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On Generalized Quasi-conformally 2-recurrent Riemannian Manifolds

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Abstract

Objectives: The object of the present paper is to study a type of Riemannian manifold called generalized quasi-conformally 2-recurrent Riemannian manifold, it is denoted by $G\left\{\widetilde{C}\left({}^2K_n\right)\right\}$. **Methods**: Differentiable manifold was defined on the basis of topology, differential calculus and real analysis. Riemannian manifold is a part of differentiable manifold in which we study by tensor notation and index free notation. Using such notations, several types of recurrent manifold and their standard results are used to characterize generalized quasi-conformally 2-recurrent Riemannian manifold and also several results of tensor calculus such as contraction, covariant differentiation, divergence of tensor, Bianchi's identities, Ricci identity, Fundamental theorem of Riemannian geometry and many more are used in this paper to finding the interesting results. **Finding**: We study $G\left\{\widetilde{C}\left({}^2K_n\right)\right\}$ manifold with Einstein manifold and prove that an Einstein $G\left\{\widetilde{C}\left({}^2K_n\right)\right\}$ is a manifold of constant curvature. We also established a necessary and sufficient condition for an Einstein $G\left\{\widetilde{C}\left({}^2K_n\right)\right\}$ manifold to be a generalized 2-recurrent Riemannian manifold $G\left\{\widetilde{C}\left({}^2K_n\right)\right\}$. In the last, we used vanishes Ricci tensor and shows that $G\left\{\widetilde{C}\left({}^2K_n\right)\right\}$ manifold reduces to$G\left\{\widetilde{C}\left({}^2K_n\right)\right\}$. **Novelty**: Recurrent manifold and generalized 2-recurrent Riemannian manifold has already been studied in published literature. In this paper we study generalized 2-recurrent Riemannian manifold with quasi-conformal curvature tensor and find interesting results.

Keywords: Recurrent manifold; Ricci tensor; generalized 2-recurrent manifold; quasi-conformal curvature tensor; Einstein manifold

1 Introduction

The notion of recurrent manifold was introduced by Walker [1]. A non-flat Riemannian manifold is said to be recurrent if the curvature tensor R satisfies the condition

$$(\nabla_U R)(X,Y)Z = A(U)R(X,Y)Z \tag{1}$$

where A is non-zero 1-form associated with ρ and defined as$A(U) = g(U,\rho)$, for all X, Y, Z, & $U \in \chi(M^n)$, the set of all vector field of the manifold M^n and ∇ is the Levi-Civita connection of the manifold. Such a manifold is denoted by $\{K_n\}$. If the 1-form vanishes identically, then the manifold is said to be a symmetric manifold [2].

Lichnerowics [3] introduced the 2-recurrent Riemannian manifold which is defined as: A non-flat Riemannian manifold is called 2-recurrent Riemannian manifold if the Riemannian curvature tensor R satisfies the condition

$$(\nabla_V \nabla_U R)(X,Y)Z = B(U,V)R(X,Y)Z \tag{2}$$

where B is a non-zero 2-form i.e. $(0,\ 2)$ tensor, such a manifold is denoted by $\{\,^2K_n\}$. In the continuous study of this recurrent manifold Ray [4] defined generalized 2-recurrent Riemannian manifold as: A non-flat Riemannian manifold is called generalized 2-recurrent Riemannian manifold if the Riemannian curvature tensor R satisfies the condition

$$(\nabla_V \nabla_U R)(X,Y)Z = A(V)(\nabla_U R)(X,Y)Z + B(U,V)R(X,Y)Z \tag{3}$$

where A is 1-form and B is a non-zero 2-form. Such a manifold is denoted by $G\{\,^2K_n\}$. The generalized 2-recurrent Riemannian manifold was studied by many authors [5–7]. In particular if $A=0$ and $B \neq 0$ then manifold reduce to 2-recurrent Riemannian manifold $\{\,^2K_n\}$ defined by Lichnerowics [3].

It is known that the quasi-conformal curvature tensor $\widetilde{C}$ of the Riemannian manifold is defined by Yano and Sawaki [8] as

$$\widetilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y] \tag{4}$$

where a and b are constants, S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature of the manifold M^n. A Riemannian manifold (M^n, g) $(n>3)$ is called quasi-conformally flat if the quasi-conformal curvature tensor $\widetilde{C}=0$. If $a=1$ and $b=-\frac{1}{n-2}$, then the quasi-conformal curvature tensor is reduced to the conformal curvature tensor C defined by Eisenhart [9] as

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}(g(Y,Z)X - g(X,Z)Y) \tag{5}$$

In tuned with the definition given by Ray [4], a Riemannian manifold (M^n, g) $(n>3)$ is called a generalized quasi-conformal 2-recurrent Riemannian manifold if $\widetilde{C}$ admits the relation

$$\left(\nabla_V \nabla_U \widetilde{C}\right)(X,Y)Z = A(V)\left(\nabla_U \widetilde{C}\right)(X,Y)Z + B(U,V)\widetilde{C}(X,Y)Z, \tag{6}$$

where A and B are stated earlier, such a manifold is denoted by $G\{\widetilde{C}(\,^2K_n)\}$. In particular if $A=0$ and $B \neq 0$ then manifold reduce to quasi-conformal 2-recurrent Riemannian manifold and it is denoted by $\widetilde{C}\{\,^2K_n\}$.

Definitions:

$$R(X,Y,Z,W) = g(R(X,Y)Z,W) \tag{7}$$

$$\widetilde{C}(X,Y,Z,W) = g(\widetilde{C}(X,Y)Z,W) \tag{8}$$

$$S(X,W) = g(QX,W) = \textstyle\sum_{i=1}^{n} R(X,e_i,e_i,W) \tag{9}$$

where $\{e_i\}$ being an orthonormal basis at a point in the tangent space, Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, r is the scalar curvature defined as [10]

$$r = \textstyle\sum_{i=1}^{n} S(e_i,e_i) \tag{10}$$

$$B(X,Y) \overset{def}{=} g(X,LY) \tag{11}$$

where L is the linear transformation from the tangent space at P:$TP(M^n) \to TP(M^n)$.

An Einstein manifold is defined by

$$S(X,Y) = \frac{r}{n}g(X,Y) \tag{12}$$

for which scalar curvature r is constant.

In this paper, section 1 is the introduction of recurrent and generalized 2-recurrent Riemannian manifold. In section 2, we study generalized 2-recurrent Riemannian manifold with Einstein manifold. In next section 3, we first obtain a necessary and sufficient condition for an Einstein $G\{\widetilde{C}(\,^2K_n)\}$ to be a $G\{\,^2K_n\}$. Next it is shown that if the Ricci tensor vanishes then $G\{\widetilde{C}(\,^2K_n)\}$ reduces to $G\{\,^2K_n\}$.

2 Methodology

Riemannian manifold is a part of differentiable manifold which we study by index free notation and tensor notation. Different type of differentiable manifold and their standard results used to characterize the recurrent manifolds. The fundamental theorem of Riemannian geometry, Ricci Identity, Bianchi first and second Identity, Contraction method and Levi-Civita connection are used to finding the results in this paper. Standard techniques and methods in the field of differential geometry developed by investigators such as Jaiswal and Ojha[(11)], De and Sarkar[(12)], Bagewadi and Ingalahalli[(13)] and their recent work are used and extended in this paper.

3 Results and Discussion

We study $G\left\{\widetilde{C}(\,^2K_n)\right\}$ manifold with Einstein manifold and prove that an Einstein $G\left\{\widetilde{C}(\,^2K_n)\right\}$ is a manifold of constant curvature. We also prove that an Einstein $G\left\{\widetilde{C}(\,^2K_n)\right\}$ manifold to be a generalized 2-recurrent Riemannian manifold $G\left\{\widetilde{C}(\,^2K_n)\right\}$ if and only if the scalar curvature is zero. In the last, we prove that $G\left\{\widetilde{C}(\,^2K_n)\right\}$ manifold reduces to $G\left\{\widetilde{C}(\,^2K_n)\right\}$ if the Ricci tensor vanishes.

Einstein $G\{\widetilde{C}(\,^2K_n)\}$

Let us assume that the $G\{\widetilde{C}(2K_n)\}$ is an Einstein manifold, then the quasi-conformal curvature tensor (5) takes the form

$$\widetilde{C}(X,Y)Z = aR(X,Y)Z - \frac{r}{n}\left(\frac{a}{n-1}\right)[g(Y,Z)X - g(X,Z)Y] \tag{13}$$

where r is the scalar curvature of the manifold.

From (13) it follows that

$\left(\nabla_U \widetilde{C}\right)(X,Y)Z = a(\nabla_U R)(X,Y)Z.$

So using Bianchi's identity we find that

$$\left(\nabla_U \widetilde{C}\right)(X,Y)Z + \left(\nabla_X \widetilde{C}\right)(U,Y)Z + \left(\nabla_Y \widetilde{C}\right)(X,U)Z = 0 \tag{14}$$

Covariant differentiation of (14) gives

$$\left(\nabla_V \nabla_U \widetilde{C}\right)(X,Y)Z + \left(\nabla_V \nabla_X \widetilde{C}\right)(U,Y)Z + \left(\nabla_V \nabla_Y \widetilde{C}\right)(X,U)Z = 0. \tag{15}$$

By virtue of (6) and (14), it follows from (15) that

$$B(U,V)\widetilde{C}(X,Y)Z + B(X,V)\widetilde{C}(U,Y)Z + B(Y,V)\widetilde{C}(X,U)Z = 0. \tag{16}$$

From (16) on contraction we get

$$B\left(\widetilde{C}(X,Y)Z,V\right) = 0. \tag{17}$$

From (13) and (8), we get

$$\begin{aligned}\widetilde{C}(X,Y,Z,W) &= -\widetilde{C}(X,Y,W,Z)\\ &= -\widetilde{C}(Y,X,Z,W)\\ &= \widetilde{C}(Z,W,X,Y)\end{aligned} \tag{18}$$

Taking $U = LV$ and using the definition (11), the equation (16) takes the form

$$g(LV,LV)\widetilde{C}(X,Y)Z + g(X,LV)\widetilde{C}(LV,Y)Z + g(Y,LV)\widetilde{C}(X,LV)Z = 0. \tag{19}$$

On using (17) and (18) in equation (15), we get

$$g(LV,LV)\widetilde{C}(X,Y)Z = 0,$$

i.e. $\widetilde{C}(X,Y)Z=0$ since $g(LV,LV)\neq 0$, which give the manifold is quasi-conformally flat.

Hence from (15) we get

$$R(X,Y)Z=\frac{r}{n(n-1)}\left[g(Y,Z)X-g(X,Z)Y\right].$$

That is, the manifold is of constant curvature. Hence we have the following theorem:

Theorem 1: An Einstein $G\{\widetilde{C}({}^{2}K_n)\}$ is a manifold of constant curvature.

Necessary and sufficient condition for an Einstein $G\{\widetilde{C}({}^{2}K_n)\}$ to be a $G\{{}^{2}K_n\}$

Now, we assume that an Einstein $G\{\widetilde{C}({}^{2}K_n)\}$ is a $G\{{}^{2}K_n\}$. Then we have

$(\nabla_V\nabla_U R)(X,Y)Z=A^*(V)(\nabla_U R)(X,Y)Z+B^*(U,V)R(X,Y)Z,$

where A^* is a 1-form and B^* is a non-zero (0, 2) tensor. On contraction we have

$$(\nabla_V\nabla_U S)(Y,Z)=A^*(V)(\nabla_U S)(Y,Z)+B^*(U,V)S(Y,Z) \tag{20}$$

where S is the Ricci tensor.

Since the manifold is Einstein so from (20), we have

$B^*(U,V)S(Y,Z)=0,$

i.e. $S(Y,Z)=0,$

which is equivalent to an Einstein manifold to $r=0$.

Conversely, suppose that an Einstein $G\{\widetilde{C}({}^{2}K_n)\}$ satisfying $r=0$. From (4) and (6) we get

$$(\nabla_V\nabla_U R)(X,Y)Z=A(V)(\nabla_U R)(X,Y)Z+B(U,V)\left(R(X,Y)Z-\frac{r}{n(n-1)}\{g(Y,Z)X-g(X,Z)Y\}\right. \tag{21}$$

Since in an Einstein manifold scalar curvature r is constant. Substituting $r=0$ in (21), we get

$(\nabla_V\nabla_U R)(X,Y)Z=A(V)(\nabla_U R)(X,Y)Z+B(U,V)R(X,Y)Z.$

Thus we have the following theorem:

Theorem 2 : An Einstein $G\{\widetilde{C}({}^{2}K_n)\}$ is a $G\{{}^{2}K_n\}$ iff the scalar curvature is zero.

Next, if the Ricci tensor i.e. $S(Y,Z)=0$, then by the use of (5) and (6), $G\{\widetilde{C}({}^{2}K_n)\}$ reduce to the form

$(\nabla_V\nabla_U R)(X,Y)Z=A(V)(\nabla_U R)(X,Y)Z+B(U,V)R(X,Y)Z,$

which is $G\{{}^{2}K_n\}$. Hence we have the following theorem:

Theorem 3: If the Ricci tensor vanishes then a $G\{\widetilde{C}({}^{2}K_n)\}$ reduces to a $G\{{}^{2}K_n\}$.

4 Conclusion

In this article we study generalized 2-recurrent Riemannian manifold with quasi-conformal curvature tensor which is denoted by $G\{\widetilde{C}({}^{2}K_n)\}$. The properties of $G\{\widetilde{C}({}^{2}K_n)\}$ have studied with Einstein manifold and generalized 2-recurrent Riemannian manifold.

5 Declaration

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