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RESEARCH ARTICLE

Unrestricted Mersenne and Mersenne-Lucas Hybrid Sequences

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Abstract

Objectives: The creation of Unrestricted Mersenne and Mersenne-Lucas Hybrid Sequences is the goal. **Methods:** Consider Mersenne and Mersenne-Lucas sequences associated with hybrid numbers. Then choosing the coefficients of hybrid numbers as arbitrary to find recurrence relations, generating functions and Binet formulas for the above sequences, and verifying the same through well-known identities. **Findings:** An infinite number of terms of the unrestricted Mersenne and Mersenne-Lucas hybrid sequences are found. We have verified these sequences through some well-known identities. **Novelty:** In contrast to the study of Mersenne and Mersenne-Lucas hybrid sequences, where the coefficients of the ordered basis of the subsequent components of the sequences were chosen, here we choose arbitrary coefficients for these sequences.

Keywords: Mersenne Sequence; MersenneLucas Sequence; Hybrid numbers; Unrestricted Sequences; Binet Formula

1 Introduction

A number of the form $M_n = 2^n - 1$, where n is an integer, was first introduced in 1644 by French mathematician Marin Mersenne. The Mersenne sequences have been the subject of numerous investigations. The Mersenne-Lucas sequences are defined as $ML_n = 2^n + 1,\ n \geq 2$ with $ML_0 = 2,\ ML_1 = 3$. At (1,2), we outlined the Mersenne Lucas sequences, which include its generating functions and Binet formulas. In 2018, Ozdemir proposed the hybrid number, which comprises real, complex, dual, and hyperbolic numbers. It is of the form $H = z_0 + z_1 i + z_2 \varepsilon + z_3 h$, where $z_0, z_1, z_2, z_3 \in R$ and $i,\ \varepsilon,\ h$ are operators such that $i^2 = -1,\ \varepsilon^2 = 0,\ h^2 = 1,\ ih = -hi = i + \varepsilon$.

A number of authors delved into hybrid numbers with coefficients that are familiar sequences, such as Leonardo (3), Horadam (4), Fibonacci (5), Lucas (6), Pell (7), Jacobsthal (8,9), Mersenne (10), and Mersenne-Lucas (11). The authors in the studies above select the ordered basis coefficients for successive sequence elements, while we are using arbitrary Mersenne and Mersenne-Lucas sequences that serve as the basis for hybrid numbers. To learn more regarding this topic, see (12–15).

2 Method of Analysis

For any integers $p,\ q,\ v$ and non-negative integer n, unrestricted Mersenne and Mersenne-Lucas hybrid sequences are

$$M_n^{(p,\,q,\,v)} = M_n + M_{n+p}i + M_{n+q}\varepsilon + M_{n+v}h$$

$$ML_n^{(,\,q,\,v)} = ML_n + ML_{n+p}i + ML_{n+q}\varepsilon + ML_{n+v}h$$

where M_n and ML_n are the n^{th} Mersenne and Mersenne-Lucas sequences respectively.

Unrestricted Mersenne and Mersenne-Lucas hybrid sequences satisfy the recurrence relations
$M_n^{(p,\,q,\,v)} = 3M_{n-1}^{(p,\,q,\,v)} - 2M_{n-2}^{(p,\,q,\,v)}$ and
$ML_n^{(p,\,q,\,v)} = 3ML_{n-1}^{(p,\,q,\,v)} - 2ML_{n-2}^{(p,\,q,\,v)}$

3 Result and discussion

3.1 Generating Function

The generating functions for the sequences $\left\{M_n^{(p,\,q,\,v)}\right\}_{n=0}^{\infty}$ and $\left\{ML_n^{(p,\,q,\,v)}\right\}_{n=0}^{\infty}$ are
$M(x) = \frac{M_0^{(p,\,q,\,v)} + \left(M_1^{(p,\,q,\,v)} - 3M_0^{(p,\,q,\,v)}\right)x}{1-3x+2x^2}$ and

$$ML(x) = \frac{ML_0^{(p,\,q,\,v)} + \left(ML_1^{(p,\,q,\,v)} - 3ML_0^{(p,\,q,\,v)}\right)x}{1-3x+2x^2}$$

Proof

Let us define $M(x) = \sum_{n=0}^{\infty} M_n^{(p,\,q,\,v)} x^n$
$M(x) = M_0^{(p,\,q,\,v)} + xM_1^{(p,\,q,\,v)} + \sum_{n=2}^{\infty} M_n^{(p,\,q,\,v)} x^n$
Multiply this equation by $-3x$ and $2x^2$, we obtain

$$-3M(x)\,x = -3xM_0^{(p,\,q,\,v)} - 3\sum_{n=2}^{\infty} M_{n-1}^{(p,\,q,\,v)} x^n$$

$$2M(x)\,x^2 = 2\sum_{n=2}^{\infty} M_{n-2}^{(p,\,q,\,v)} x^n$$

Adding these three equations, we get

$$\left(1-3x+2x^2\right)M(x)$$

$$= M_0^{(p,\,q,\,v)} + x\left(M_1^{(p,\,q,\,v)} - 3M_0^{(p,\,q,\,v)}\right) + \sum_{n=2}^{\infty}\left(M_n^{(p,\,q,\,v)} - 3M_{n-1}^{(p,\,q,\,v)} + 2M_{n-2}^{(p,\,q,\,v)}\right] x^n$$

Therefore, $M(x) = \frac{M_0^{(p,\,q,\,v)} + \left(M_1^{(p,\,q,\,v)} - 3M_0^{(p,\,q,\,v)}\right)x}{1-3x+2x^2}$.

In a similar way, we obtain $ML(x) = \frac{ML_0^{(p,\,q,\,v)} + \left(ML_1^{(p,\,q,\,v)} - 3ML_0^{(p,\,q,\,v)}\right)x}{1-3x+2x^2}$.

3.2 Binet formula

For any integers $p,\ q,\ v$ and n, the unrestricted Mersenne and Mersenne-Lucas hybrid sequences are

$M_n^{(p,\,q,\,v)} = 2^n a - b$ and $ML_n^{(p,\,q,\,v)} = 2^n a + b$

where $a = 1 + 2^p i + 2^q \varepsilon + 2^v h,\ \ b = 1 + i + \varepsilon + h.$

Proof

$$M_n^{(p,\,q,\,v)} = M_n + M_{n+p} i + M_{n+q}\varepsilon + M_{n+v} h$$

$$= (2^n - 1) + \left(2^{n+p} - 1\right) i + \left(2^{n+q} - 1\right)\varepsilon + \left(2^{n+v} - 1\right)$$

$$= 2^n \left(1 + 2^p i + 2^q \varepsilon + 2^v h\right) - (1 + i + \varepsilon + h)$$

$= 2^n a - b.$

$$ML_n^{(p,\,q,\,v)} = ML_n + ML_{n+p} i + ML_{n+q}\varepsilon + ML_{n+v} h$$

$$= (2^n + 1) + \left(2^{n+p} + 1\right) i + \left(2^{n+q} + 1\right)\varepsilon + \left(2^{n+v} + 1\right) h$$

$$= 2^n \left(1 + 2^p i + 2^q \varepsilon + 2^v h\right) + (1 + i + \varepsilon + h)$$

$= 2^n a + b.$

3.3 Vajda Identity

For any integers $p,\ q,\ v, m, n$ and k, we have

$$M_{m+n}^{(p,\,q,\,v)} M_{m+k}^{(p,\,q,\,v)} - M_m^{(p,\,q,\,v)} M_{m+n+k}^{(p,\,q,\,v)} = 2^m M_n \left(2^k ba - ab\right)$$

$$ML_{m+n}^{(p,\,q,\,v)} ML_{m+k}^{(p,\,q,\,v)} - ML_m^{(p,\,q,\,v)} ML_{m+n+k}^{(p,\,q,\,v)} = 2^m M_n \left(ab - 2^k ba\right)$$

Proof

$$M_{m+n}^{(p,\,q,\,v)} M_{m+k}^{(p,\,q,\,v)} - M_m^{(p,\,q,\,v)} M_{m+n+k}^{(p,\,q,\,v)}$$

$$= \left(2^{m+n} a - b\right)\left(2^{m+k} a - b\right) - (2^m a - b)(2^{m+n+k} a - b)$$

$$= 2^{m+k} ba\,(2^n - 1) - 2^m ab(2^n - 1)$$

$= 2^m M_n \left(2^k ba - ab\right).$

$$ML_{m+n}^{(p,\,q,\,v)} ML_{m+k}^{(p,\,q,\,v)} - ML_m^{(p,\,q,\,v)} ML_{m+n+k}^{(p,\,q,\,v)}$$

$$= \left(2^{m+n} a + b\right)\left(2^{m+k} a + b\right) - (2^m a + b)(2^{m+n+k} a + b)$$

$$= -2^{m+k} ba\,(2^n - 1) + 2^m ab(2^n - 1)$$

$= 2^m M_n \left(ab - 2^k ba\right).$

If we substitute $k \to -n$ in the Vajda identities, we obtain the Catalan's identities.

3.4 Catalan's identity

For any integers $p,\ q,\ v, m$ and $n,$ we have

$$M_{m+n}^{(p,\,q,\,v)} M_{m-n}^{(p,\,q,\,v)} - \left(M_m^{(p,\,q,\,v)}\right]^2 = 2^{m-n} M_n\left(ba - 2^n ab\right)$$

$$ML_{m+n}^{(p,\,q,\,v)} ML_{m-n}^{(p,\,q,\,v)} - \left(ML_m^{(p,\,q,\,v)}\right]^2 = 2^{m-n} M_n\left(2^n ab - ba\right)$$

If we substitute $n \to 1$ in Catalan's identities, we get Cassini's identities.

3.5 Cassini's identity

For any integers $p,\ q,\ v$ and $m,$ we have

$$M_{m+1}^{(p,\,q,\,v)} M_{m-1}^{(p,\,q,\,v)} - \left(M_m^{(p,\,q,\,v)}\right]^2 = 2^{m-1}\left(ba - 2ab\right)$$

$$ML_{m+1}^{(p,\,q,\,v)} ML_{m-1}^{(p,\,q,\,v)} - \left(ML_m^{(p,\,q,\,v)}\right]^2 = 2^{m-1}\left(2ab - ba\right)$$

3.6 D'Ocagne's identity

For any integers $p,\ q,\ v, m$ and $n,$ we have

$$M_m^{(p,\,q,\,v)} M_{n+1}^{(p,\,q,\,v)} - M_{m+1}^{(p,\,q,\,v)} M_n^{(p,\,q,\,v)} = 2^m ab - 2^n ba$$

$$ML_m^{(p,\,q,\,v)} ML_{n+1}^{(p,\,q,\,v)} - ML_{m+1}^{(p,\,q,\,v)} ML_n^{(p,\,q,\,v)} = 2^n ba - 2^m ab$$

Proof

$$M_m^{(p,\,q,\,v)} M_{n+1}^{(p,\,q,\,v)} - M_{m+1}^{(p,\,q,\,v)} M_n^{(p,\,q,\,v)}$$

$$= \left(2^m a - b\right)\left(2^{n+1} a - b\right) - \left(2^{m+1} a - b\right)\left(2^n a - b\right)$$

$$= 2^n ba\left(1 - 2\right) - 2^m ab(1 - 2)$$

$= 2^m ab - 2^n ba.$

$$ML_m^{(p,\,q,\,v)} ML_{n+1}^{(p,\,q,\,v)} - ML_{m+1}^{(p,\,q,\,v)} ML_n^{(p,\,q,\,v)}$$

$$= \left(2^m a + b\right)\left(2^{n+1} a + b\right) - \left(2^{m+1} a + b\right)\left(2^n a + b\right)$$

$$= 2^n ba\left(2 - 1\right) - 2^m ab(2 - 1)$$

$= 2^n ba - 2^m ab.$

3.7 Honsberger Identity

For any integers $p,\ q,\ v, m$ and $n,$ we have

i. $M_{m-1}^{(p,\,q,\,v)} M_n^{(p,\,q,\,v)} + M_m^{(p,\,q,\,v)} M_{n+1}^{(p,\,q,\,v)}$

$$= 2^{m+n-1} a^2 ML_2 - 2^n baML_1 - 2^{m-1} abML_1 + 2b^2$$

ii. $ML_{m-1}^{(p,\,q,\,v)} ML_n^{(p,\,q,\,v)} + ML_m^{(p,\,q,\,v)} ML_{n+1}^{(p,\,q,\,v)}$

$$= 2^{m+n-1} a^2 ML_2 + 2^n baML_1 + 2^{m-1} abML_1 + 2b^2$$

Proof

i.

$$M_{m-1}^{(p,\,q,\,v)} M_n^{(p,\,q,\,v)} + M_m^{(p,\,q,\,v)} M_{n+1}^{(p,\,q,\,v)}$$

$$= \left(2^{m-1} a - b\right)\left(2^n a - b\right) + \left(2^m a - b\right)\left(2^{n+1} a - b\right)$$

$$= 2^{m+n-1} a^2 \left(2^2 + 1\right) - 2^n ba\left(2 + 1\right) - 2^{m-1} ab\left(2 + 1\right) + 2b^2$$

$= 2^{m+n-1} a^2 ML_2 - 2^n baML_1 - 2^{m-1} abML_1 + 2b^2.$

ii. $ML_{m-1}^{(p,\,q,\,v)} ML_n^{(p,\,q,\,v)} + ML_m^{(p,\,q,\,v)} ML_{n+1}^{(p,\,q,\,v)}$

$$= \left(2^{m-1} a + b\right)\left(2^n a + b\right) + \left(2^m a + b\right)\left(2^{n+1} a + b\right)$$

$$= 2^{m+n-1} a^2 \left(2^2 + 1\right) + 2^n ba\left(2 + 1\right) + 2^{m-1} ab\left(2 + 1\right) + 2b^2$$

$= 2^{m+n-1} a^2 ML_2 + 2^n baML_1 + 2^{m-1} abML_1 + 2b^2.$

3.8 Theorem

For any integers $p,\ q,\ v$ and $n,$ we have

i. $M_n^{(p,\,q,\,v)} ML_n^{(p,\,q,\,v)} = 2M_n^{(p,\,q,\,v)} - M_{2n} - M_{2(n+p)} + M_{2(n+q)} + 2^{n+q+1} M_{v-q}\varepsilon - 2^{n+p+1} M_{q-p}\left(1 + h\right) + 2^{n+p+1} M_{v-p}(i + \varepsilon)$

ii. $M_n^{(p,\,q,\,v)} + ML_n^{(p,\,q,\,v)} = 2^{n+1} a$

iii. $M_n^{(p,\,q,\,v)} - ML_n^{(p,\,q,\,v)} = -2b$

Proof

i. $M_n^{(p,\,q,\,v)} ML_n^{(p,\,q,\,v)}$

$$= \left(M_n + M_{n+p} i + M_{n+q}\varepsilon + M_{n+v} h\right)\left(ML_n + ML_{n+p} i + ML_{n+q}\varepsilon + ML_{n+v} h\right)$$

$= M_{2n} - M_{2(n+p)} - M_{n+q} ML_{n+p} + M_{n+p} ML_{n+q} + M_{2(n+v)} + i\left(M_{n+p} ML_n + M_n ML_{n+p} - M_{n+v} ML_{n+p} + M_{n+p} ML_{n+v}\right] +$
$\varepsilon\left(M_{n+q} ML_n - M_{n+v} ML_{n+p} + M_n ML_{n+q} + M_{n+v} ML_{n+q} + M_{n+p} ML_{n+v} - M_{n+q} ML_{n+v}\right] +$
$h\left(M_{n+v} ML_n + M_{n+q} ML_{n+p} - M_{n+p} ML_{n+q} + M_n ML_{n+v}\right]$

$= M_{2n} - M_{2(n+p)} + M_{2(n+v)} - \left(2^{n+p+1} M_{q-p}\right) + i\left(2M_{2n+p} - 2^{n+p+1} M_{v-p}\right] + \varepsilon\left(2M_{2n+q} - 2^{n+p+1} M_{v-p} + 2^{n+q+1} M_{v-q}\right] +$
$h\left(2M_{2n+v} + 2^{n+p+1} M_{q-p}\right]$

$$= \left(M_{2n} + iM_{2n+p} + \varepsilon M_{2n+q} + hM_{2n+\upsilon}\right) - M_{2(n+p)} + M_{2(n+\upsilon)} + iM_{2n+p} + \varepsilon M_{2n+q} + hM_{2n+\upsilon} - 2^{n+p+1}\left(M_{q-p} - M_{\upsilon-p}i - M_{\upsilon-p}\varepsilon + M_{q-p}h\right] + 2^{n+w+1}M_{\upsilon-q}\varepsilon$$

$$= M_{2n}^{(p,\,q,\,v)} + \left(M_{2n}^{(p,\,q,\,v)} - M_{2n}\right) - M_{2(n+p)} + M_{2(n+\upsilon)} - 2^{n+p+1}\left(M_{q-p} - M_{\upsilon-p}i - M_{\upsilon-p}\varepsilon + M_{q-p}h\right] + 2^{n+q+1}M_{\upsilon-q}\varepsilon$$

$= 2M_n^{(p,\,q,\,v)} - M_{2n} - M_{2(n+p)} + M_{2(n+\upsilon)} + 2^{n+q+1}M_{\upsilon-q}\varepsilon - 2^{n+p+1}M_{q-p}(1+h) + 2^{n+p+1}M_{\upsilon-p}(i+\varepsilon).$

ii. $M_n^{(p,\,q,\,v)} + ML_n^{(p,\,q,\,v)}$

$$= (M_n + M_{n+p}i + M_{n+q}\varepsilon + M_{n+\upsilon}h) + (ML_n + ML_{n+p}i + ML_{n+q}\varepsilon + ML_{n+\upsilon}h)$$

$$= (M_n + ML_n) + i\,(M_{n+p} + ML_{n+p}) + \varepsilon\,(M_{n+q} + ML_{n+q}) + h\,(M_{n+\upsilon} + ML_{n+\upsilon})$$

$$= 2^{n+1}(1 + i2^p + \varepsilon 2^q + h2^v)$$

$= 2^{n+1}a.$

iii. $M_n^{(p,\,q,\,v)} - ML_n^{(p,\,q,\,v)}$

$$= (M_n + M_{n+p}i + M_{n+q}\varepsilon + M_{n+\upsilon}h) - (ML_n + ML_{n+p}i + ML_{n+q}\varepsilon + ML_{n+\upsilon}h)$$

$$= (M_n - ML_n) + i\,(M_{n+p} - ML_{n+p}) + \varepsilon\,(M_{n+q} - ML_{n+q}) + h\,(M_{n+\upsilon} - ML_{n+\upsilon})$$

$$= -2(1 + i + \varepsilon + h)$$

$= -2b.$

3.9 Theorem

For any integers $p,\ q,\ v$ and n, we have

i. $M_{n+1}^{(p,\,q,\,v)} + M_n^{(p,\,q,\,v)} = 3\,(2^n)\,a - 2b$

ii. $ML_{n+1}^{(p,\,q,\,v)} + ML_n^{(p,\,q,\,v)} = 3\,(2^n)\,a + 2b$

Proof

i. $M_{n+1}^{(p,\,q,\,v)} + M_n^{(p,\,q,\,v)}$

$$= (M_{n+1} + M_n) + i\,(M_{n+p+1} + M_{n+p}) + \varepsilon\left(M_{n+q+1} + M_{n+q}\right) + h\,(M_{n+\upsilon+1} + M_{n+\upsilon})$$

$$= 3\,(2^n)\,(1 + i2^p + \varepsilon 2^q + h2^v] - 2(1 + i + \varepsilon + h)$$

$= 3\,(2^n)\,a - 2b.$

ii. $ML_{n+1}^{(p,\,q,\,v)} + ML_n^{(p,\,q,\,v)}$

$$= (ML_{n+1} + ML_n) + i\,(ML_{n+p+1} + ML_{n+p}) + \varepsilon\left(ML_{n+q+1} + ML_{n+q}\right) + h\,(ML_{n+\upsilon+1} + ML_{n+\upsilon})$$

$$= 3\,(2^n)\,(1 + i2^p + \varepsilon 2^q + h2^v] + 2(1 + i + \varepsilon + h)$$

$= 3\,(2^n)\,a + 2b.$

3.10 Theorem

For any integers $p,\ q,\ v$ and $n,$ we have

i. $3M_{n+1}^{(p,\,q,\,v)} - 2M_{n}^{(p,\,q,\,v)} = M_{n+2}^{(p,\,q,\,v)}$

ii. $M_{n}^{(p,\,q,\,v)} + M_{n+p}^{(p,\,q,\,v)} i + M_{n+q}^{(p,\,q,\,v)} \varepsilon + M_{n+\upsilon}^{(p,\,q,\,v)} h$

$$= 2M_{n}^{(p,\,q,\,v)} - M_{n} - M_{n+2p} + M_{n+2\upsilon}$$

Proof

i. $3M_{n+1}^{(p,\,q,\,v)} - 2M_{n}^{(p,\,q,\,v)}$

$$= (3M_{n+1} - 2M_{n}) + i\,(3M_{n+p+1} - 2M_{n+p}) + \varepsilon\left(3M_{n+q+1} - 2M_{n+q}\right) + h\,(3M_{n+\upsilon+1} - 2M_{n+\upsilon})$$

$$= M_{n+2} + M_{n+p+2} i + M_{n+q+2}\varepsilon + M_{n+\upsilon+2} h$$

$= M_{n+2}^{(p,\,q,\,v)}.$

ii. $M_{n}^{(p,\,q,\,v)} + M_{n+p}^{(p,\,q,\,v)} i + M_{n+q}^{(p,\,q,\,v)} \varepsilon + M_{n+\upsilon}^{(p,\,q,\,v)} h$

$= (M_{n} + M_{n+p} i + M_{n+q}\varepsilon + M_{n+\upsilon} h) + (M_{n+p} + M_{n+2p} i + M_{n+p+q}\varepsilon + M_{n+p+\upsilon} h)\, i + \left(M_{n+q} + M_{n+p+q} i + M_{n+2q}\varepsilon + M_{n+q+\upsilon} h\right)\varepsilon +$
$(M_{n+\upsilon} + M_{n+p+\upsilon} i + M_{n+q+\upsilon}\varepsilon + M_{n+2\upsilon} h)\, h$

$= M_{n}^{(p,\,q,\,v)} + (M_{n+p} i - M_{n+2p} + M_{n+p+q} - M_{n+p+q} h + M_{n+p+\upsilon}\varepsilon + M_{n+p+\upsilon} i] + (M_{n+q}\varepsilon + M_{n+p+q} h - M_{n+p+q} - M_{n+q+\upsilon}\varepsilon] +$
$(M_{n+\upsilon} h - M_{n+p+\upsilon}\varepsilon - M_{n+p+\upsilon} i + M_{n+q+\upsilon}\varepsilon + M_{n+2\upsilon}]$

$$= M_{n}^{(p,\,q,\,v)} - M_{n+2p} + M_{n+2\upsilon} + M_{n+p} i + M_{n+q}\varepsilon + M_{n+\upsilon} h$$

$= 2M_{n}^{(p,\,q,\,v)} - M_{n} - M_{n+2p} + M_{n+2\upsilon}.$

3.11 Theorem

For any integers $p,\ q,\ v$ and $n,$ we have

i. $(M_{n}^{(p,\,q,\,v)})^2 + (M_{n+1}^{(p,\,q,\,v)})^2 + (M_{n+2}^{(p,\,q,\,v)})^2 = 21\left(2^{2n}\right)a^2 + 3b^2 - 7(2^n)(ba + ab)$

ii. $(ML_{n+1}^{(p,\,q,\,v)})^2 - (ML_{n}^{(p,\,q,\,v\upsilon)})^2 = 2^{2n}a^2 M_2 + 2^n(ab + ba)$

Proof

i. $(M_{n}^{(p,\,q,\,v)})^2 = 2^{2n}a^2 - 2^n ba - 2^n ab + b^2$

$$(M_{n}^{(p,\,q,\,v)})^2 = 2^{2n}a^2 - 2^n ba - 2^n ab + b^2$$

$$(M_{n+2}^{(p,\,q,\,v)})^2 = 2^{2n+4}a^2 - 2^{n+2} ba - 2^{n+2} ab + b^2$$

$$(M_{n}^{(p,\,q,\,v)})^2 + (M_{n+1}^{(p,\,q,\,v)})^2 + (M_{n+2}^{(p,\,q,\,v)})^2$$

$$= 2^{2n}a^2(1 + 2^2 + 2^4) - 2^n ba(1 + 2 + 2^2) - 2^n ab(1 + 2 + 2^2) + 3b^2$$

$= 21\left(2^{2n}\right)a^2 + 3b^2 - 7(2^n)(ba + ab).$

ii. $(ML_n^{(p,\,q,\,v)})^2 = 2^{2n}a^2 + 2^n ba + 2^n ab + b^2$

$$(ML_{n+1}^{(p,\,q,\,v)})^2 = 2^{2n+2}a^2 + 2^{n+1} ba + 2^{n+1} ab + b^2$$

$$(ML_{n+1}^{(p,\,q,\,v)})^2 - (ML_n^{(p,\,q,\,v)})^2$$

$$= 2^{2n}a^2(2^2-1) + 2^n ba(2-1) + 2^n ab(2-1)$$

$= 2^{2n}a^2 M_2 + 2^n(ab+ba).$

4 Conclusion

An infinite number of terms of the unrestricted Mersenne and Mersenne-Lucas hybrid sequences are found. One can generate various number patterns whose characteristics satisfying unrestricted hybrid sequences.

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