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RESEARCH ARTICLE

A Study on Edge Pebbling Number, Covering Cover Edge Pebbling Number of Friendship Graphs, Odd Path and Even Path

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Abstract

Objectives: To find the edge pebbling number and covering cover edge pebbling number of friendship graphs. **Methods:** The possible minimum edge covering set of the friendship graph is considered and the set with the minimum pebble requirement covering all vertices is selected. **Findings:** Obtained the modified result of edge pebbling number of friendship graph, defined the covering cover edge pebbling number of a graph G, and covering cover edge pebbling number for friendship graphs, odd path and even path is found. Also found that friendship graphs are edge demonic. **Novelty**: Finding the covering cover edge pebbling number of a graph plays a vital role in optimization problems. Since all the vertices are covered, for instance the locations of mobile towers can be considered as edges and residential areas as vertices, and network signals can be modified according to the number of pebbles.

Keywords: Friendship Graphs; Odd Path; Even Path; Minimum Edge Covering Set; Edge Pebbling Number; Edge Demonic; Covering Cover Edge Pebbling Number

1 Introduction

Let G be a simple connected graph with p vertices and q edges. An edge pebbling move on G is defined to be the removal of two pebbles from one edge and the addition of one pebble to an adjacent edge [1]. An edge pebbling number $P_E(G)$ is defined to be the least number of pebbles such that any distribution of $P_E(G)$ pebbles on the edges of G allows one pebble to be moved to any specified, but arbitrary edge [1]. The cover edge pebbling number $CP_E(G)$ of a graph G is defined as, however the pebbles are initially placed the minimum number of pebbles required to place one pebble in all the edges through edge pebble move in all possible distribution [1]. The cover edge pebbling number for some standard graphs such as path, complete graph, friendship graph and star graphs are determined in [1]. The cover edge pebbling number for some families of graphs such as Helm graph, crown graph and pan graph are determined in [2].

A graph G is said to be edge demonic if the edge pebbling number equals the number of edges. i.e., A graph G is said to be edge demonic if $P_E(G) = q$ [3].

In section 2 the edge pebbling number of friendship graph is determined by modifying the previously published result by including a new occurrence. The covering cover edge pebbling number of a graph G is defined and the same is computed for friendship graph F_n.

2 Methodology

The possible minimum edge covering set of the graph is to be considered and the set with the minimum pebble requirement covering all vertices is selected. Then, by finding the key edges the covering cover edge pebbling number is computed.

3 Results and Discussion

Covering Cover Edge Pebbling Number

Definition 2.1 A friendship graph F_n is a simple graph with 2n + 1 vertices and 3n edges in which n copies of 3-cycles were joined in a common vertex.

Throughout this paper the edges of F_n are labelled as follows,

- The n edges not incident to the common vertex is labelled as $e_1, e_2, \ldots, e_n$ arbitrarily
- The two edges, adjacent to e_i are labelled as e_{i1} and e_{i2} in any arbitrary way, where i = 1,2, …, n.

Definition 2.2 The covering cover edge pebbling number of a graph G denoted by $C_cP_E(G)$ is defined as the minimum number of pebbles required to cover the edges that forms a covering for G in all possible configurations of pebbles.

The below mentioned theorem is a modified theorem published in [1]

Theorem 2.3 The edge pebbling number of a Friendship graph $P_E(F_n)$ = 3n +2, n>1.

Proof:

Case 1: Single pebble distribution.

The total number of edges in F_n is 3n. Distributing one pebble, on 3n – 1 edges each, leaving the target edge, edge pebble move is not possible. If one more pebble is placed on any of the 3n – 1 edge, the target edge will be reached. So altogether 3n pebbles are needed to reach any target edge.

We can conclude that in this case to reach any target edge, a maximum of 3n pebbles are needed.

Case 2: Placing pebbles on the edges, incident with the common vertex.

Subcase(i): Target edge lies within the edges incident with the common vertex.

The number of edges incident with the common vertex in F_n is 2n. Since all the edges are incident with the common vertex the distance between any of those edges is zero. Hence considering the worst case of distribution of pebbles. i.e., placing one pebble, on all the 2n-1 edges, edge pebble move is not possible. By adding one more pebble on any of the 2n-1 edges or on the target edge itself we are done. Hence if the target edge lies within those 2n edges, then it can be reached with 2n pebbles.

Subcase(ii): Target edge does not lie within the edges incident with the common vertex.

Number of edges not incident with the common vertex in F_n is n. To reach any of those n edges 2n+1 pebbles are needed by the same argument used in the previous subcase(i). Hence, in this case to reach any target edge in all possible distribution of pebbles a maximum of 2n+1 pebbles are needed.

Case 3: Placing pebbles on the edges, not incident with the common vertex.

Subcase(i): Target edge lies within the edges which are incident with the common vertex

Number of edges not incident with the common vertex in F_n is n. If we distribute one pebble each on those n edges then to reach any edge incident with the common vertex is not possible. If we distribute two pebbles each, on those n edges then any edge incident with the common vertex is possible to reach. In this case 2n pebbles are needed to reach the target edge.

Subcase(ii): Target edge does not lie within the edges which are incident with the common vertex

The target edge can be any of the n edges from the n copies of 3-cycles. Suppose if the 3n pebbles found in **case 1** are distributed among all the n edges there are two possibilities, in which, the target edge could not be reached. If we place 7 pebbles on one edge, 2 pebbles on another edge and 3 pebbles each, on the remaining n-3 edges the target edge could not be reached. Also, if we place 6 pebbles on one edge and 3 pebbles each on the remaining n-2 edges, the target edge could not be reached. In both the cases only, if we add two more pebbles, we can reach the target edge. Hence in this case we need 3n+2 pebbles to reach any arbitrary edge.

By the above three cases we can conclude that, $P_E(F_n)$ = 3n+2. This completes the proof.

Theorem 2.4 The friendship graph F_n is not edge demonic.

Proof:

The number of edges of a friendship graph is 3n. By the above theorem edge pebbling number of a friendship graph is 3n+2. Hence, by the definition of edge demonic graph [2] we can conclude that friendship graph is not edge demonic.

Theorem 2. 5 The covering cover edge pebbling number of a friendship graph $C_cP_E(F_2) = 9$.

Proof: The minimum edge covering of F_2 are,

$E_1 = \{e_1, e_{21}, e_{22}\}$

$E_2 = \{e_2, e_{11}, e_{12}\}$

$E_3 = \{e_1, e_2\} \cup \{e_{i1} \text{ or } e_{i2}\}$, i = 1,2

Case 1: Consider E_1

Let us place 9 pebbles on e_1. After a number of edge pebble move from e_1 we can place one pebble on each edge e_{21} and e_{22} by using 4+4 pebbles. The remaining one pebble will cover e_1. So, in this configuration of pebbles all the vertices of F_2 will be incident to at least one edges of E_1.

Consider placing 8 pebbles on e_1. After a number of edge pebble move from e_1 we can place one pebble on each edge e_{21} and e_{22} by using 4+4 pebbles or we can place one pebble on each edge e_{11}, e_{12} and e_{21} (or e_{22}) by using 2+2+4 pebbles or we can place one pebble on e_2. In either case we could not place a pebble on each edge of E_1.

Consider placing 8 pebbles on more than one edge, clearly all the edges in E_1 can be pebbled.

Hence, placing all pebbles on e_1 is the worst initial configuration.

Case 2: Consider E_2

It can be dealt as in case 1 and placing all pebbles on e_2 is the worst initial configuration.

Case 3: Consider E_3

Let us place 9 pebbles on e_1 (or e_2). After a number of edge pebble move from e_1 (or e_2) we can place one pebble on e_2 (or e_1) by using 2^3 pebbles but to pebble e_{i1} or e_{i2} we need at least 2 more pebbles. In this case at least 11 pebbles are to be placed in e_1 or e_2. Since the number of pebbles increase in this case than case(i), we can exclude this minimum edge covering.

Altogether, we can conclude that $C_cP_E(F_2) = 9$

Result 2.6 Since the pattern of $F_3, F_4, \ldots, F_n$ remains the same as F_2 in all the graphs the worst initial configuration is placing all the pebbles on their key edges. With this result we can prove the following theorem.

Theorem 2.7 The covering cover edge pebbling number of a friendship graph $C_cP_E(F_n) = 8n - 7$, $n > 1$.

Proof:

In F_n the key edges are those edges which are not incident with the common vertex. Choose such an edge say e_1.

Consider the $2n - 2$ edges that are incident with the common vertex and not adjacent to the key edge e_1. The distance between any of the $2n - 2$ edges and the key edge e_1 is 2. Hence to pebble all those edges $2^2(2n - 2) = 8n - 8$ pebbles are to be placed on e_1. Now to pebble e_1 one more pebble is to be placed. Obviously, the pebbled edges will form an edge covering.

Hence altogether we can conclude that the covering cover edge pebbling number of a friendship graph F_n is $8n - 7$, $n > 1$.

Theorem 2.8 The covering cover edge pebbling number of a path $P_4 = 5$.

Proof :

Consider a path $P_4 = (v_1, e_1, v_2, e_2, v_3, e_3, v_4)$ with usual notations.

The minimum edge covering of P_4 is,

$E = \{e_1, e_3\}$

Let us place 5 pebbles on e_1 or e_3. To reach e_3 from e_1 four pebbles are to be placed on e_1 (viceversa). So, in this configuration of pebbles all the vertices of P_4 will be incident to at least one edges of E_1.

Consider placing 4 pebbles on e_1 or e_3. After a number of edge pebble move from e_1 or e_3 we can place one pebble only on the edge e_3 or e_1. In either case we could not place a pebble on all edges of E.

Consider placing 4 pebbles on more than one edge, clearly all the edges in E can be pebbled.

Hence, placing all pebbles on e_1 or e_2 is the worst initial configuration.

Hence, we can conclude that $C_cP_E(P_4) = 5$.

Result 2.9 Since the pattern of $P_6, \ldots, P_{2n}$ remains the same as P_4, in all the graphs the worst initial configuration is placing all the pebbles on their key edges. With this result we can prove the following theorem.

Theorem 2. 10 The covering cover edge pebbling number of an even path is $\frac{2^{2n}-1}{3}$. i.e., $C_cP_E(P_{2n}) = \frac{2^{2n}-1}{3}$.

Proof: Consider the path $P_{2n} = (v_1, e_1, v_2, e_2, \ldots, v_{2n-1}, e_{2n-1}, v_{2n})$ with usual notations.

The minimum edge covering of P_{2n} is,

$E = \{e_1, e_3, e_5, \ldots, e_{2n-1}\}$

The key edges for P_{2n} are e_1 and e_{2n-1}. Let us place the pebbles on either of the key edges. Arbitrarily choose e_1 as the edge to place pebbles.

Now,

$d(e_1, e_3) = 1$, $d(e_1, e_5) = 3$, $d(e_1, e_7) = 5$, …, $d(e_1, e_{2n-1}) = 2n-3$. So, the number of pebbles to be placed for the above considerations respectively is $2^2, 2^4, 2^6, \ldots, 2^{2n-2}$. One more pebble is to be placed on e_1 to cover it.

Hence altogether the number of pebbles needed to cover the edges in $E = 1 + 2^2+2^4+2^6+\ldots+2^{2n-2} = \frac{2^{2n}-1}{3}$.

i.e., $C_cP_E(P_{2n}) = \frac{2^{2n}-1}{3}$.

Theorem 2.11 The covering cover edge pebbling number of a path $P_5 = 11$.

Proof :

Consider a path $P_5 = (v_1,e_1,v_2, e_2, v_3, e_3, v_4, e_4, v_5)$ with usual notations.

The minimum edge covering of P_5 are,

$E_1 = \{e_1, e_2, e_4\}$

$E_2 = \{e_1, e_3, e_4\}$

Case(i) Consider E_1

Subcase(i)

Let us place 11 pebbles on e_1. To reach e_2 from e_1 two pebbles are to be placed on e_1 and to reach e_4 from e_1 eight pebbles are to be placed on e_1. The remaining one pebble will cover e_1. So, in this configuration of pebbles all the vertices of P_5 will be incident to at least one edges of E_1.

Consider placing 10 pebbles on e_1. After a number of edge pebble move from e_1 we could not place a pebble on all edges of E_1.

Consider placing 10 pebbles on more than one edge, clearly all the edges in E_1 can be pebbled.

Subcase(ii)

Consider placing pebbles on e_4. To reach e_2 from e_4 four pebbles are to be placed on e_4 and to reach e_1 from e_4 eight pebbles are to be placed on e_4. One more pebble is to be placed on e_4 to cover it. Since the number of pebbles increase in this subcase than subcase(i), we can exclude this configuration of pebbles.

Case(ii) Consider E_2

Subcase(i)

Let us place 11 pebbles on e_4. To reach e_3 from e_4 two pebbles are to be placed on e_4 and to reach e_1 from e_4 eight pebbles are to be placed on e_4. The remaining one pebble will cover e_4. So, in this configuration of pebbles all the vertices of P_5 will be incident to at least one edges of E_1.

Consider placing 10 pebbles on e_4. After a number of edge pebble move from e_4 we could not place a pebble on all edges of E_1.

Consider placing 10 pebbles on more than one edge, clearly all the edges in E_1 can be pebbled.

Subcase(ii)

Consider placing pebbles on e_1. To reach e_3 from e_1 four pebbles are to be placed on e_1 and to reach e_4 from e_1 eight pebbles are to be placed on e_1. One more pebble is to be placed on e_1 to cover it. Since the number of pebbles increase in this subcase(ii) than subcase(i), we can exclude this configuration of pebbles.

Hence, by case(i) placing all pebbles on e_1 is the worst initial configuration and by case (ii) placing all pebbles on e_4 is the worst initial configuration. And it is also clear that in both the cases the number of pebbles required is same —— (*)

Hence, we can conclude that $C_cP_E(P_5) = 11$.

Result 2.12 Since the pattern of $P_7, \ldots, P_{2n+1}$ remains the same as P_5, in all the graphs the worst initial configuration is placing all the pebbles on their key edges according to the minimum edge covering. With this result we can prove the following theorem.

Theorem 2. 13 The covering cover edge pebbling number of an even path is $\frac{2^{2n+1}+1}{3}$.

i.e., $C_cP_E(P_{2n+1}) = \frac{2^{2n+1}+1}{3}$.

Proof: Consider the path $P_{2n+1} = (v_1,e_1,v_2, e_2, \ldots, v_{2n-1}, e_{2n-1}, v_{2n}, e_{2n}, v_{2n+1})$ with usual notations.

The minimum edge covering of P_{2n+1} are,

$E_1 = \{e_1, e_2, e_4, e_6, \ldots, e_{2n-2}, e_{2n}\}$

$E_2 = \{e_1, e_3, e_5, e_7, \ldots, e_{2n-1}, e_{2n}\}$

The key edges for P_{2n+1} are e_1 and e_{2n}.

By previous theorem (*) , it is clear that we can consider either of the cases.

Let us consider E_1.

The key edges for P_{2n+1} are e_1 and e_{2n}. Let us place the pebbles on either of the key edges. Arbitrarily choose e_1 as the edge to place pebbles.

Now,

$d(e_1, e_2) = 0$, $d(e_1, e_4) = 2$, $d(e_1, e_6) = 4$, …, $d(e_1, e_{2n}) = 2n - 2$. So, the number of pebbles to be placed for the above considerations respectively is 2, 2^3, 2^5, 2^7, …, 2^{2n-1}. One more pebble is to be placed on e_1 to cover it.

Hence altogether the number of pebbles needed to cover the edges in $E_1 = 1 + 2^3+2^5+2^7+\ldots+2^{2n-1} = \frac{2^{2n+1}+1}{3}$.

i.e., $C_cP_E(P_{2n+1}) = \frac{2^{2n+1}+1}{3}$.

4 Conclusion

This paper shows that the covering cover edge pebbling number can be determined by finding the minimum edge covering number and key edges. In addition, it has many applications to optimization problems.

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