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RESEARCH ARTICLE

A Common Fixed Point Theorem in Cone Rectangular Metric Space Under Expansive Type Condition

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Abstract

Objectives: In this paper, we have to establish a generalized common fixed point theorem in cone rectangular metric spaces. **Methods:** In this paper, we use the Banach contraction principle technique to establish the generalized fixed point theorem. **Findings:** The paper presents a unique common fixed point theorem for two weakly compatible self-maps satisfying expansive type mapping in cone rectangular metric space without assuming the normality condition of a cone. Our result extends and supplements some well-known results in cone rectangular metric spaces. **Novelty:** The main novelty of this article is to prove a common fixed theorem under expansive type conditions. As a direct consequence, we give an example to illustrate our obtained result.

Keywords: Normal Cone; Cone Rectangular Metric Space; Coincidence Point; Common Fixed Point; Weakly Compatible; Expansive Type Mappings

1 Introduction

In recent times fixed point theorems have gained importance because of their numerous applications. Fixed point theorems have many applications in various fields differential equations, topology, functional analysis, integral equations, operator theory, game theory, computer science, logic programming, artificial intelligence, applied Engineering, Telecommunications, Physics, Economics and Management.

In [1], introduced the concept of metric spaces. It is well-known that the classical Banach Contraction Principle [2] is the fixed point theorem. Let X be a non-empty set. An element $x \in T$ is said to be a fixed point of a self-map $T : X \to X$ if $T(x) = x$. A result giving a set of conditions on T and X under which T has a fixed point is known as a fixed point theorem. Several researchers proved the fixed point theorems in Metric Spaces, Banach Spaces, Topological spaces, Fuzzy metric spaces and cone metric spaces based on the Banach Contraction Principle.

In [3], introduced a class of generalized (rectangular) metric spaces by replacing the triangular inequality of metric spaces with a similar one which involves four or more points instead of three points. The author also improved Banach Contraction Principle in such spaces. Recently many authors [4,5] proved the existence and uniqueness of a fixed point for different types of mappings.

In [6], introduced the concept of cone metric spaces by replacing the real number system in the definition of metric space with an ordered Banach space. Recently many authors [7–9] have obtained coincidence and common fixed point results for self-maps satisfying contractive conditions in cone metric-type spaces. In [10], introduced the concept of quasi-cone metric spaces and established some Fixed Point Results in Quasi-cone metric spaces. Recently many authors [11–13] have obtained coincidence and common fixed point results for self-maps satisfying contractive conditions in cone metric spaces.

In [14], introduced the concept of cone rectangular metric space by replacing the triangular inequality in the definition of cone metric space with a rectangular inequality and proved the Banach contraction principle on these spaces. Recently many authors [15–17] have obtained some fixed point results in cone rectangular metric spaces. In this paper, we prove a common fixed point theorem for two weakly compatible self-maps under an expansive type condition in cone rectangular metric spaces with a suitable example to illustrate our obtained result. Our main result generalizes many known results [15] in cone rectangular metric spaces.

2 Preliminaries

Definition 2.1. [14] Let X be a non-empty set. The mapping $d_E : X \times X$ said to be cone rectangle metric space if it satisfies:

(1) $\theta < d(x,y)$, for all $x, y \in X$ with $x \neq y$ and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(3) $d_{\mathrm{E}}(x,y) \leq d_{\mathrm{E}}(x,w) + d_{\mathrm{E}}(w,z) + d_{\mathrm{E}}(z,y)$,

for all x, y $\in$ X and for all distinct points $u, v \in X - \{x,y\}$.

Definition 2.2. [6] Let P be a subset of a real Banach space $E$and θ is the zero vector of E.P is said to be a cone in E if it satisfies the following properties:

(i) P is non-empty, closed and $P \neq \{\theta\}$;

(ii) **x**, **y** $\in P$ implies a**x** + b**y** $\in P$, where a and b are positive real numbers;

(iii) The intersection of P and –P is $\{\theta\}$

Definition 2.3. [6] A cone P is said to be a solid cone if an interior of P is a non-empty subset of E

Definition 2.4. [6] A partial order relation $\leq$ with respect to a solid cone $P \subseteq E$ is defined as $\mathbf{x} \leq \mathbf{y}$ if **y** – **x** $\in P$, for x, y $\in E$.

Definition 2.5. [6] A cone P is called a normal cone if there is a number k >1 such that for all **x**, **y** $\in$ X, $\theta \leq x \leq y$ implies that $||\mathbf{x}|| \leq k||\mathbf{y}||$.

Definition 2.6. [14] Let(X, d_E) be a cone rectangular metric space and P be a solid cone in E. Then the sequence $\{x_n\}$ is said to converge to x if $d_E(x_n,\, x) \to 0,\ as\, n \to \infty$.

Definition 2.7 [14] Let (X, d_E) be a cone rectangular metric space and P be a solid cone in E. Then the sequence $\{x_n\}$ is said to be Cauchy if for all p > 0 we have $d_E\,(x_n,\, x_{n+p}) \to 0,\ as\ n \to \infty$.

Throughout this paper, P is not necessarily a normal cone in E, the relation $x \ll y$ stands for $y\,x$ belongs to an interior of P and R denotes the set of all Real numbers.

3 Main Results

Theorem 3.1: Let (X, d_E) be a cone rectangular metric space. If the mappings S and T: $X \to X$ satisfy the following:

$$d_E\left(Sx,Sy\right) \geq\ \lambda_1 d_E\left(Tx,Ty\right) + \lambda_2 d_E\left(Sx,Tx\right) + \lambda_3 d_E\left(Sy,Tx\right) + \lambda_4 d_E\left(Sy,Ty\right), \tag{3.1}$$

for all x, y $\in X$, where $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \in R$ such that $\lambda_1 >\ 1$ and $0\ < \lambda_2,\ \lambda_3,\ \lambda_4 < 1$.

If T(X) $\subseteq S(X)$ and either of T(X) or $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Further, if T and S are weakly compatible self-maps then they have a unique common fixed point in X.

Proof . We start the proof with an arbitrary point $x_0 \in X$. Since T(X) $\subseteq S\,(X)$and let $x_1 \in$ X be such that $Tx_0 = Sx_1$. Continuing this process, we can construct a sequence $\{y_n\}$ in X such that $y_n = Sx_n = Tx_{n\,-\,1}$, for all n$\geq$ 1.

If $y_{m\,-\,1} = y_m$, for some m$\geq$ 1, then $y_{m\,-\,1} = Sx_{m\,-\,1} = Tx_{m\,-\,1}$. That is, S and T have a coincidence point $x_{m-\,1}$ in X. Assume $y_{n\,-\,1} \neq y_n$, for all n $\geq$ 1. Then from (3.1) it follows that,

$$d_E\left(y_{n-1},\, y_n\right) = d_E\left(Sx_{n-1},\, Sx_n\right)$$

$$\geq \lambda_1 d_E\left(Tx_{n-1},Tx_n\right) + \lambda_2 d_E\left(Sx_{n-1},Tx_{n-1}\right) + \lambda_3 d_E\left(Sx_n,Ix_{n-1}\right) + \lambda_4 d_E\left(Sx_n,Tx_n\right),$$

$$= \lambda_1 d_E\left(y_n,y_{n+1}\right) + \lambda_2 d_E\left(y_{n-1},y_n\right) + \lambda_3 d_E\left(y_n,y_n\right) + \lambda_4 d_E\left(y_n,y_{n+1}\right),$$

$$\geq (\lambda_1 + \lambda_4)\, d_E\,(y_n, y_{n+1}) + \lambda_2 d_E\,(y_{n-1}, y_n)\,,$$

which implies that,

$d_E\,(y_n,\, y_{n+1}) \leq \left(\frac{1-\lambda_2}{\lambda_1+\lambda_4}\right] d_E\,(y_{n-1},\, y_n)\,,$

Hence,

$d_{\mathrm{E}}\,(y_n, y_{n+1}) \leq \lambda d_{\mathrm{E}}\,(y_{n-1}, y_n)\,, \forall n \geq 1, where \lambda = \frac{1-\lambda_2}{\lambda_1+\lambda_4} < 1\,(\,as\ \lambda_1 + \lambda_4 + \lambda_2 > 1)\,.$

By induction for all n $\geq$ 0,

$$d_E\,(y_n,\, y_{n+1}) \leq \lambda^n d_E\,(y_0, y_1)\,, \tag{3.2}$$

where 0 <λ< 1.

Using (3.1), (3.2), rectangular inequality and the facts that,

$\lambda_1 > 1, \lambda_2 < 1, 0 < \lambda_3 < 1$ and $0 < \lambda_4 < 1$, i.e., $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 > 1$ and $0 < \lambda < 1$, we get,

$$d_E\,(y_{n-1},\, y_{n+1}) = d_E\,(Sx_{n-1}, Sx_{n+1})$$

$$\geq \lambda_1 d_E\,(Tx_{n-1}, Tx_{n+1}) + \lambda_2 d_E\,(Sx_{n-1}, Tx_{n-1}) + \lambda_3 d_E\,(Sx_{n+1}, Tx_{n-1}) + \lambda_4 d_E\,(Sx_{n+1}, Tx_{n+1})$$

$\geq \lambda_1 d_E\,(y_n, y_{n+2}) + \lambda_2 d_E\,(y_{n-1}, y_n) + \lambda_3 d_E\,(y_{n+1}, y_n) + \lambda_4 d_E\,(y_{n+1}, y_{n+2})$

Therefore,

$$\begin{aligned}\lambda_1 d_{\mathrm{E}}\,(y_n, y_{n+2}) &\leq d_{\mathrm{E}}\,(y_{n-1}, y_{n+1}) - \lambda_2 d_{\mathrm{E}}\,(y_{n-1}, y_n) - \lambda_3 d_{\mathrm{E}}\,(y_{n+1}, y_n) - \lambda_4 d_{\mathrm{E}}\,(y_{n+1}, y_{n+2})\\ &\leq [d_{\mathrm{E}}\,(y_{n-1}, y_n) + d_{\mathrm{E}}\,(y_n, y_{n+2}) + d_{\mathrm{E}}\,(y_{n+2}, y_{n+1})]\\ &\quad - \lambda_2 d_{\mathrm{E}}\,(y_{n-1}, y_n) - \lambda_3 d_{\mathrm{E}}\,(y_{n+1}, y_n) - \lambda_4 d_{\mathrm{E}}\,(y_{n+1}, y_{n+2})\\ &\leq d\,(y_n, y_{n+2}) + (1-\lambda_2)\, d_{\mathrm{E}}\,(y_{n-1}, y_n) - \lambda_3 d_{\mathrm{E}}\,(y_{n+1}, y_n) + (1-\lambda_4)\, d_{\mathrm{E}}\,(y_{n+1}, y_{n+2})\end{aligned}$$

which implies that,

$$\begin{aligned}d_{\mathrm{E}}\,(y_n, y_{n+2}) &\leq \left(\frac{1-\lambda_2}{\lambda_1-1}\right) d_{\mathrm{E}}\,(y_{n-1}, y_n) - \left(\frac{\lambda_3}{\lambda_1-1}\right) d_{\mathrm{E}}\,(y_n, y_{n+1}) + \left(\frac{1-\lambda_4}{\lambda_1-1}\right) d_{\mathrm{E}}\,(y_{n+1}, y_{n+2})\\ &\leq \left(\frac{1-\lambda_2}{\lambda_1-1}\right) \lambda^{n-1} d_{\mathrm{E}}\,(y_0, y_1) - \left(\frac{\lambda_3}{\lambda_1-1}\right) \lambda^n d_{\mathrm{E}}\,(y_0, y_1) + \left(\frac{1-\lambda_4}{\lambda_1-1}\right) \lambda^{n+1} d_{\mathrm{E}}\,(y_0, y_1)\\ &= \left(\frac{1-\lambda_2}{\lambda_1-1} - \lambda\frac{\lambda_3}{\lambda_1-1} + \lambda^2\frac{1-\lambda_4}{\lambda_1-1}\right) \lambda^{n-1} d_{\mathrm{E}}\,(y_0, y_1)\\ &= \left(\frac{1-\lambda_2}{\lambda_1-1} - \frac{\lambda_3}{\lambda_1-1} + \frac{1-\lambda_4}{\lambda_1-1}\right) \lambda^{n-1} d_{\mathrm{E}}\,(y_0, y_1)\\ &\leq \left(\frac{2-(\lambda_2+\lambda_4+\lambda_3)}{\lambda_1-1}\right) \lambda\lambda^{n-1} d_{\mathrm{E}}\,(y_0, y_1)\\ &\leq \left(\frac{1+\lambda_1}{\lambda_1-1}\right) \lambda^n d_{\mathrm{E}}\,(y_0, y_1)\end{aligned}$$

Hence

$$d_{\mathrm{E}}\,(y_n, y_{n+2}) \leq \alpha\lambda^n d_{\mathrm{E}}\,(y_0, y_1) \tag{3.3}$$

For the sequence {y_n} we consider $d_E(y_n, y_{n+p})$ in two cases.

If p is odd say 2m +1, for m$\geq$ 1, then by using rectangular inequality and (3.2) we get,

$$\begin{aligned}
d_{\mathrm{E}}(y_n, y_{n+2m+1}) &\le d_{\mathrm{E}}(y_{n+2m+1}, y_{n+2m}) + d_{\mathrm{E}}(y_{n+2m}, y_{n+2m-1}) + d_{\mathrm{E}}(y_{n+2m-1}, y_n) \\
&\le d_{\mathrm{E}}(y_{n+2m}, y_{n+2m+1}) + d_{\mathrm{E}}(y_{n+2m-1}, y_{n+2m}) + d_{\mathrm{E}}(y_{n+2m-1}, y_{n+2m-2}) \\
&+ d_{\mathrm{E}}(y_{n+2m-2}, y_{n+2m-3}) + \cdots + d_{\mathrm{E}}(y_{n+2}, y_{n+1}) + d_{\mathrm{E}}(y_{n+1}, y_n) \\
&= d_{\mathrm{E}}(y_n, y_{n+1}) + d_{\mathrm{E}}(y_{n+1}, y_{n+2}) + \cdots + d_{\mathrm{E}}(y_{n+2m-1}, y_{n+2m}) + d_{\mathrm{E}}(y_{n+2m}, y_{n+2m+1}) \\
&\le \lambda^n d_{\mathrm{E}}(y_0, y_1) + \lambda^{n+1} d_{\mathrm{E}}(y_0, y_1) + \cdots + \lambda^{n+2m-1} d_{\mathrm{E}}(y_0, y_1) + \lambda^{n+2m} d_{\mathrm{E}}(y_0, y_1) \\
&\le \left[1 + \lambda + \lambda^2 + \lambda^3 + \cdots\right] \lambda^n d_{\mathrm{E}}(y_0, y_1) \\
&\le \frac{\lambda^n}{1-\lambda} d_{\mathrm{E}}(y_0, y_1) \\
&\le \left[\alpha + \frac{1}{1-\lambda}\right] \lambda^n d_{\mathrm{E}}(y_0, y_1)
\end{aligned}$$

Hence,

$$d_{\mathrm{E}}(y_n, y_{n+2m+1}) \le \left[\alpha + \frac{1}{1-\lambda}\right] \lambda^n d_{\mathrm{E}}(y_0, y_1) \tag{3.4}$$

for all $n \ge 1, m \ge 1 and \alpha = \frac{1+\lambda_1}{\lambda_1 - 1} \ge 0$.

If p is even say 2m, for m$\ge$ 1, then by using rectangular inequality, (3.2), (3.3) and the fact that 0 <λ <1 we get,

$$\begin{aligned}
d_{\mathrm{E}}(y_n, y_{n+2m}) &\le d_{\mathrm{E}}(y_{n+2m}, y_{n+2m-1}) + d_{\mathrm{E}}(y_{n+2m-1}, y_{n+2m-2}) + d_{\mathrm{E}}(y_{n+2m-2}, y_n) \\
&\le d_{\mathrm{E}}(y_{n+2m-1}, y_{n+2m}) + d_{\mathrm{E}}(y_{n+2m-2}, y_{n+2m-1}) + \cdots + d_{\mathrm{E}}(y_{n+4}, y_{n+3}) \\
&+ d_{\mathrm{E}}(y_{n+3}, y_{n+2}) + d_{\mathrm{E}}(y_{n+2}, y_n) \\
&= d_{\mathrm{E}}(y_n, y_{n+2}) + d_{\mathrm{E}}(y_{n+2}, y_{n+3}) + d_{\mathrm{E}}(y_{n+3}, y_{n+4}) + \cdots \\
&+ d_{\mathrm{E}}(y_{n+2m-2}, y_{n+2m-1}) + d_{\mathrm{E}}(y_{n+2m-1} y_{n+2m}) \\
&\le \mu \lambda^n d_{\mathrm{E}}(y_0, y_1) + \left[\lambda^{n+2} d_{\mathrm{E}}(y_0, y_1) + \lambda^{n+3} d_{\mathrm{E}}(y_0, y_1)\right. \\
&\left. + \cdots + \lambda^{n+2m-2} d_{\mathrm{E}}(y_0, y_1) + \lambda^{n+2m-1} d_{\mathrm{E}}(y_0, y_1)\right] \\
&= \alpha \lambda^n d_{\mathrm{E}}(y_0, y_1) + \left[\lambda^2 + \lambda^3 + \ldots + \lambda^{2m-1}\right] \lambda^n d_{\mathrm{E}}(y_0, y_1) \\
&\le \alpha \lambda^n d_{\mathrm{E}}(y_0, y_1) + \left[1 + \lambda + \lambda^2 + \lambda^3 + \ldots\right] \lambda^n d_{\mathrm{E}}(y_0, y_1) \\
&\le \alpha \lambda^n d_{\mathrm{E}}(y_0, y_1) + \frac{\lambda^n}{1-\lambda} d_{\mathrm{E}}(y_0, y_1)
\end{aligned}$$

Hence

$$d_{\mathrm{E}}(y_n, y_{n+2m}) \le \left[\alpha + \frac{1}{1-\lambda}\right] \lambda^n d_{\mathrm{E}}(y_0, y_1) \tag{3.5}$$

for all $n \ge 1, m \ge 1$ and $\alpha = \frac{1+\lambda_1}{\lambda_1 - 1} \ge 0$

From (3.4) and (3.5) we have,

$d_{\mathrm{E}}(y_n, y_{n+p}) \le \left[\alpha + \frac{1}{1-\lambda}\right] \lambda^n d_{\mathrm{E}}(y_0, y_1)$, for all $n \ge 1, m \ge 1$ and $\alpha = \frac{1+\lambda_1}{\lambda_1 - 1} \ge 0$

Assume that $\theta \ll k$. Since

$$\left[\alpha + \frac{1}{1-\lambda}\right] \lambda^n d_E(y_0, y_1) \to \theta,\ n \to \infty$$

Therefore for any k belongs to an interior of P, we can find a natural number N_1such that for each n > N_1, we have,

$\left[\alpha + \frac{1}{1-\lambda}\right] \lambda^n d_E(y_0, y_1) \ll k$, for all n >$N_1$ and p $\ge$ 1 and hence, $d_E(y_n, y_{n+p}) \ll k$.

Therefore, $\{y_n\}$ is a Cauchy sequence in X. Since $T(X)$ is a complete subspace of X, then there exists a point $z \in T(X) \subseteq S(X)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n-1} = z.$$

Also, we can find x $\in X$ such that z = Sx.

Let $\theta \ll k$ be given, we can choose natural numbers N_2 and N_3 such that $d_E(z, y_{n-1}) \ll \frac{\lambda_1 k}{2(\lambda_1+1)}$, for all n >$N_2$ and $d_E(y_{n-1}, y_n) \ll \frac{k}{2}$, for all n > N_3. Let N= max {N_2, N_3}.

We have by (3.1),

$$d_E(y_{n-1}, z) = d_E(Sx_{n-1}, Sx)$$

$$\geq \lambda_1 d_E(Tx_{n-1}, Tx) + \lambda_2 d_E(Sx_{n-1}, Tx_{n-1}) + \lambda_3 d_E(Sx, Tx_{n-1}) + \lambda_4 d_E(Sx, Tx)$$

$$\geq \lambda_1 d_E(y_n, Tx) + \lambda_2 d_E(y_{n-1}, y_n) + \lambda_3 d_E(z, y_n) + \lambda_4 d_E(z, Tx)$$

$$\geq \lambda_1 d_E(y_n, Tx)$$

Hence, $d_E(y_n, Tx) \leq \frac{1}{\lambda_1} d_E(y_{n-1}, z)$

Using rectangular inequality we have,

$$\begin{aligned} d_{\mathrm{E}}(z, Tx) &\leq d_{\mathrm{E}}(z, y_{n-1}) + d_{\mathrm{E}}(y_{n-1}, y_n) + d_{\mathrm{E}}(y_n, Tx) \\ &\leq d_{\mathrm{E}}(z, y_{n-1}) + d_{\mathrm{E}}(y_{n-1}, y_n) + \frac{1}{\lambda_1} d_{\mathrm{E}}(y_{n-1}, z) \\ &= \left[1 + \frac{1}{\lambda_1}\right] d_{\mathrm{E}}(z, y_{n-1}) + d_{\mathrm{E}}(y_{n-1}, y_n) \end{aligned}$$

Hence,

$$d_E(z, Tx) \ll \frac{k}{2} + \frac{k}{2} = k, \ i.e., \ d_E(z, Tx) = \theta$$

Therefore, $Sx = Tx = z$.

That is, z is a point of coincidence of S and T. If z* is another point of coincidence of S and T, then Sy = Ty = z*, for some y $\in X$ then

$$d_E(z, z*) = d_E(Sx, Sy)$$

$$\geq \lambda_1 d_E(Tx, Ty) + \lambda_2 d_E(Sx, Tx) + \lambda_3 d_E(Sy, Tx) + \lambda_4 d_E(Sy, Ty)$$

$$\geq \lambda_1 d_E(z, z*) + \lambda_2 d_E(z, z) + \lambda_3 d_E(z*, z) + \lambda_4 d_E(z*, z*)$$

$$= (\lambda_1 + \lambda_3) d_E(z, z*)$$

Hence,

$d_E(z, z*) \leq \frac{1}{\lambda_1+\lambda_4} d_E(z, z*)$

Since, $\lambda_1 + \lambda_4 > 1$, we have, $d_E(z, z*) = \theta, \ i.e., \ z = z*$.

That is, S and T have a unique point of coincidence in X. Suppose S and T are weakly compatible mappings, then we have, Sz = STx = TSx = Tz. Therefore, Sz = Tz = w (say). This shows that w is another point of coincidence between S and T. Therefore, by the uniqueness of the point of coincidence, we must have z = w. Hence, z is a unique common fixed point of S and T in X. Similarly,we can prove S and T have a unique common fixed point in X if $S(X)$ is a complete subspace of X.

The following example supports Theorem 3.1.

Example 3.2: Let= $\left(\frac{1}{n} : \ n \in (1, \ 2, \ 3, \ 4\}\right\}$, E= $\mathbb{R}^2$ and P $= \{((x,y) : x, y \geq 0)\} is a cone in$ E.

Define $d_E : X \times X$ as follows:

$$\begin{cases} d_{\mathrm{E}}(x,x) = (0,0), \text{ for all } x \in X, \\ d_{\mathrm{E}}\left(1, \frac{1}{2}\right) = (3,6), \\ d_{\mathrm{E}}\left(1, \frac{1}{3}\right) = d_{\mathrm{E}}\left(\frac{1}{2}, \frac{1}{3}\right) = (1,2), \\ d_{\mathrm{E}}\left(1, \frac{1}{4}\right) = d_{\mathrm{E}}\left(\frac{1}{2}, \frac{1}{4}\right) = d_{\mathrm{E}}\left(\frac{1}{3}, \frac{1}{4}\right) = (2,4), \\ d_{\mathrm{E}}(x,y) = d_{\mathrm{E}}(y,x), \text{ for all } x, y \in X. \end{cases}$$

Then it is clear that $(X, d_E)$is a complete cone rectangular metric space but not a cone metric space, since it does not satisfy triangular inequality:

$$d_E\left(1,\frac{1}{2}\right) = (3,\ 6) > d_E\left(1,\frac{1}{3}\right) + d_E\left(\frac{1}{3},\frac{1}{2}\right) = (1,\ 2) + (1,\ 2) = (2,\ 4),\ as\ (3,\ 6)\ (2,\ 4) = (1,\ 2) \in ,$$

Now we define the mappings S and T: $X \to X$ as follows:

$$\mathrm{S}(x) = \begin{cases} x & \text{if } x \in \left\{\frac{1}{3},\frac{1}{4}\right\} \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } x = \frac{1}{2} \end{cases}$$

and $T(x) = \begin{cases} \frac{1}{3} & \text{if } x \neq \frac{1}{4} \\ 1 & \text{if } x = \frac{1}{4} \end{cases}$

Now consider,

$d_{\mathrm{E}}(\mathrm{S}(1/2), \mathrm{S}(1)) \geq \lambda_1 d_{\mathrm{E}}(T(1/2), T(1)) + \lambda_2 d_{\mathrm{E}}(\mathrm{S}(1/2), T(1/2)) + \lambda_3 d(\ \mathrm{S}(1), T(1/2)) + \lambda_4 d_{\mathrm{E}}(\mathrm{S}(1), T(1))$

i.e., $d_{\mathrm{E}}(1, 1/2) \geq \lambda_1 d_{\mathrm{E}}(1/3, 1/3)) + \lambda_2 d_{\mathrm{E}}(1, 1/3) + \lambda_3 d_{\mathrm{E}}(1/2, 1/3) + \lambda_4 d_{\mathrm{E}}(1/2, 1/3)$

i.e., $(3,6) \geq \lambda_2(1,2) + \lambda_3(1,2) + \lambda_4(1,2)$

i.e. $(3,6) \geq (\lambda_2 + \lambda_3 + \lambda_4)(1,2)$

$d_{\mathrm{E}}(\mathrm{S}(1), \mathrm{S}(1/3)) \geq \lambda_1 d_{\mathrm{E}}(T(1), T(1/3)) + \lambda_2 d_{\mathrm{E}}(\mathrm{S}(1), T(1)) + \lambda_3 d_{\mathrm{E}}(\mathrm{S}(1/3), T(1)) + \lambda_4 d_{\mathrm{E}}(\mathrm{S}(1/3), T(1/3))$

i.e., $d_{\mathrm{E}}(1/2, 1/3) \geq \lambda_1 d_{\mathrm{E}}(1/3, 1/3)) + \lambda_2 d_{\mathrm{E}}(1/2, 1/3) + \lambda_3 d_{\mathrm{E}}(1/3, 1/3) + \lambda_4 d_{\mathrm{E}}(1/3, 1/3)$

i.e., $(1,2) \geq \lambda_2(1,2)$

$d_{\mathrm{E}}(\mathrm{S}(1/3), \mathrm{S}(1)) \geq \lambda_1 d_{\mathrm{E}}(T(1/3), T(1)) + \lambda_2 d_{\mathrm{E}}(\mathrm{S}(1/3), T(1/3)) + \lambda_3 d_{\mathrm{E}}(\mathrm{S}(1), T(1/3)) + \lambda_4 d_{\mathrm{E}}(\mathrm{S}(1), T(1))$

i.e., $d_{\mathrm{E}}(1/3, 1/2) \geq \lambda_1 d_{\mathrm{E}}(1/3, 1/3)) + \lambda_2 d_{\mathrm{E}}(1/3, 1/3) + \lambda_3 d_{\mathrm{E}}(1/2, 1/3) + \lambda_4 d_{\mathrm{E}}(1/2, 1/3)$

i.e., $(1,2) \geq \lambda_3(1,2) + \lambda_4(1,2)$

i.e.,

$d_{\mathrm{E}}(\mathrm{S}(1/2), \mathrm{S}(1/3)) \geq \lambda_1 d_{\mathrm{E}}(T(1/2), T(1/3)) + \lambda_2 d_{\mathrm{E}}(\mathrm{S}(1/2), T(1/2)) + \lambda_3 d_{\mathrm{E}}(\mathrm{S}(1/3), T(1/2))$
$+ \lambda_4 d_{\mathrm{E}}(S(1/3), T(1/3))$

i.e., $d_{\mathrm{E}}(1, 1/3) \geq \lambda_1 d_{\mathrm{E}}(1/3, 1/3)) + \lambda_2 d_{\mathrm{E}}(1, 1/3) + \lambda_3 d_{\mathrm{E}}(1/3, 1/3) + \lambda_4 d_{\mathrm{E}}(1/3, 1/3)$

i.e., $(1,2) \geq \lambda_2(1,2)$

$d_{\mathrm{E}}(\mathrm{S}(1/3), \mathrm{S}(1/2)) \geq \lambda_1 d_{\mathrm{E}}(T(1/3), T(1/2)) + \lambda_2 d_{\mathrm{E}}(\mathrm{S}(1/3), T(1/3)) + \lambda_3 d_{\mathrm{E}}(\mathrm{S}(1/2), T(1/3))$
$+ \lambda_4 d_{\mathrm{E}}(\mathrm{S}(1/2), T(1/2))$

i.e., $d_{\mathrm{E}}(1/3, 1) \geq \lambda_1 d_{\mathrm{E}}(1/3, 1/3)) + \lambda_2 d_{\mathrm{E}}(1/3, 1/3) + \lambda_3 d_{\mathrm{E}}(1, 1/3) + \lambda_4 d_{\mathrm{E}}(1, 1/3)$

i.e., $(1,2) \geq \lambda_3(1,2) + \lambda_4(1,2)$

i.e., $(1,2) \geq (\lambda_3 + \lambda_4)(1,2)$

$d_{\mathrm{E}}(\mathrm{S}(1), \mathrm{S}(1/4)) \geq \lambda_1 d_{\mathrm{E}}(T(1), T(1/4)) + \lambda_2 d_{\mathrm{E}}(\mathrm{S}(1), T(1)) + \lambda_3 d_{\mathrm{E}}(\mathrm{S}(1/4), T(1)) + \lambda_4 d_{\mathrm{E}}(\mathrm{S}(1/4), T(1/4))$

i.e., $d_{\mathrm{E}}(1/2, 1) \geq \lambda_1 d_{\mathrm{E}}(1/3, 1)) + \lambda_2 d_{\mathrm{E}}(1/2, 1/3) + \lambda_3 d_{\mathrm{E}}(1/4, 1/3) + \lambda_4 d_{\mathrm{E}}(1/4, 1)$

i.e., $(3,6) \geq \lambda_1(1,2) + \lambda_2(1,2) + \lambda_3(2,4) + \lambda_4(2,4)$

i.e., $(3,6) \geq (\lambda_1 + \lambda_2)(1,2) + (\lambda_3 + \lambda_4)(2,4)$

$d_{\mathrm{E}}(\mathrm{S}(1/4), \mathrm{S}(1)) \geq \lambda_1 d_{\mathrm{E}}(T(1/4), T(1)) + \lambda_2 d_{\mathrm{E}}(\mathrm{S}(1/4), T(1/4)) + \lambda_3 d_{\mathrm{E}}(\mathrm{S}(1), T(1/4)) + \lambda_4 d_{\mathrm{E}}(\mathrm{S}(1), t(1))$

i.e., $d_{\mathrm{E}}(1/4, 1/2) \geq \lambda_1 d_{\mathrm{E}}(1, 1/3)) + \lambda_2 d_{\mathrm{E}}(1/4, 1) + \lambda_3 d_{\mathrm{E}}(1/2, 1) + \lambda_4 d_{\mathrm{E}}(1/2, 1/3)$

i.e., $(2,4) \geq (\lambda_1 + \lambda_4)(1,2) + \lambda_2(2,4) + \lambda_3(3,6)$

$$d_E(S(1/2),S(1/4)) \geq \lambda_1 d_E(T(1/2),T(1/4)) + \lambda_2 d_E(S(1/2),T(1/2)) + \lambda_3 d_E(S(1/4),T(1/2)) + \lambda_4 d_E(S(1/4),T(1/4))$$

i.e., $d_E(1,1/4) \geq \lambda_1 d_E(1/3,1)) + \lambda_2 d_E(1,1/3) + \lambda_3 d_E(1/4,1/3) + \lambda_4 d_E(1/4,1)$

i.e., $(2,4) \geq \lambda_1(1,2) + \lambda_2(1,2) + \lambda_3(2,4) + \lambda_4(2,4)$

i.e., $(2,4) \geq (\lambda_1+\lambda_2)(1,2) + (\lambda_3+\lambda_4)(2,4)$

$$d_E(S(1/4),S(1/2)) \geq \lambda_1 d_E(T(1/4),T(1/2)) + \lambda_2 d_E(S(1/4),T(1/4)) + \lambda_3 d_E(S(1/2),T(1/4)) + \lambda_4 d_E(S(1/2),T(1/2))$$

i.e., $d_E(1/4,1) \geq \lambda_1 d_E(1,1/3)) + \lambda_2 d_E(1/4,1) + \lambda_3 d_E(1,1) + \lambda_4 d_E(1,1/3)$ *i.e.*, $(2,4) \geq \lambda_1(1,2) + \lambda_2(2,4) + \lambda_4(1,2)$ *i.e.*, $(2,4) \geq (\lambda_1+\lambda_4)(1,2) + \lambda_2(2,4)$

$$d_E(S(1/3),S(1/4)) \geq \lambda_1 d_E(T(1/3),T(1/4)) + \lambda_2 d_E(S(1/3),T(1/3)) + \lambda_3 d_E(S(1/4),T(1/3)) + \lambda_4 d_E(S(1/4),T(1/4))$$

i.e., $d_E(1/3,1/4) \geq \lambda_1 d_E(1/3,1)) + \lambda_2 d_E(1/3,1/3) + \lambda_3 d_E(1/4,1/3) + \lambda_4 d_E(1/4,1)$

i.e., $(2,4) \geq \lambda_1(1,2) + \lambda_3(2,4) + \lambda_4(2,4)$

i.e., $(2,4) \geq \lambda_1(1,2) + (\lambda_3+\lambda_4)(2,4)$

$$d_E(S(1/4),S(1/3)) \geq \lambda_1 d_E(T(1/4),\Im(1/3)) + \lambda_2 d_E(S(1/4),T(1/4)) + \lambda_3 d_E(S(1/3),T(1/4)) + \lambda_4 d_E(S(1/3),T(1/3))$$

i.e., $d_E(1/4,1/3) \geq \lambda_1 d_E(1,1/3)) + \lambda_2 d_E(1/4,1) + \lambda_3 d_E(1/3,1) + \lambda_4 d_E(1/3,1/3)$

i.e., $(2,4) \geq \lambda_1(1,2) + \lambda_2(2,4) + \lambda_3(1,2)$

i.e., $(2,4) \geq (\lambda_1+\lambda_3)(1,2) + \lambda_2(2,4)$

Then for every x, y $\in$ X, the inequality (3.1) of **Theorem 3.1** holds for $\lambda_1 = \frac{5}{4}$, $\lambda_2 = \lambda_3 = \frac{1}{8}$ and $\lambda_4 = \frac{1}{16}$. Since, $S\left(\frac{1}{3}\right) = T\left(\frac{1}{3}\right) = \frac{1}{3}$. It is clear that, $T(X) \subseteq S(X)$ and we have $ST\left(\frac{1}{3}\right) = S\left(T\left(\frac{1}{3}\right)\right) = S\left(\frac{1}{3}\right) = \frac{1}{3}$ and $TS\left(\frac{1}{3}\right) = T\left(S\left(\frac{1}{3}\right)\right) = T\left(\frac{1}{3}\right) = \frac{1}{3}$. Therefore $ST\left(\frac{1}{3}\right) = TS\left(\frac{1}{3}\right)$. That is, $S$and T are commutes at coincidence point $\frac{1}{3}$. Therefore S and T are weakly compatible mappings. Therefore there exists a unique common fixed point $\frac{1}{3}$ of S and T.

With the suitable value of λ_1, λ_2, λ_3, and λ_4 we obtain the following results of[15] on cone rectangular metric spaces.

Corollary 3.3.[15] If the two self-maps S and T: X$\to X$ defined on cone rectangular metric space (X, d_E) satisfy the following:

$$d_E(Sx,Sy) \geq \lambda_1 d_E(Tx,Ty) + \lambda_2 d_E(Sx,Tx) + \lambda_4 d_E(Sy,Ty),$$

for all x, y $\in$ X, *where* $\lambda_1, \lambda_2, \lambda_4 \in R$ such that $\lambda_1 > 1$ and $0 < \lambda_2, \lambda_4 < 1$.

If $T(X) \subseteq S(X)$ and either of $T(X)$ or $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Further, if T and S are weakly compatible self-maps then they have a unique common fixed point in X.

Corollary 3.4.[15] If the two self-maps S and T: X$\to X$ defined on cone rectangular metric space(X, d_E) satisfy the following:

$$d_E(Sx,Sy) \geq \lambda_1 d_E(Tx,Ty), \ for\ all\ x,y \in X,\ where\ \lambda_1 > 1.$$

If T(X) $\subseteq$ S(X) and either of T(X) or S(X) is a complete subspace of X, then T and S have a unique coincidence point in X. Further, if T and S are weakly compatible self-maps then they have a unique common fixed point in X.

4 Conclusion

Finally, we have obtained a most generalized unique common fixed point theorem for two weakly compatible self-maps under an expansive type condition in a cone rectangular metric space without assuming the normality condition of a cone. Our result is generalized results of[15] and supported by a suitable example.

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