong identity and X be any non empty subset of R. Then X has a GCRD.

Proof. Let $X = \{x, y\}$. if $x = y = 0$ then 0 is a $GCRD$ of $\{x, y\}$, so the result is obvious. So, without loss of generality, let $y \neq 0$. Since R is a (PLKI) $\Gamma-$ semiring, so by Theorem 4.3.10, that there exists a left Euclidean norm f defined on R satisfying the condition that if $x = q\alpha y + r$ for $r \in R \setminus \{0\}, \alpha \in \Gamma$ and $f(r) < f(y)$ then $x \notin R\Gamma y$. By repeated application of f we can find elements $q_1, \ldots, q_{n+1}$ and $r_1, \ldots, r_n$ of $R \setminus \{0\}$ and $\alpha, \alpha_1 \ldots \alpha_n \in \Gamma$ such that $x = q_1\alpha y + r_1, y = q_2\alpha_1 r_1 + r_2, \ldots, r_{n-2} = q_n\alpha_{n-1}r_{n-1} + r_n, r_{n-1} = q_{n+1}\alpha_n r_n$ and $f(y) > f(r_1) > \cdots > f(r_n)$ (The process of selecting the q_i and r_i must terminate after finitely many steps, since there are no infinite decreasing sequences of elements of N), working backward, we have $r_{n-2} = (q_n\alpha_{n-1}q_{n+1} + 1]\,\alpha_n r_n, r_{n-3} = (q_{n-1}\alpha_{n-2}q_n\alpha_{n-1}q_{n+1} + (q_{n-1} + q_{n+1}]\,\alpha_n r_n$ and so on until we establish that $r_n \in CRD(\{x, y\})$. Conversely, let $\in CRD(\{x, y\})$. By Theorem 4.3.10, we have $m \in RD(r_1), m \in RD(r_2) \ldots, m \in RD(r_n)$. So $RD(r_n) = CRD(\{x, y\})$. Hence, r_n is a $GCRD$ of $\{x, y\}$.

4 Conclusion

Not all the results that hold good for ideals in rings may be true for the ideals in semirings as well as in $\Gamma - s$emiring$s$ and even for the $k-$ ideals in $\Gamma - s$emiring$s$. Various mathematicians while generalizing the Lasker−Noether's Theorem for semirings ignored this fact. For $k-$ideals in semirings, Lescot[5] established weak primary decomposition as a result in 2015. Following this, here we includes $k-$irreducible ideals, common right divisors and Euclidean $\Gamma - s$emiring and characterize some fundamental results by using the conditions like commutativity, simple, semi subtractive, centreless, multiplicative $\Gamma-$idempotent, strong multiplicative $\Gamma-$idempotent and additively cancellative etc. We define several conclusions of irreducibility in ideals, common right divisors and Euclidean norms in relation to these various conditions. The ideas described in this article have a lot of potential for nourishing and one can investigate them further in polynomial $\Gamma-$semirings, the primary ideals of $\Gamma-$semirings, and by utilizing the concept of Dale Norm in antisimple Γ-semirings.

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