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RESEARCH ARTICLE

Minimum Pendant Dominating Partition Energy of a Graph

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Abstract

Objectives: To define and study the minimum pendant dominating partition energy of some standard graphs. Further we also establish upper and lower bounds for minimum pendant dominating partition energy of a graph G. **Methods:** To establish the upper and lower bounds for the energy of graphs we employ the Standard methods of proofs namely direct methods and using Matlab to compute the minimum pendant dominating partition eigen values of a graph G. **Findings:** In this paper we calculated the minimum pendant dominating partition energy of some standard family of graphs and also find the properties of minimum pendant dominating partition eigen values of of G. Some bounds are established. **Novelty:** In this article we found some interesting properties of minimum pendant dominating partition eigen values.

Keywords: Dominating set; Pendant Dominating set; Minimum Pendant Dominating set; Partition Energy; Minimum Partition Energy

1 Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges. The degree of v_i written by $d(v_i)$ is the number of edges incident with v_i. The maximum vertex of degree is denoted by $\Delta(G)$ and minimum vertex of degree is denoted by $\delta(G)$. The adjacency matrix $A_{pe}(G)$ of G is defined by its entries as $a_{ij} = 1\ if\ v_iv_j \in E(G)$ or $v_i \in D$if $(i = j)$ where D is a pendant dominating set of G and 0 otherwise. The eigen values of graph G are the eigenvalues of its adjacency matrix $A_{pe}(G)$, denoted by $\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_n$. A graph G is said to be singular if at least one of its eigenvalues is equal to zero. For singular graphs, evidently, $detA = 0$. A graph is nonsingular if all its eigenvalues are different from zero.

The energy of a graph G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. This concept was introduced by I. Gutman in[1] initially, the graph energy concept did not attract any noteworthy attention of mathematicians, but later they did realize its value and worldwide mathematical research of graph energy started. Nowadays, in connection with graph energy, energy like quantities was considered also for other matrices. In this paper, we are defining a matrix, called the minimum pendant dominating k- partition matrix denoted by $P_kA_{pe}(G)$ and we study its eigenvalues and the energy. Further, we study the mathematical aspects of the minimum pendant dominating partition energy of a graph. It may be possible that the minimum pendant dominating partition energy which we are considering in this paper have applications in other areas of science such as chemistry and so on. The graphs we are considering are assumed to be definite, simple, undirected having no isolated vertices, and of order at least two. A subset S of $V\ (G)$ is a dominating set of G if each vertex $u \in V\ - S$ is adjacent to a vertex in S. The least cardinality of a dominating set is called the domination number of G and is usually denoted by $\gamma(G)$. A dominating set S is called a pendant dominating set if $< S >$ contains at least one pendant vertex. The least cardinality of a pendant dominating set in G is called the pendant domination number of G, denoted by $\gamma_{pe}(G)$. For more details about pendant domination parameter refer [2,3].

1.1 Partition Energy

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and edge set E. Let $P_k\ = \{V_1,\ V_2, V_3, \ldots, V_K\}$ be a partition of a vertex set V. The partition matrix of G is the $n\ \times n$ matrix is defined by $A(G)\ = (a_{ij})$, where

$$a_{ij} = \begin{cases} 2, & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_{i_i}, v_j \in V_r \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non adjacent where } v_i, v_j \in V_r \\ 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets} \\ 0, & V_r \text{ and } V_s \text{ for } r \neq s \text{ where } v_i \in V_r \text{ and } v_j \in V_s \\ 0 & \text{otherwise} \end{cases}$$

The eigen values of this matrix are called k-partition eigenvalues of G. The k-partition energy $P_kE(G)$ is defined as the sum of the absolute values of k-partition eigenvalues of G. For details about partition energy refer [4]

1.2 Minimum Pendant Dominating Partition Energy

Let G be a simple graph of order n with vertex set $V\ = \{v_1, v_2, v_3, \ldots, v_n\}$ and edge set E. Let $P_k\ = \{V_1,\ V_2, V_3, \ldots, V_K\}$ be a partition of a vertex set V. A subset D of $V\ (G)$ is a pendant dominating set of G if induced subgraph of D contains at least one pendant vertex. The least cardinality of a pendant dominating set is called the pendant domination number of G and is usually denoted by $\gamma_{pe}(G)$. The minimum pendant dominating k-partition matrix of G is the $n \times n$ matrix defined by$P_kA_{pe}(G) = (a_{ij})$,

Where

$$a_{ij} = \begin{cases} 2, & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_{i_i}, v_j \in V_r \\ -1, & v_i \text{ and } v_j \text{ are non adjacent where } v_i, v_j \in V_r \\ 1, & \text{if i} = \text{j}, v_i \in \text{D or } v_i \text{ and } v_j \text{ are adjacent between the sets} \\ & V_r \text{ and } V_s \text{ for } r \neq s \text{ where } v_i \in V_r \text{ and } v_j \in V_s \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_{pe}(G)$ is denoted by $f_n(G,\lambda) =\ det\left(\lambda I -\ A_{pe}(G)\right)$. The minimum pendant dominating k-partition eigenvalues of the graph G are the eigenvalues of $A_{pe}(G)$. Since $A_{pe}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non- increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum pendant dominating k-partition energy of G is defined as $P_kE_{pe}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid$.

For details about minimum dominating energy of a graph refer [5]

2 Methodology

Here, we are presenting the methods which are planning to use in our research for proving the result or to explore new ideas or concept in our research area. To establish the upper and lower bounds for the energy of graphs we employ the Standard methods of proofs namely direct methods, by induction or by the contradiction method.

3 Results and Discussion

3.1 The Minimum Pendant Dominating Partition Energy of a Graph

Example 3.1: Consider a graph as shown in Figure 1. The possible minimum pendant dominating sets are:

$(i)D_1 = \{v_1, v_2, v_4\}\ (ii)D_2 = \{v_2, v_3, v_5\}\ (iii)D_3 = \{v_2, v_3, v_6\}\ (iv)D_4 = \{v_2, v_3, v_4\}\ (v)D_5 = \{v_1, v_4, v_3\}\ (vi)D_6 = \{v_5, v_6, v_2\}$.

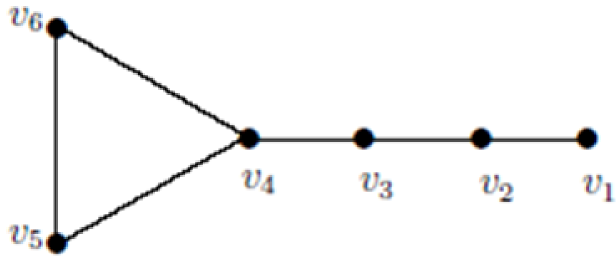

Fig 1. Simple Graph with 6 vertices

$$A_{pe,D_1}(G) = \left(\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 1 & 2 & -1 & -1 & -1 & -1 \\ v_2 & 2 & 1 & 2 & -1 & -1 & -1 \\ v_3 & -1 & 2 & 0 & 2 & -1 & -1 \\ v_4 & -1 & -1 & 2 & 1 & 2 & 2 \\ v_5 & -1 & -1 & -1 & 2 & 0 & 2 \\ v_6 & -1 & -1 & -1 & 2 & 2 & 0 \end{array}\right)$$

So, $f_n(G,\lambda) = \lambda^6 - 3\lambda^5 - 27\lambda^4 + 38\lambda^3 + 126\lambda^2 - 3\lambda - 68 = 0$.

The minimum pendant dominating partition eigen values are $\lambda_1 \approx 4.8060$, $\lambda_2 \approx 1.3834$, $\lambda_3 \approx 1$, $\lambda_4 \approx 0.7115$, $\lambda_5 \cong 2.8732$, $\lambda_6 \approx 5.8847$. The minimum pendant

domination partition energy, $P_1 A_{D_1}(G) = 16.6588$.

Now, suppose if we choose another pendant dominating set, $D_2 = (v_2, v_3, v_5\}$. Then

$$A_{pe,D_2}(G) = \left(\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 2 & -1 & -1 & -1 & -1 \\ v_2 & 2 & 1 & 2 & -1 & -1 & -1 \\ v_3 & -1 & 2 & 1 & 2 & -1 & -1 \\ v_4 & -1 & -1 & 2 & 0 & 2 & 2 \\ v_5 & -1 & -1 & -1 & 2 & 1 & 2 \\ v_6 & -1 & -1 & -1 & 2 & 2 & 0 \end{array}\right)$$

Clearly, $f_n(G,\lambda) = \lambda^6 - 3\lambda^5 - 30\lambda^4 + 33\lambda^3 + 198\lambda^2 + 42\lambda - 151 = 0$.

The minimum pendant dominating partition eigen values are $\lambda_1 \approx -3.8859$, $\lambda_2 \approx -1.7760$, $\lambda_3 \approx -1.4548$, $\lambda_4 \approx 0.7623$, $\lambda_5 \cong 3.2116$, $\lambda_6 \approx 6.1428$.

The minimum pendant domination partition energy, $P_1 A_{D_2}(G) \approx 17.2334$. Therefore, it is clear from the above example that the minimum pendant dominating partition energy of a graph G depends on the minimum pendant dominating set of a graph G.

Theorem 3.1.[6] Let G be a graph with vertex set, edge set E and a minimum pendant dominating set D. Let $f_n(G,\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0$ be the

characterstic polynomial of G. Then,

1. $a_0 = 1$
2. $a_1 = -\mid D \mid$

Theorem 3.2. Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ edge set E and $D = \{u_1, u_2, \ldots, u_m\}$ be a minimum pendant dominating set. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a minimum pendant dominating partition matrix $A_{pe}(G)$ then,

1. $\sum_{i=1}^{n} \lambda_i = \gamma_{pe}(G)$
2. $\sum_{i=1}^{n} \lambda_i^2 = 2m + \gamma_{pe}(G)$

3.2 The Minimum Pendant Dominating Partition Energy of Some Standard Graphs

Theorem 3.2.1. For $n \geq 2$, the minimum pendant dominating 2-partition energy of star graph $K_{1,n-1}$ in which the vertex of degree $n-1$ is in one partition and vertices of degree 1 are in the another partition is $(n-1)+\sqrt{n^2+2n-7}$.

Proof. Consider the star graph $K_{1,n-1}$ with vertex set $V=\{v_1, v_2, \ldots, v_n\}$. The minimum pendant dominating set $D=\{v_1, v_2\}$. The 2-partition minimum pendant dominating adjacency matrix is

$$P_2A_{pe}(K_{1,n-1}) = \begin{pmatrix} & v_1 & v_2 & v_3 & \ldots & v_{n-2} & v_{n-1} & v_n \\ v_1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ v_2 & 1 & 1 & -1 & \ldots & -1 & -1 & -1 \\ v_3 & 1 & -1 & 0 & \ldots & -1 & -1 & -1 \\ . & . & . & . & \ldots & . & . & . \\ . & . & . & . & \ldots & . & . & . \\ . & . & . & . & \ldots & . & . & . \\ v_{n-2} & 1 & -1 & -1 & \ldots & 0 & -1 & -1 \\ v_{n-1} & 1 & -1 & -1 & \ldots & -1 & 0 & -1 \\ v_n & 1 & -1 & -1 & \ldots & -1 & -1 & 0 \end{pmatrix}$$

characteristics equation is $(\lambda-1)^{n-3}(\lambda-2)(\lambda^2+(n-3)\lambda-2(n-2))=0$.

The minimum pendant dominating 2-partition eigen values are:

$$\lambda = 1\,((n-3)\,times],\ \lambda = 2\,((1\ time)],\ \lambda = \frac{(n-3)\pm\sqrt{(n^2+2n-7)}}{2}[one\ time\ each]$$

Minimum pendant dominating 2-partition energy of $K_{1,n-1}$ is

$$P_2E_{pe}(K_{1,n-1}) = \mid 1 \mid (n-3) + \mid -2 \mid + \left(\frac{(n-3)+\sqrt{(n^2+2n-7)}}{2} \right| + \left(\frac{(n-3)-\sqrt{(n^2+2n-7)}}{2} \right|$$

$$P_2E_{pe}(K_{1,n-1}) = (n-1)+\sqrt{(n^2+2n-7)}$$

Theorem 3.2.2. For $n \geq 3$, the minimum pendant dominating 1-partition energy of complete graph K_n is equal to $(2n-5)+\sqrt{(4n^2-4n+17)}$.

Proof. Let K_n be the complete graph with vertex set $V=\{v_1, v_2, v_3, .., v_n\}$. The minimum pendant dominating set is $D=\{v_1, v_2\}$. Then the minimum pendant dominating 1-partition matrix of complete graph is

$$P_1A_{pe}(K_n) = \begin{pmatrix} & v_1 & v_2 & v_3 & \ldots & v_{n-2} & v_{n-1} & v_n \\ v_1 & 1 & 2 & 2 & \ldots & 2 & 2 & 2 \\ v_2 & 2 & 1 & 2 & \ldots & 2 & 2 & 2 \\ v_3 & 2 & 2 & 0 & \ldots & 2 & 2 & 2 \\ . & . & . & . & \ldots & . & . & . \\ . & . & . & . & \ldots & . & . & . \\ . & . & . & . & \ldots & . & . & . \\ v_{n-2} & 2 & 2 & 2 & \ldots & 0 & 2 & 2 \\ v_{n-1} & 2 & 2 & 2 & \ldots & 2 & 0 & 2 \\ v_n & 2 & 2 & 2 & \ldots & 2 & 2 & 0 \end{pmatrix}$$

Characteristics equation is $(\lambda+1)(\lambda+2)^{n-3}(\lambda^2-(2n-3)\lambda-(2n+2))=0$.

The minimum pendant dominating 1-partition eigen values are:

$$\lambda = -1\,(1\ time],\ \lambda = -2\,((n-3)times],\ \lambda = \frac{(2n-3)\pm\sqrt{(4n^2-4n+17)}}{2}[one\ time\ each]$$

Minimum pendant dominating 1-partition energy of $K_n$is

$$P_1E_{pe}(K_n) = |-1| + |-2|(n-3) + \left(\frac{(2n-3)+\sqrt{(4n^2-4n+17)}}{2}\right| + \left(\frac{(2n-3)-\sqrt{(4n^2-4n+17)}}{2}\right|$$

$$P_1E_{pe}(K_n) = (2n-5)+\sqrt{(4n^2-4n+17)}$$

Theorem 3.2.3. For $n \geq 3$, the minimum pendant dominating n-partition energy of complete graph K_n is equal to $(n-3) + \sqrt{(n^2-2n+9)}$.

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, v_3, .., v_n\}$. The minimum pendant dominating set is $D = \{v_1, v_2\}$. Then the minimum pendant dominating n-partition matrix of complete graph is

$$P_nA_{pe}(K_n) = \left(\begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & \ldots & v_{n-2} & v_{n-1} & v_n \\ \hline v_1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ v_2 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ v_3 & 1 & 1 & 0 & \ldots & 1 & 1 & 1 \\ . & . & . & . & \ldots & . & . & . \\ . & . & . & . & \ldots & . & . & . \\ . & . & . & . & \ldots & . & . & . \\ v_{n-2} & 1 & 1 & 1 & \ldots & 0 & 1 & 1 \\ v_{n-1} & 1 & 1 & 1 & \ldots & 1 & 0 & 1 \\ v_n & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \end{array}\right)$$

Characteristics equation is $\lambda(\lambda+1)^{n-3}(\lambda^2-(n-1)\lambda-2) = 0$.

The minimum pendant dominating n-partition eigen values are:

$$\lambda = 0\,(1\ time],\ \lambda = -1\,((n-3)times],\ \lambda = \frac{(n-1)\pm\sqrt{(n^2-2n+9)}}{2}[one\ time\ each]$$

Minimum pendant dominating n-partition energy of $K_n$is

$$P_nE_{pe}(K_n) = 0 + |-1|(n-3) + \left(\frac{(n-1)+\sqrt{(n^2-2n+9)}}{2}\right| + \left(\frac{(n-1)-\sqrt{(n^2-2n+9)}}{2}\right|$$

$$P_nE_{pe}(K_n) = (n-3) + \sqrt{(n^2-2n+9)}.$$

Theorem 3.2.4. For $n \geq 3$, the minimum pendant dominating 2-partition energy of complete bipartite graph $K_{n,n}$ is equal to $(2n-1) + \sqrt{(4n^2+4n-7)}$.

Proof. Consider the complete bipartite graph $K_{n,n}$whose vertex set is partitioned into $V_1 = \{v_1, v_2, v_3, .., v_n\}$, $V_2 = \{u_1, u_2, u_3, .., u_n\}$. The minimum pendant dominating set is $D = \{v_1, u_1\}$. The minimum pendant dominating 2-partition matrix is

$$P_2A_{pe}(K_{n,n}) = \begin{pmatrix} & v_1 & v_2 & v_3 & \ldots & v_n & u_1 & u_2 & u_3 & \ldots & u_n \\ \hline v_1 & 1 & -1 & -1 & \ldots & -1 & 1 & 1 & 1 & \ldots & 1 \\ v_2 & -1 & 0 & -1 & \ldots & -1 & 1 & 1 & 1 & \ldots & 1 \\ v_3 & -1 & -1 & 0 & \ldots & -1 & 1 & 1 & 1 & \ldots & 1 \\ . & . & . & . & \ldots & . & . & . & . & \ldots & . \\ . & . & . & . & \ldots & . & . & . & . & \ldots & . \\ . & . & . & . & \ldots & . & . & . & . & \ldots & . \\ v_n & -1 & -1 & -1 & \ldots & 0 & 1 & 1 & 1 & \ldots & 1 \\ u_1 & 1 & 1 & 1 & \ldots & 1 & 1 & -1 & -1 & \ldots & -1 \\ u_2 & 1 & 1 & 1 & \ldots & 1 & -1 & 0 & -1 & \ldots & -1 \\ u_3 & 1 & 1 & 1 & \ldots & 1 & -1 & -1 & 0 & \ldots & -1 \\ . & . & . & . & \ldots & . & . & . & . & \ldots & . \\ . & . & . & . & \ldots & . & . & . & . & \ldots & . \\ . & . & . & . & \ldots & . & . & . & . & \ldots & . \\ . & . & . & . & \ldots & . & . & . & . & \ldots & . \\ u_n & 1 & 1 & 1 & \ldots & 1 & -1 & -1 & -1 & \ldots & 0 \end{pmatrix}$$

Characteristics equation is $(\lambda-1)^{2n-3}(\lambda-2)(\lambda^2+(2n-3)\lambda-(4n-4))=0$.

The minimum pendant dominating 2-partition eigen values are:

$$\lambda = 1\,((2n-3)\ times],\ \lambda = 2\,(1\ time],\ \lambda = \frac{-(2n-3)\pm\sqrt{(4n^2+4n-7)}}{2}[one\ time\ each]$$

Minimum pendant dominating 2-partition energy of $K_{n,n}$is

$$P_2E_{pe}(K_{n,n}) = |\,1\,||\,(2n-3)\ +(2|+\left(\frac{-(2n-3)+\sqrt{(4n^2+4n-7)}}{2}\right| + \left(\frac{-(2n-3)-\sqrt{(4n^2+4n-7)}}{2}\right|$$

$$P_2E_{pe}(K_{n,n}) = (2n-1)\ +\sqrt{(4n^2+4n-7)}.$$

3.3 Properties of Minimum Pendant Dominating Partition Eigenvalues

Let $G=(V,E)$ be a graph with n vertices and $P_k=\{v_1,v_2,\ldots,v_k\}$ be a partitions of G. For $1\le\ i\le\ k$, let b_i denote the total number of edges joining the vertices of V_i and c_i be the total number of edges joining the vertices from V_i to V_j for $i\neq j$, $1\le\ i\le\ k$ and d_i be the number of non adjacent pairs of vertices within V_i.

Let $m_1=\sum_{i=1}^{k}|\,b_i\,|,\quad m_2=\sum_{i=1}^{k}|\,c_i\,|,\quad and\ m_3=\sum_{i=1}^{k}|\,d_i\,|,$

Theorem 3.3.1. Let G be a simple graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$, edge set E. $P_k=\{v_1,v_2,\ldots,v_k\}$ be partition of V and D be a minimum pendant dominating set. If $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the minimum pendant dominating k-partition eigenvalues of minimum pendant dominating k-partition matrix $A_{pe}(G)$ then

1. $\sum_{i=1}^{n}\lambda_i=|\,D\,|$
2. $\sum_{i=1}^{n}\lambda_i^2=|\,D\,|+2(4m_1+m_2+m_3)$

Proof. (i) We know that the sum of the k- partition eigenvalues of $A_{pe}(G)$ is the trace of $A_{pe}(G)$. Therefore $\sum_{i=1}^{n}\lambda_i=a_{ii}=|\,D\,|$ Similarly the sum of squares of the k-partition eigenvalues of A_{pe}(G is trace of $[A_{pe}(G)]^2$. Therefore

$$\sum_{i=1}^{n}\lambda_i^2=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a_{ij}$$

$$=\sum_{i=1}^{n}(a_{ii})^2+\sum_{i\neq j}^{n}a_{ij}a_{ij}$$

$$= \sum_{i=1}^{n} (a_{ii})^2 + 2\sum_{i \neq j}^{n} (a_{ii})^2$$

$$= | D | + 2(4m_1 + m_2 + m_3)$$

3.4 Bounds For Minimum Pendant Dominating Partition Energy of a Graph

Theorem 3.4.1. Let $G = (V,E)$ be a graph with n vertices and $P_k = \{v_1, v_2, \ldots, v_k\}$ be a partition of V . Then $P_k E_{pe}(G) \leq \sqrt{n[| D | + 2(4m_1 + m_2 + m_3)]}$

where m_1, m_2, m_3 are as defined above for G

Proof. Cauchy-Schwartz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

If we $a_i = 1, b_i = | \lambda_i |$ then

$$\left(\sum_{i=1}^{n} | \lambda_i |\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \lambda_i^2\right)$$

$$\Longrightarrow (P_k E_{pe}(G))^2 \leq n[| D | + 2(4m_1 + m_2 + m_3)]$$

Theorem 3.4.2. Let G be a simple graph with n vertices and m edges. $P_k = \{v_1, v_2, \ldots, v_k\}$ be a partition of V . If D is the minimum pendant dominating set and $P = | detA_{pe}(G) |$ then

$$\sqrt{| D | + 2(4m_1 + m_2 + m_3) + n(n-1)P^{\frac{2}{n}}} \leq P_k E_{pe}(G) \leq \sqrt{n[| D | + 2(4m_1 + m_2 + m_3)]}$$

Proof. Cauchy-Schwartz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

If we $a_i = 1, b_i = | \lambda_i |$ then

$$\left(\sum_{i=1}^{n} | \lambda_i |\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \lambda_i^2\right)$$

$$(P_k E_{pe}(G))^2 \leq n[| D | + 2(4m_1 + m_2 + m_3)]$$

$$\Longrightarrow (P_k E_{pe}(G))^2 \leq n[| D | + 2(4m_1 + m_2 + m_3)]$$

Now by arithmetic mean and geometric mean inequality

$$\frac{1}{n(n-1)} \sum_{i \neq j} | \lambda_i || \lambda_j | \geq \left(\prod_{i \neq j} | \lambda_i || \lambda_j |\right]^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{i \neq j} | \lambda_i |^{2(n-1)} \right]^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{i \neq j} | \lambda_i | \right]^{\frac{2}{n}}$$

$$= | detA_{pe}(G) |^{\frac{2}{n}} = P^{\frac{2}{n}}$$

$$\sum_{i \neq j} |\lambda_i| \left|\lambda_j\right| \geq n(n-1)P^{\frac{2}{n}} \tag{5.1}$$

Now consider,

$$(P_k E_{pe}(G)]^2 = \left(\sum_{i=1}^{n} | \lambda_i | \right)^2$$

$$= \left(\sum_{i=1}^{n} | \lambda_i | \right)^2 + \sum_{i \neq j} | \lambda_i || \lambda_j |$$

$\therefore$ $(P_k E_{pe}(G)]^2 \geq | D | + 2(4m_1 + m_2 + m_3) + n(n-1)P^{\frac{2}{n}}$ from (5.1)

$$i.e., P_k E_{pe}(G) \geq \sqrt{| D | + 2(4m_1 + m_2 + m_3) + n(n-1)P^{\frac{2}{n}}}$$

Theorem 3.4.3 Let G be a graph with n vertices and m edges with $2(2m_1 + m_2 + m_3) + | D | \geq n$ and $(4m_1 + 2m_2 - 2m_3 + | D |)^2 - n(8m_1 + 2m_2 + 2m_3 + | D |) \geq 0$ then $P_k E_{pe}(G) \leq \frac{2(2m_1+m_2+m_3)+|D|}{n} + \sqrt{(n-1)(2(2m_1 + m_2 + m_3) + | D |] - \left(\frac{2(2m_1+m_2+m_3)+|D|}{n}\right)^2}$

Proof. Cauchy-Schwartz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right)$$

If we $a_i = 1, b_i = | \lambda_i |$ then

$$\left(\sum_{i=2}^{n} | \lambda_i | \right)^2 \leq \left(\sum_{i=2}^{n} 1 \right) \left(\sum_{i=2}^{n} \lambda_i^2 \right)$$

$$\implies (P_k E_{pe}(G)]^2 \leq (n-1)\left(2(4m_1 + m_2 + m_3) + | D | - \lambda_1^2\right)$$

$$\implies (P_k E_{pe}(G)] \leq \lambda_1 + \sqrt{(n-1)\left(2(4m_1 + m_2 + m_3) + | D | - \lambda_1^2\right)}$$

Let

$f(x) = x + \sqrt{(n-1)\left(2\left(4m_1+m_2+m_3\right)+|D|-x^2\right)}$ for decreasing function

$$f'(x) \leq 0 \Longrightarrow 1 - \frac{x(n-1)}{\sqrt{(n-1)\left(2\left(4m_1+m_2+m_3\right)+|D|-x^2\right)}} \leq 0$$

$$\Longrightarrow x \geq \sqrt{\frac{2\left(2m_1+m_2-m_3\right)+|D|}{n}}$$

Since
$\left(4m_1+2m_2-2m_3+|D|\right)^2 - n\left(8m_1+2m_2+2m_3+|D|\right) \geq 0$, we have

$$\sqrt{\frac{2\left(2m_1+m_2-m_3\right)+|D|}{n}} \leq \frac{2\left(2m_1+m_2-m_3\right)+|D|}{n} \leq \lambda_1$$

Therefore $f(\lambda_1) \leq f\left(\frac{2(2m_1+m_2-m_3)+|D|}{n}\right)$

$$i.e.,\ P_kE_{pe}(G) \leq f(\lambda_1) \leq f\left(\frac{2\left(2m_1+m_2-m_3\right)+|D|}{n}\right)$$

$$i.e.,\ P_kE_{pe}(G) \leq f\left(\frac{2\left(2m_1+m_2-m_3\right)+|D|}{n}\right)$$

$$i.e., P_kE_{pe}(G) \leq f\left(\frac{2\left(2m_1+m_2-m_3\right)+|D|}{n}\right) + \sqrt{(n-1)\left(2\left(2m_1+m_2+m_3\right)+|D|\right] - \left(\frac{2\left(2m_1+m_2+m_3\right)+|D|}{n}\right)^2}$$

4 Conclusion

Nowadays, the study of theory of domination and energy of graph is an important area in Graph theory and also remarkable research is going on in this area. In recent years many scholars are working in this area and also they are introducing new domination parameters. In this way, we are now introduced a new domination invariant called pendant domination in graphs. In this paper we have initiated the study of this parameter and extend this parameter to energy of graph. We have calculated the energies for some standard family graphs and we have established some bounds for this parameter. Further, we have studied some important properties of minimum pendant dominating partition eigenvalues

References

1) Gutman I. The Energy of a Graph: Old and New Results. In: Algebraic Combinatorics and Applications. Springer Berlin Heidelberg. 2001;p. 196–211. Available from: https://doi.org/10.1007/978-3-642-59448-9_13.
2) Nayaka SR, Purushothama P, S. Pendant Domination in Graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*. 2020;112:219–230. Available from: https://www.researchgate.net/publication/342158953_PENDANT_DOMINATION_IN_GRAPHS.
3) Nayaka SR, Puttaswamy, Purushothama S. Pendant domination in generalized graphs. *International Journal of Scientific Engineering and Science*. 2017;1(7):2456–7361. Available from: https://www.researchgate.net/publication/339644820_Pendant_Domination_in_Some_Generalised_Graphs.
4) Sriraj MA, Shwetha BC, Veena CR, Roopa SV. Partition energy of some lexicographic product of two graphs. *Proceedings of the Jangjeon Mathematical Society*. 2022;25(4):435–454. Available from: http://dx.doi.org/10.17777/pjms2022.25.4.435.
5) Kanna1 MRR, Dharmendra BN, Sridhara G. Minimum Dominating Energy of a Graph. *International Journal of Pure and Applied Mathematics*. 2013;85(4):1314–3395. Available from: http://dx.doi.org/10.12732/ijpam.v85i4.7.
6) Purushothama S, Puttaswamy, Nayaka SR. Minimum Pendant Dominating Energy of a Graph. *American International Journal of Research in Science*. 2019;p. 2328–3629. Available from: https://www.researchgate.net/publication/335291794_MINIMUM_PENDANT_DOMINATING_ENERGY_OF_A_GRAPH.