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Some Kinds of Separation Axioms in Kasaj Topological Spaces

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Abstract

Objectives: The separation axioms are about the use of Kasaj topology to distinguish disjoint sets and distinct points. **Methods:** Any two Kasaj topologically distinguishable points must be distinct, and any two separated points must be Kasaj topologically distinguishable. **Findings:** New classes of separation axioms in Kasaj topological space namely, KS_g, KS_{gs} and KS_{sg} spaces by utilizing KS_g, KS_{gs} and KS_{sg}- open and closed sets are introduced and studied. Several of their fundamental characterizations and their relationships with other corresponding kinds of spaces are discussed. **Novelty:** In our paper work using the relation among the spaces are graphically illustrated as direct graphs.
AMS Subject Classifications: 54A05, 54B05, 54A99

Keywords: KS-semi-Ti($i \leq 4$) spaces; KSα-Ti($i \leq 4$) spaces; KSg-Ti($i \leq 4$) spaces; KSgs-Ti($i \leq 4$) spaces; KSsg -Ti($i \leq 4$) spaces

1 Introduction

In 2020, Kashyap G. Rachchh and Sajeed I. Ghanchi[1] introduced a partial extension of microtopological space, namely Kasaj topological spaces. In the same year, Rachchh K. G., Ghanchi S. I., Soneji A. A., and Ghanchi S. I.[2] established the concepts of kasaj-closure and kasaj-interior and defined kasaj semi-closed and kasaj generalized closed sets in Kasaj topological spaces. In 2022, Prakash et al.[3] defined and studied the notions of KS_{gs}-closed and KS_{sg}-closed sets in Kasaj topological space. The classes KS_g, KS_{gs} and KS_{sg}-open and closed sets play a vital role in the development of the generalization of Kasaj topological spaces. To further initiate new ideas on this paper, we refer to Abdelwaheb Mhemdi, Tareq M. Al-shami[4] which is introduced in year 2021, the concepts of functionally T_i space for i = 0, 1, 2. They study their main properties, especially those are related to product spaces and topological and hereditary properties. In 2022, Ferit Yalaz, Aynur Keskin ka ymakci[5], initiated the concept of σ-R_0-space and σKer_Γ by using the σ-topology produced with the help of the ideal and local closure function. This investigation has made significant contributions to the theory of separation axioms in Kasaj topological spaces. The main goal of the present paper is to consider and study the new classes of spaces called KS_g, KS_{gs} and KS_{sg} spaces by using the respective Kasaj open and closed sets, respectively. Several properties concerning these kinds of spaces were presented. Also, the relationships among these spaces were investigated graphically, and examples were presented wherever necessary.

Preliminaries:

Definition: [1]

The Kasaj topology is defined by $KS_R(X) = \{(K \cap S) \cup (K' \cap S') : K, K' \in \tau_R(X)$, fixed $S, S^{'} \notin \tau_R(X), S \cup S' = U\}$

The Kasaj topology $KS_R(X)$ satisfies the following postulates:

1. $U,\ \phi \in KS_R(X)$
2. The union of elements of any sub collection of $KS_R\ (X)$ *is in* $KS_R\ (X)$
3. The intersection of any finite sub collection of elements of $KS_R\ (X)$ *is in* $KS_R\ (X)$

Then $(U, \tau_R(X), KS_R(X))$ is called Kasaj Topological Spaces and the members of $KS_R(X)$ are called Kasaj-open (KS-open) set and the complement of a Kasaj-open set is called a Kasaj-closed (KS-closed) set.

Definition: [1]

The Kasaj closure and the Kasaj interior of a set P is denoted by KS_{cl}(P) and KS_{int}(P) respectively. It is denoted by KS_{cl}(P) = $\bigcap${Q: P ⊆ Q, Q is KS-closed} and KS_{int}(P) = $\bigcup${Q: Q ⊆ P, Q is KS-open}

Definition: [1]

For any two subsets P, Q of U in a Kasaj topological space (U, τ_R(X), KS_R(X))

(i) P is a Kasaj-closed set iff KS_{cl}(P) = P

(ii) P is a Kasaj-open set iff KS_{int}(P) = P

(iii) If P ⊆ Q, then KS_{int}(P) ⊆ KS_{int}(Q) and KS_{cl}(P) ⊆ KS_{cl}(Q)

(iv) KS_{cl}(KS_{cl}(P)) = KS_{cl}(P) and KS_{int}(KS_{int}(P)) = KS_{int}(P)

(v) KS_{cl}(P) $\bigcup$ KS_{cl}(Q) ⊆ KS_{cl}(P$\bigcup$Q)

(vi) KS_{int}(P) $\bigcup$ KS_{int}(Q) ⊆ KS_{int}(P$\bigcup$Q)

(vii) KS_{cl}(P$\bigcap$Q) ⊆ KS_{cl}(P) $\bigcap$ KS_{cl}(Q)

(viii) KS_{int}(P$\bigcap$Q) ⊆ KS_{int}(P) $\bigcap$ KS_{int}(Q)

(ix) KS_{cl}(P) = $[KS_{int}(P)]^c$

(x) KS_{int}(P) = $[KS_{cl}(P)]^c$

2 Methodology

In our paper work, the following methodologies are used: KSg- T0, KSgs- T0, KSsg- T0 spaces and KSg- T1, KSgs- T1, KSsg- T1 spaces. If we pair distinct points and then there exist KSg, KSgs, KSsg -open sets such that the distinct points belong to any one of the KSg, KSgs, KSsg -open sets in KSg- T0, KSgs- T0, KSsg- T0 spaces and KSg- T1, KSgs- T1, KSsg- T1 spaces. But in KSg- T2, KSgs- T2, KSsg- T2 spaces, if we take each pair of distinct points, and then there exists KSg, KSgs, KSsg -open sets such that the distinct points belong to any one of the KSg, KSgs, KSsg -open sets and so the intersection of two KSg, KSgs, KSsg -open sets is empty. Likewise, KSg- T3, KSgs- T3, KSsg- T3 spaces, if given an element and KSg, KSgs, KSsg -closed sets belong to a Kasaj topological space but element do not belong to KSg, KSgs, KSsg -closed sets and there exist distinct KSg, KSgs, KSsg -open sets, two of which contain elements in one KSg, KSgs, KSsg -open set and KSg, KSgs, KSsg -closed set in another KSg, KSgs, KSsg -open set. For KSg- T4, KSgs- T4, KSsg- T4 spaces, if given a pair of distinct KSg, KSgs, KSsg -closed sets, there exists disjoint KSg, KSgs, KSsg -open sets contained in a Kasaj topological space. This implies that KSg, KSgs, KSsg -closed sets contained in KSg, KSgs, KSsg -open sets.

3 Result and discussion

3.1 KS_g -T_i(i=0,1,2,3,4) spaces

In this section, we define and discuss some properties of KS_g -T_0, KS_g -T_1, KS_g -T_2, KS_g -T_3 and KS_g -T_4 spaces in kasaj topological spaces and obtain some of their basic properties.

Definition 3. 1.1. A KS topological space$(U, \tau_R(X), KS_R(X))$ is said to be

(i) KS_g-T_0 spaces if given a pair of distinct points $x, y \in U$ either ∃ a KS_g-open set G $\in KS_R(X) \ni x \in$ G, y ∉ G or ∃ a KS_g-open set $H \in KS_R(X) \ni y \in H, x \notin H$.

(ii) KS_g-T_1 spaces if given a pair of distinct points $x, y \in U$ with $x \neq y$, ∃ a KS_g-open set $G, H \in KS_R(X) \ni x \in G$, $y \notin G; y \in H, x \notin H$.

(iii) KS_g-T_2 spaces if given a pair of distinct points $x, y \in U$ with $x \neq y$, $\exists$ a KS_g-open set $G, H \in KS_R(X) \ni x \in G, y \in H$; $G \cap H = \phi$.

(iv) KS_g-T_3 spaces if given an element $x \in U$ and a KS_g-closed set $F \subseteq U \ni x \notin F$, $\exists$ disjoint KS_g-open sets$G_1, G_2 \subseteq U \ni x \in G_1,\ F \subseteq G_2$.

(v) KS_g-T_4 spaces if given a pair of disjoint KS_g-closed sets $C_1, C_2 \subseteq U$, $\exists$ disjoint KS_g-open sets $C_1, C_2 \subseteq U \ni C_1 \subseteq G_1 C_2 \subseteq G_2$.

Example 3. 1.2. Let U = {a,b,c,d,e} with U/R = {{c,d},{b,e},{a}} and X = {a,b}$\subseteq$ U. Then $\tau_R(X)$ = { ϕ, U, {a}, {a, b, e}, {b, e}}. If we consider S={e}, S$'$ = {a,b,c,d} then $KS_R(X)$ = { ϕ, {a}, {b},{e}{a, b}, {a, e}, {b, e}, {a,b,e},{a,b,c,d},U} and KS_g-open ={ ϕ,{a},{b},{c},{d},{e},{a,b},{a,c},{a,d},{a,e},{b,c},{b,d},{b,e},{c,e},{d,e},{a,b,c},{a,b,d},{a,b,e},{a,c,e},{a,d,e},{b,c,e},{b,d,e},{a,b,c,d}, {a,b,c,e},{a,b,d,e}}

(i) Let d, e $\in$ U, d $\neq$ e $\exists$ a KS_g -open set ={a,d}$\ni$ d $\in$ {a, d} and e$\notin${a, d}.

(ii) From Exemplar 3.1.2(i) and $\exists$ a KS_g -open set ={b,e}$\ni$ e $\in$ {b, d} and d $\notin$ {b, e}.

(iii) From Exemplar 3.1.2 ((i)&(ii)) $\exists$ {a, d} $\cap$ {b, e} = ϕ.

(iv) Let e $\in$ U, {a,d}= KS_g -closed sets and e$\notin${a, d} $\exists$ {c,e} and {a,b,d}= KS_g -open sets $\ni$e $\in$ {c, e} and {a, d} $\subseteq$ {a, b, d}.

(v) Let {c} and {d} = KSg-closed sets where {c}\neq{d} $\exists$ {a,c} and {b,d}=KS_g-open sets where {a, c} = {b, d}$\ni${c} $\subseteq$ {a, c} and {d} $\subseteq$ {b, d}.

Theorem 3. 1.3. If U is KS_g-T_0 space and V is a subspace of U then V is also KS_g-T_0 space.

Proof: Let U be KS_g-T_0 space and V be a subspace of U. To show that V is KS_g-T_0 space, let x, y $\in$ V with $x \neq y$. Since V $\subseteq$ U, we have x, y $\in$ U. But U is KS_g-T_0space. So $\exists$ a KS_g-open set $G \ni G$ contains only one point x $\in$ G and $y \notin G$ then V $\cap$G is a KS_g-open set in $V \ni x\in$ V $\cap$G and $y \notin$V$\bigcap$G. Hence V is KS_g-T_0 space.

Theorem 3.1.4. A KS topological space U is a KS_g-T_0 space iff KS_g-closure of distinct points are distinct.

Proof: Let x and y be distinct points of U. Since U is KS_g-T_0 space there exist a KS_g-open set $G \ni x \in G$ and $y \notin G$. Consequently, U $-$ G is a KS_g-closed set containing y but not x. But KS_{gcl}(y) is the intersection of all a KS_g-closed set containing y. Hence, y $\in$ KS_{gcl}(y). But $x \notin KS_{gcl}(y)$ as $x \notin$(U$-$G). Therefore, KS_{gcl}(x)$\neq$$KS_{gcl}$(y). Conversely, let KS_{gcl}(x) $\neq$$KS_{gcl}$(y) for x$\neq$y. Then $\exists$ at least one point z $\in$ U $\ni$ z $\in$ KS_{gcl}(x) but z$\notin$$KS_{gcl}$(y). We claim x $\notin$ KS_{gcl}(y) because if x $\in$ KS_{gcl}(y), x $\subseteq$ KS_{gcl}(y) implies KS_{gcl}(x) $\subseteq$ KS_{gcl}(y). So, z $\in$ KS_{gcl}(y), which is a contradiction. Hence x$\notin$$KS_{gcl}$(y), which implies x $\in$ U-KS_{gcl}(y), which is a KS_g-open set containing x but not y. Hence U is a KS_g-T_0 space.

Theorem 3.1.5. A KS topological space U is KS_g-T_1 space if and only if each one point set is KS_g-closed.

Proof: Assume that U is KS_g-T_1 space. Let x $\in$ U. Then for each y $\in$ U $-$ {x} $\exists$ a KS_g-open set U $\ni$y $\in$ U and $x \notin U$. Since, $x \notin U$ the sets {x} and U are disjoint. i e, {x} $\cap$ U = ϕ that implies U $\subseteq$ U $-$ {x}. Thus, y $\in$ U $\subseteq$ U $-$ x that implies U $-$ {x} is a KS_g-open set that implies {x} is a KS_g-closed set. Conversely, assume that each point set is KS_g-closed. Let x, y $\in$ U with $x \neq y$. So, U $-$ {x} is a KS_g-open set containing y and but not x. Also, U $-$ {y} is KS_g-open containing x but not y. So U is KS_g-T_1 space.

Theorem 3.1.6. Every subspace of KS_g-T_1 space is KSg-T_1 space.

Proof: Let $(U, \tau_R(X), KS_R(X))$ be KS_g-T_1 space. Let $(V, \tau_R(Y), KS_R^*(X))$ be a subspace of U.

Let x, y $\in$ V with $x \neq y$. Since V $\subseteq$ U, we have x, y $\in$ U. Since U is KS_g-T_1 space $\exists$ KS_g-open sets G and H such that x $\in$ G, y $\notin$ G and y $\in$ H, x $\notin$ H.Let I = V $\cap$ G and J = V $\cap$ H. Then I and J are KS_g-open sets in V. Also, x $\in$ I, y $\notin$ I and y $\in$ J, x $\notin$ J. So, V is KS_g-T_1 space.

Theorem 3.1.7. A KS topological space U is KS_g-T_1 space iff every finite subset of U is KS_g-closed in U.

Proof: Assume that U is KS_g-T_1 space. Let G be a finite subset of U. Let G = {x_1, x_2, ...x_n}. Then G = {x_1} $\cup$ {x_2} $\cup$ ...{x_n} which is KS_g-closed, begin a finite union of KS_g-closed sets. Conversely, let each finite subset of U is KS_g-closed in U. Then {x} is KS_g-closed since it is finite. Since each singleton is KS_g-closed, U is KS_g-T_1 space.

Theorem 3.1.8. A KS_g-closed subspace of KS_g -T_4 space is KS_g -T_4 space.

Proof: Let V be a KSg-closed subspace of KS_g -T_4 space. Let C_1 and C_2 are disjoint KS_g-closed subsets of V. Since V is KS_g -closed in U, C_1 and C_2 are also KS_g -closed in U. There exist disjoint KS_g -open sets G and H in U such that C_1 $\subseteq$ G and C_2 $\subseteq$ H. Since V contains both C_1 and C_2, we have C_1 $\subseteq$ V $\cap$G, C_2 $\subseteq$ V $\cap$ H and (V $\cap$ G) $\cap$ (V $\cap$ H) = ϕ. Since, G and H are KS_g -open in X, (V $\cap$ G) and (V $\cap$ H) are KS_g -open in V. Thus, in the subspace V, we have disjoint KS_g -open sets (V $\cap$ G) containing C_1 and (V $\cap$ H) containingC_2. Hence the subspace V is KS_g -T_4 space.

3.2 KS_{gs}-T_i, (i=0,1,2,3,4) spaces

In this section, we define and discuss some properties of KS_{gs}-T_0, KS_{gs}-T_1, KS_{gs}-T_2, KS_{gs}-T_3 and KS_{gs}-T_4 spaces in kasaj topological spaces and obtain some of their basic properties.

Definition 3. 2.1. A KS topological space$(U, \tau_R(X), KS_R(X))$ is said to be

(i) KS_{gs}-T_0 spaces if given a pair of distinct points x, y ∈ U either ∃ a KS_{gs}-open set G ∈ $KS_R(X)$ ∋x ∈ G, y ∉ G or ∃ a KS_{gs}-open set H ∈$KS_R(X)$∋ y ∈ H, x ∉H.

(ii) KS_{gs}-T_1 spaces if given a pair of distinct points x, y ∈ U with x≠y, ∃ a KS_{gs}-open set G, H ∈ $KS_R(X)$ ∋x ∈ G, y∉G; y ∈ H, x∉H.

(iii) KS_{gs}-T_2 spaces if given a pair of distinct points x, y ∈ U with x≠ y, ∃ a KS_{gs}-open set G, H ∈ $KS_R(X)$∋ x ∈ G, y ∈ H, G ∩ H = ϕ.

(iv) KS_{gs}-T_3 spaces if given an element x ∈ U and a KS_{gs}-closed set F ⊆ U∋ x∉ F, ∃ disjoint KS_{gs}-open sets G_1, G_2 ⊆ U ∋ x ∈ G_1, F⊆ G_2.

(v) KS_{gs}-T_4 spaces if given a pair of disjoint KSgs-closed sets C_1, C_2 ⊆ U, ∃ disjoint KS_{gs}-open sets G_1, G_2 ⊆ U∋C_1 ⊆ G_1, C_2 ⊆ G_2.

Example 3. 2.2. Let U = {a,b,c,d,e} with U/R = {{c,e},{a,b},{d}} and X = {a,b,c}⊆U. Then $\tau_R(X)$= { ϕ, U, {a, b}, {c, e}, {a, b, c, e}}. If we consider S={a,d}, $S^{'}$={b,c,e} then $KS_R(X)$= {ϕ, {a},{b},{a, b}, {a, d}, {c, e},{a, b, d}, {a, c, e}, {b, c, e}, {a, b, c, e}, {a, c, d, e}, U } and KS_{gs}-open = {ϕ,{a},{b},{c},{e},{a,b},{a,c},{a,d},{a,e},{b,c},{b,e},{c,e},{a,b,c},{a,b,d},{a,b,e},{a,c,d},{a,c,e},{a,d,e},{b,c,d},{b,c,e}, {b,d,e},{a,b,c,d},{a,b,c,e},{a,b,d,e},{a,c,d,e}}

(i) Let b, e ∈ U, b≠e ∃ a KS_{gs}-open set ={b,c}∋b ∈ {b, c}and e∉{b, c}.

(ii) From Exemplar 3.2.2(i) and ∃ a KS_{gs}-open set = {a,e}∋e ∈ {a, e} and b∉{a, e}.

(iii) From Exemplar 3.2.2((i)&(ii)) ∃ {b, c} ∩ {a, e} = ϕ.

(iv) Let c ∈ U, {a,e}= KS_{gs}-closed sets and c∉{a, e} ∃ {b,c}and{a,d,e} = KS_{gs}-open sets ∋ c ∈ {b, c} and {a, e} ⊆ {a, d, e}.

(v) Let {a,d} and {b,e} = KS_{gs}-closed sets where {a, d}≠{b, e} ∃ {a,d} and {b,c,e} = KSgs-open sets where {a, d}≠{b, c, e}∋{a, d} ⊆ {a, c} and {b, e} ⊆ {b, c, e}.

Theorem 3. 2.3. Let U is KS_{gs}-T_1 space iff the intersection of all KS_{gs}-neighborhoods of any point x in U is the singleton{x}.

Proof: Assume that U is KS_{gs}-T_1 space. Let x ∈ U. Let G be the intersection of all KS_{gs}-neighborhoods of x. Let y be any point in U different from x. Since the space U is KS_{gs}-T_1 space. ∃ a KS_{gs}-neighborhood N of x∋ y∉N. Since y∉N, we have y∉G since G is the intersection of all KS_{gs}-neighborhoods of x. Since y∉G, no point different from x is in G. Hence, G ={x}.

Conversely, assume that the intersection of all KS_{gs}-neighborhoods of p in U is p. To prove that, U is KS_{gs}-T_1 space, let x, y ∈ U with x ≠y. Let G = ∩{all KS_{gs}-neighborhoods of x in U}. Then G ={x}. Let H = ∩{all KS_{gs}-neighborhoods of y in U}. Then G ={y}. Since y≠x, we have y∉G that implies ∃ a KS_{gs}-neighborhood P of x with y∉U. Since x≠y, x∉H that implies a KS_{gs}-neighborhood Q∋ y with x∉Q. Hence U is KS_{gs}-T_1 space.

Theorem 3. 2.4. For a KS topological space U, each of the following statement are equivalent:

(i) U is KS_{gs}-T_1 space.

(ii) The intersection of all KS_{gs} -open sets containing the set G is G.

(iii) The intersection of all KS_{gs} -open sets containing the point x ∈ U is {x}.

Proof: (i)⇒ (ii) Suppose U is KS_{gs} -T_1 space. By Theorem (3.2.3) each singleton set is KS_{gs} -closed in U. Let G ⊆ U. Then for each x ∈ U − G, {x} is KS_{gs}-closed in U and hence U − {x} is KS_g -open. Clearly, G ⊆ U − {x} for each x ∈ U − G. Therefore G ⊆ ∩{U − {x} : x ∈ U − G}. On the other hand, if y∉G then y ∈ U − G and y∉U − {y}. Therefore, y∉∩{U − {x} : x ∈ U − G} ⊆ G. Therefore, the intersection of all KS_{gs}-open sets containing the set G is G.

(ii)⇒ (iii) Suppose the intersection of all KS_{gs} -open sets containing the set G is G. Take G = {x}. Then G = {x} = ∩{H: H is KS_{gs} -open and x ∈ H}. Therefore, the intersection of all KS_{gs} -open sets containing the point x ∈ U is {x}.

(iii)⇒ (i) Let x, y ∈ U and y≠x. Then y∉{x} = ∩{H: H is KS_{gs} -open and x∈ H}. Hence, ∃ a KS_{gs} -open set H containing x but not y. Similarly, ∃ a KS_{gs} -open set H containing y but not x. Thus, U is KS_{gs} -T_1 space.

Theorem 3. 2.5. Each singleton set in space is KS_{gs}-closed.

Proof: Let U be KS_{gs} -T_2 space. Since U is KS_{gs} -T_2 space. This ⇒ U is KS_{gs} -T_1space.

⇒ {x} is KS_{gs} -closed for x ∈ U. Hence, each singleton set in KS_{gs}-T_2space is KSgs-closed.

Theorem 3.2.6. A subspace of KS_{gs} -T_2 space is KS_{gs} -T_2 space.

Proof: Let V be a subspace of KS_{gs} -T_2 space U. Let p, q ∈ V with $p \neq q$. Then p, q ∈ U.

Since U is KS_{gs} -T_2 space, ∃ KS_{gs} -open sets G and H such that p ∈ G, q ∈ H and G ∩ H = ϕ. Thus, we have G ∩ H, H ∩ V are KS_{gs} -open in V, (G ∩ V) ∩ (H ∩ V) = ϕ. p ∈ G ∩ V and q ∈ H ∩ V. Hence V is KS_{gs} -T_2 space.

Theorem 3. 2.7. In any KS topological space, the following are equivalent:

(i) U is KS_{gs} -T_2 space.

(ii) For each x≠y, ∃ a KS_{gs} -open set G ∋ x ∈ G and y∉ KS_{gscl}(G).

(iii) For each x ∈ U, {x} = ∩{KS_{gscl} (U): U is a KS-open set in U and x ∈ U}

Proof:

(i)⇒(ii) Assume (i) holds. Let x, y ∈ U and x≠y, then ∃ disjoint KS_{gs} -open sets G and H ∋ x ∈ G and y ∈ H. Clearly U − H is a KS_{gs} -closed set. Since G ∩ H = ϕ, G⊆U–H. Therefore, KS_{gscl} ⊆ KS_{gscl} (U − H) = U − H. Now y∉ U − H that implies y∉ KS_{gscl} (G).

(ii)⇒(iii) For each x≠y ∃ a KS_{gs}-open set G ∋ x ∈ G and y∉KS_{gscl} (G). So y∉ ∩{ KS_{gscl} (G): G is a KS_{gs}-open set in U and x ∈ G}={x}.

(iii)⇒(i) Let x, y ∈ U and x≠y. By hypothesis ∃ a KS_{gs}-open set G∋ x ∈ G and KS_{gs}(G). This implies ∃ a KS_{gs}-closed set H ∋ y∉H. Therefore, y ∈ U − H and U − H is a KS_{gs}-open set. Thus, ∃ two disjoint a KS_{gs} -open set G and U − H ∋ x ∈ G and y ∈ U − H. Therefore, U is a KS_{gs}-T_2 space.

Theorem 3. 2.8. A space U is KS_{gs}-T_4 space iff for any a KS_{gs}-open set A containing a KS_{gs}-closed set F ∃ a KS_{gs}-open set G F ⊆ G ⊆ KS_{gscl}(G) ⊆ A.

Proof: Assume that U is KS_{gs} -T_4 space. Since F and A^c are disjoint and KS_{gs} -closed sets in U, ∃ disjoint KS_{gs} -open sets G and H ∋ F ⊆ G and A^c ⊆ H. Since G and H are disjoint, G ⊆ H^c, we have KS_{gscl}(G) ⊆ H^c ⊆ A. Thus, we have a KS_{gs}-open set G ∋ F ⊆ G ⊆ KS_{gscl}(G) ⊆ A. Conversely, assume that the condition holds. Let A and B be disjoint KS_{gs}-closed sets in U. Since B^c is KS_{gs} -open and contains a KS_{gs}-closed set A by assumption, there is a KS_{gs}-open set V ∋ A ⊆ V ⊆ KS_{gscl}(V) ⊆ B^c. Thus, we have a KS_{gs}-open set V ⊇ A and $[KS_{gscl}(V)]^c$ ⊇ B. so U is KS_{gs}-T_4 space.

3.3 KS_{sg} -T_i(i=0,1,2,3,4) spaces

In this section, we define and discuss some properties of KS_{sg} -T_0, KS_{sg} -T_1, KS_{sg} -T_2, KS_{sg}-T_3 and KS_{sg}-T_4 spaces in kasaj topological spaces and obtain some of their basic properties.

Definition 3. 3.1. A KS topological space $(U, \tau_R(X), KS_R(X))$ is said to be

(i) KS_{sg}-T_0 spaces if given a pair of distinct points x, y ∈ U either ∃ a KS_{sg}-open set G ∈ $KS_R(X)$∋ x ∈ G, y∉G or ∃ a KS_{sg}-open set H ∈$KS_R(X)$∋ y ∈ H, x∉H.

(ii) KS_{sg}-T_1 spaces if given a pair of distinct points x, y ∈ U with x≠y, ∃ a KS_{sg}-open set G, H ∈ $KS_R(X)$ ∋ x ∈ G, y∉ G; y ∈ H, x∉ H.

(iii) KS_{sg}-T_2 spaces if given a pair of distinct points x, y ∈ U with x≠y, ∃ a KS_{sg}-open set G, H ∈$KS_R(X)$∋ x ∈ G, y ∈ H, G ∩ H = ϕ.

(iv) KS_{sg}-T_3 spaces if given an element x ∈ U and a KS_{sg}-closed set F ⊆ U ∋ x∉ F, ∃ disjoint KS_{sg}-open sets G_1, G_2 ⊆ U ∋ x ∈ G_1, F ⊆ G_2.

(v) KS_{sg}-T_4 spaces if given a pair of disjoint KS_{sg}-closed sets C_1, C_2 ⊆ U, ∃ disjoint KS_{sg}-open sets G_1, G_2 ⊆ U ∋ C_1 ⊆ G_1, C_2 ⊆ G_2.

Example 3. 3.2. Let U = {a,b,c,d,e} with U/R = {{c,d},{b,e},{a}} and X = {a,e}⊆ U. Then $\tau_R(X)$= { ϕ, U, {a}, {a, b, e}, {b, e}}. If we consider S={a,e}, S' ={b,c,d} then $KS_R(X)$={ ϕ, {a}, {b},{a, b}, {a, e}, {a, b, e},{b, c, d}, {a, b, c, d}, U } and KS_{sg}open={ϕ,{a},{b},{a,b},{a,e},{b,c},{b,d},{a,b,c},{a,b,d},{a,b,e},{a,c,e},{a,d,e},{b,c,d},{b,c,e},{b,d,e},{a,b,c,d},{a,b,c,e},{a,b,d,e}}

(i) Let a, b ∈ U, a≠b ∃ a KS_{sg} -open set ={a,e} ∋ a ∈ {a, e} and b∉{a, e}.

(ii) From Exemplar 3.3.2(i) and ∃ a KS_{sg}-open set ={b,c}∋ b ∈ {b, c} and a∉{b, c}.

(iii) From Exemplar 3.3.2((i)&(ii)) ∃ {a, e} ∩ {b, c} = ϕ.

(iv) Let d ∈ U,{a,c}= KS_{sg}-closed sets and d∉{a, c} ∃ {b,d} and {a,c,e} = KS_{sg}-open sets ∋d ∈ {b, d} and {a, c} ⊆ {a, c, e}.

(v) Let {c} and {e} = KS_{sg}-closed sets where {c}≠{e} ∃ {b,c} and {a,e} = KS_{sg}-open sets where {b, c}≠{a, e} ∋{c} ⊆ {b, c} and {e} ⊆ {a, e}.

Result 3.3.3. From the definitions which is defined above, we observe that

(i) Every KS-T_0, KS-semi-T_0, KS_α -T_0, KS_g -T_0 spaces are KS_{gs}(KS_{sg})-T_0 space.

(ii) Every KS-semi-T_1, KS_α -T_1, KS_g -T_1 spaces are KS_{gs}(KS_{sg})-T_0 and KS_{gs}(KS_{sg})-T_1 spaces.

(iii) Every KS-semi-T_2, KS_α - T_2, KS_g -T_2 spaces are KS_{gs}(KS_{sg})-T_0, KS_{gs}(KS_{sg})-T_1 and KS_{gs}(KS_{sg})-T_2 spaces.

(iv) Every KS-semi-T_3, KS_α -T_3, KS_g -T_3 spaces are KSgs(KSsg)-T_0, KSgs(KSsg)-T_0, KSgs(KSsg)-T_2 and KSgs(KSsg)-T_3 spaces.

(v) Every KS-semi-T_4, KSα -T_4, KSg -T_4 spaces are KSgs(KSsg)-T_4, KSgs(KSsg)-T_1, KSgs(KSsg)-T_2, KSgs(KSsg)-T_3 and KSgs(KSsg)-T_4 spaces.

Remark 3. 3.4. The following graphs are directed graphs obtained from the above results in which the directed arc U → V denotes that every U space is V space.

Theorem 3. 3.5. Let the topological space U is sg-T_3 space iff KS topological space U is KS_{sg}-T_3 space.

Proof: Suppose U is sg-T_3 space. Let x ∈ U and A ⊆ U is KS_{sg}-closed x ∈ U −A. Therefore, x ∈ U and A ⊆ U. Since U is sg-T_3 space, ∃ disjoint sg-open sets G, H ∈ U. x ∈ G and A ⊆ H. This implies that x ∈ G and A ∈ H. Since G and H are disjoint sg-open sets, we have G∩H = ϕ. Thus, G∩H = ϕ. Hence, G and H are disjoint KS_{sg}-open sets. This implies that U is KS_{sg}-T_3

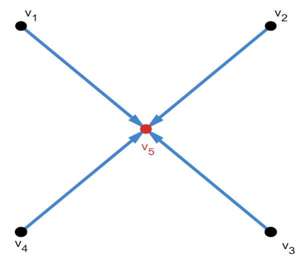

Fig 1. Vi (i = 1, 2, 3, 4) ie., V1 $\Rightarrow$ KS-T0, V2 $\Rightarrow$ KS-semi-T0, V3 $\Rightarrow$ KSα-T0, V4 $\Rightarrow$ KSg-T0 V_n(n = 5) ie., 5 $\Rightarrow$ $KS_{gs}(KS_{sg})$-T_0

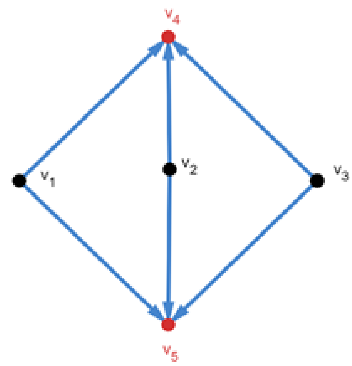

Fig 2. V_i (i = 1, 2, 3) ie.,$V_1 \Rightarrow$ KS-semi-T_1, $V_2 \Rightarrow KS_\alpha$ -T_1, $V_3 \Rightarrow KS_g$-T_1 V_n (n = 4, 5) ie.,$V_4 \Rightarrow KS_{gs}(KS_{sg}$)-T_0, $V_5 \Rightarrow KS_{gs}(KS_{sg})$-$T_1$

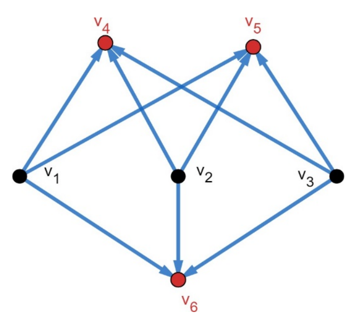

Fig 3. V_i (i = 1, 2, 3) ie., $V_1 \Rightarrow$ KS-semi-T_2, $V_2 \Rightarrow KS_\alpha$ -T_2, $V_3 \Rightarrow KS_g$-T_2 V_n (n = 4, 5, 6) ie., $V_4 \Rightarrow KS_{gs}(KS_{sg})$-$T_0$, $V_5 \Rightarrow KS_{gs}(KS_{sg})$-$T_1$, $V_6 \Rightarrow KS_{gs}(KS_{sg})$-$T_2$

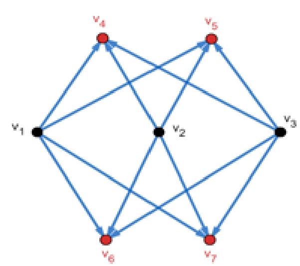

Fig 4. V_i (i = 1, 2, 3) ie., $V_1 \Rightarrow$ KS-semi-T_3, $V_2 \Rightarrow KS_\alpha$ -T_3, $V_3 \Rightarrow KS_g$-T_3 V_n (n = 4, 5, 6, 7) ie., $V_4 \Rightarrow KS_{gs}(KS_{sg}$)-T_0, $V_5 \Rightarrow KS_{gs}(KS_{sg})$-$T_1$, $V_6 \Rightarrow KS_{gs}(KS_{sg})$-$T_2$, $V_7 \Rightarrow KS_{gs}(KS_{sg})$-$T_3$

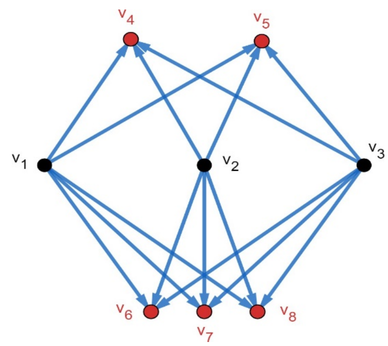

Fig 5. V_i (i = 1, 2, 3) ie., $V_1 \Rightarrow$ KS-semi-T_4, $V_2 \Rightarrow KS_\alpha$ -T_4, $V_3 \Rightarrow KS_g$-T_4 V_n (n = 4, 5, 6, 7, 8) ie., $V_4 \Rightarrow KS_{gs}(KS_{sg})$-$T_0$, $V_5 \Rightarrow KS_{gs}(KS_{sg})$-$T_1$, $V_6 \Rightarrow KS_{gs}(KS_{sg})$-$T_2$, $V_7 \Rightarrow KS_{gs}(KS_{sg})$-$T_3$, $V_8 \Rightarrow KS_{gs}(KS_{sg})$-$T_4$

space. Conversely, assume that U is KS_{sg}-T_3 space. Let $x \in U$ and A be sg-closed subset of U. Therefore, $x \in U$ and A is KS_{sg}-closed in U. Since U is KS_{sg}-T_3 space $\exists$ disjoint KS_{sg}-open sets G and H $\ni x \in G$ and $A \subseteq H$. Hence $x \in G$ and $A \subseteq H$. This proves that U is KS_{sg}-T_3 space.

Theorem 3.3.6. A subspace of KS_{sg}-T_3 space is KS_{sg}-T_3 space.

Proof: Let U be KS_{sg}-T_3 space and V be a subspace of U. To prove that V is KS_{sg}-T_3 space. Let $P \in V$ and F be a KS_{sg}-closed set in V such that $P \notin F$. So, $F = V \cap KS_{sgcl}(F)$. Since $P \notin F$. We see that $P \notin KS_{sgcl}(F)$. Since U is KS_{sg}-T_3 space, there exist disjoint KS_{sg}-open sets G and H in U such that $KS_{sgcl}(F) \subseteq G$, $P \in H$. Now $F \subseteq$ KSsgcl(F) $\subseteq G$. Since $F \subseteq V$, $F \subseteq V \cap G$. Since $P \in V$ and $P \in H$, $P \in V \cap H$. Further, $(V \cap G) \cap (V \cap H) = \phi$. since $G \cap H = \phi$. Thus, $V \cap G$, $V \cap H$ are KS_{sg}-open sets in V, $P \in V \cap H$, $F \subseteq V \cap G$ and $(V \cap G) \cap (V \cap H) = \phi$. Hence, V is KS_{sg}-T_3 space.

Theorem 3. 3.7. A KS topological space U is KS_{sg}-T_3 space iff for any $x \in U$ and KS_{sg}-neighborhood N of x, there is a KS_{sg}-open set $G \ni x \in G \subseteq KS_{sgcl}(G) \subseteq N$.

Proof: Assume that U is KS_{sg}-T_3 space and N is KS_{sg}-neighborhood of x. Then N^c is a KS_{sg}-closed set and $x \notin N^c$. Since U is T_3 space, there exist disjoint KS_{sg}-open sets G and H $\ni x \in G$ and $N^c \subseteq H$. So, $H^c \subseteq N$. Since $G \cap H = \phi$, $G \subseteq H^c \Rightarrow KS_{sgcl}(G) \subseteq H^c$. Since H^c is a KS_{sg}-closed set. Thus, $x \in G \subseteq KS_{sgcl}(G) \subseteq N$. Conversely, assume that the given condition is satisfied. Let F be a KS_{sg}-closed set in U and $x \notin F$. Since F^c is KS_{sg}-neighborhood of x, by assumption there is a KS_{sg}-open set $G \ni x \in G \subseteq KS_{sgcl}(G) \subseteq F^c$. Thus, the disjoint KS_{sg}-open sets G and $[KS_{sgcl}(G)]^c$ contains x and F respectively. Hence, U is KS_{sg}-T_3 space.

Theorem 3.3.8. The statements given below are equivalent

(i) U is KS_{sg}-T_3 space

(ii) For $x \in U$ and each KS_{sg}-open neighborhood U $\exists$ KS_{sg}-neighborhood of U such that $KS_{sgcl}(N) \subseteq U$.

Proof:

(i)$\Rightarrow$(ii) Let U be KS_{sg}-neighborhood of x, $\exists$ G belong to KS_{sg}-open in U $\ni x \in G \subseteq U$. Now G^c belongs to KS_{sg}-closed in U and $x \notin G^c$. From (i) $\exists$ P, Q disjoint a KS_{sg}-open set such that $G^c \subseteq P$, $x \in Q$, $P \cap Q = \phi$. So, $Q \subseteq M^c$. Now $KS_{sgcl}(Q) \subseteq KS_{sgcl}(P^c) = G^c$ and $G^c \subseteq P$. This implies $P^c \subseteq G \subseteq U$. Therefore, $KS_{sgcl}(Q) \subseteq U$.

(ii)$\Rightarrow$(i) Let KS_{sg}-closed F in U and $x \notin F$ or $x \in F^c$ and U is KS_{sg}-open and so F^c is KS_{sg}-neighborhood of x. By hypothesis, $\exists$ KS_{sg}-open neighborhood $N \ni x \in N$, $KS_{sgcl}(N) \subseteq F^c$. This implies $F \subseteq \{U - KS_{sgcl}(N)\}$ and $N \cap \{(U - KS_{sgcl}(N)\} = \phi$. Thus, U is KS_{sg}-T_3 space.

Theorem 3. 3.9. Let U is KS_{sg}-T_3 space iff for every G belongs KS_{sg}-closed in U and point $p \in (U - G)$ then $x \in U$, $G \subseteq N$ and $KS_{sgcl}(N) \cap KS_{sgcl}(U) = \phi$ where N and U are open sets.

Proof: Given that U is KS_{sg}-T_3 space. Let G belongs to KS_{sg}-closed in U and $U \notin G$. Then $p \in M$ and $G \subseteq N$ and $M \cap N = \phi$ where M and N are open sets. This implies $M \cap KS_{sgcl}(N) = \phi$. Since U is KS_{sg}-T_3 space, $p \in P$ and $KS_{sgcl}(N) \subseteq Q$, $P \cap N = \phi$ where P,Q are KS_{sg}-open. Also, $KS_{sgcl}(P) \cap Q = \phi$. Let $V = M \cap P$ then $p \in V$, $G \subseteq N$ and $KS_{sgcl}(N) \cap KS_{sgcl}(V) = \phi$ where N, V are KS_{sg}-open in U. Conversely, suppose for all G belongs to KS_{sg}-closed in U and $p \in (U - G)$, we have $p \in U$, $G \subseteq N$ and $KS_{sgcl}(N) \cap KS_{sgcl}(V) = \phi$ where N, V are KS_{sg}-open sets. This implies $p \in U$, $G \subseteq N$, and $V \cap N = \phi$. Therefore, U is KS_{sg}-T_3 space.

Theorem 3.3.10. For a KS topological space U the following are equivalent

(i) U is KS_{sg}-T_3 space

(ii) For every a KS_{sg} -open set G is a union of KS-semi clopen sets.

(iii) For every a KS_{sg} -closed set A is an intersection of KS-semi clopen sets

Proof:

(i)⇒(ii) Let G be a KS_{sg}-open set G and $x \in G$. If $A = U - G$, then A is a KS_{sg}-closed set. By assumption there exists disjoint KS-semi open subsets P and Q of U such that $x \in P$ and $A \subseteq Q$. If H = scl(P), then H is KS-semi clopen and $H \cap A \subseteq H \cap Q = \phi$. It follows that $x \in H \subseteq G$. Thus, G is a union of KS-semi clopen sets.

(ii)⇒(iii) This is obvious.

(iii)⇒(i) Let A be a KS_{sg}-closed set and $x \notin A$. By assumption, there exist KS-semi clopen set H such that $A \subseteq H$ and $x \notin H$. If $G = U - H$, then G is KS-semi open set containing x and $U \cap V = \phi$. Thus, U is KS_{sg}-T_3 space.

Theorem 3. 3.11. For KS topological space U the following are equivalent:

(i) U is a KS_{sg}-T_4 space.

(ii) For every a KS_{sg}-closed set A and every a KS_{sg}-open set G containing A. There is a KS-semi clopen set $H \ni A \subseteq H \subseteq G$.

Proof:

(i)⇒(ii) Let A be a KS_{sg}-closed set and G be a KS_{sg}-open set with $A \subseteq G$. Now we have $A \cap (U - G) = \phi$, hence ∃ disjoint KS-semi open sets V_1 and $V_2 \ni A \subseteq V1$ and $U - G \subseteq V_2$. If $H = KS_{scl}(V_1)$, then H is KS-semi clopen satisfying $A \subseteq H \subseteq G$.

(ii)⇒(i) This is obvious.

4 Conclusion

The class of KS-open and KS-closed sets has an important role to examine the separation axiom in kasaj topological space. In this work, we studied new types of separation axioms namely, KSg-Ti (i=0,1,2,3,4) spaces; KSgs-Ti, (i=0,1,2,3,4) spaces and KSsg-Ti, (i=0,1,2,3,4) spaces and the relation between spaces are discussed in new form, which are explained by direct graphs and named the vertices as V1, V2, V3, V4, V5, V6, V7, V8 in the section 3. Several characterizations and the relation between properties of the above sets of separation axioms are discussed and proved. Furthermore, useful results are investigated by comparing KSgs(KSsg)-T3 spaces and KSgs(KSsg)-T4 spaces in the context of these new concepts.

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