INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Existence and Non - Existence of Exponential Diophantine Triangles Over Triangular Numbers

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OPEN ACCESS

Received: 14-09-2023
Accepted: 24-09-2023
Published: 31-10-2023

Citation: Mahalakshmi M, Kannan J, Deepshika A, Kaleeswari K (2023) Existence and Non - Existence of Exponential Diophantine Triangles Over Triangular Numbers . Indian Journal of Science and Technology 16(41): 3599-3604. https://doi.org/10.17485/IJST/v16i41.2338

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Funding: None

Competing Interests: None

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Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objectives: The specified problem addressed here is the existence and non-existence of Exponential Diophantine triangles over triangular numbers ($t_n,\ n \in \mathbb{N}$). **Methods:** An Exponential Diophantine triangle over triangular numbers ($t_n,\ n \in \mathbb{N}$) is defined as a triangle with sides $nx+1, ny+2$ and nz where $x, y,$ and z are non - negative integers such that $t_n^x + t_{n+1}^y = z^2$. To prove the existence of such triangles, negative Pell's equation and its solutions are used along with some basic number theoretic concepts. To verify the non-existence, the well-known Catalan's conjecture, binomial expansion, and various theories concerning congruence are employed. **Findings:** Here it is proved that, for five different choices of sides, an Exponential Diophantine triangle over t_n can be constructed. In particular, infinitely many such triangles can be found. For some particular choice of sides, Python coding is displayed along with its output to verify the existence of required triangles. On the other side, another five different choices of sides are considered and it is shown that no considered type of triangles exists in these cases. **Novelty:** The idea of solving an exponential Diophantine equation and the idea of constructing triangles under some conditions using Diophantine equations already exists in the mathematical society. This article is created uniquely by combining these two concepts along with the innovative usage of exponential Diophantine equations.

Keywords: Exponential Diophantine triangle; Exponential Diophantine Triangle over triangular numbers; Exponential Diophantine equation; Triangles; Triangular numbers

1 Introduction

In this article, an Exponential Diophantine triangle over triangular numbers (t_n) is defined as a triangle with sides $nx+1, ny+2$ and nz where $x, y,$ and z are non-negative integers such that $t_n^x + t_{n+1}^y = z^2$. Hereafter, throughout this paper, such triangles are denoted as ED triangles over t_n. The research problem taken for study is the existence and non-existence of such triangles. That is, here are the situations under which an ED triangle over t_n exists and the situations under those triangles not exist are discussed.

With a deep point of view, one can observe that the research problem considered here depends mainly on solving the exponential Diophantine equation over non-negative integers. This article is motivated by the works [1,2] and can be considered as next-level research in this area.

Previous works of researchers involve solving the exponential Diophantine equations alone or construct a geometrical shape with some special conditions through Diophantine equations such as Pythagorean equations, Pell equations [3–12]. This work connects both the ideas and it is undertaken to move this type of research to a somewhat different extent.

Let us discuss some key terms used in this article. First of all, the dominating factor in this work is the study of exponential Diophantine equation $t_n^x + t_{n+1}^y = z^2$ (An exponential Diophantine equation is a type of Diophantine equation in which the exponents are known or unknown variables). To obtain the existence and non-existence, one has to solve this exponential Diophantine equation. Apart from using the usual Catalan's conjecture, basic properties of congruence, we employ certain factors in addition to solve this equation. Such factors include binomial expansion and negative Pell's equation.

Excluding introduction, conclusion, and references, this article comprises three different sections: Methodology (section 2), Existence of ED triangles over t_n (subsection 3.1) and non-existence of ED triangles over t_n (subsection 3.2). In each of these sections, the article successfully attained its aim.

2 Methodology

This section includes the necessary preliminary things needed for the main work of this article. It includes Catalan's conjecture, Negative Pell's equation, and some elementary results.

Conjecture 2.1. $(3,2,2,3)$ is a unique solution (a,b,x,y) for the Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers such that $min\{a,b,x,y\} > 1$ [12].

This conjecture is known as Catalan's conjecture.

Definition 2.2. The equation $x^2 - dy^2 = -1$ is called the Negative Pell's equation where $d \in \mathbb{N}$ and it is not a perfect square.

Definition 2.3. The equation $x^2 - dy^2 = -n$ is called the Generalized Negative Pell's equation where $d, d \in \mathbb{N}$ and d is not a perfect square.

Lemma 2.4. If l and k are positive integer solutions of the generalized negative Pell's equation $l^2 - 8k^2 = -7$, then the values of l and k (say l_r & k_r, $r = 1,2,3,\cdots$) can be obtained by the relations

$$\left.\begin{aligned} l_r &= 6l_{r-2} - l_{r-4} \\ k_r &= 6k_{r-2} - k_{r-4} \end{aligned}\right\} \tag{1}$$

for $r \geq 5$, provided $l_1 = 1, l_2 = 5, l_3 = 11, l_4 = 31, k_1 = 1, k_2 = 2, k_3 = 4$ and $k_4 = 11$.

Proof. We aim to show that Equation (1) satisfies the relation $l_r^2 - 8k_r^2 = -7$ for all $r \geq 5$. Using Equation (1), one can write

$$l_r^2 - 8k_r^2 = -259 - 12[l_{r-2}l_{r-4} - 8k_{r-2}k_{r-4}] \tag{2}$$

Let us show for $r \geq 5$ by induction that

$$l_{r-2}l_{r-4} - 8k_{r-2}k_{r-4} = -21 \tag{3}$$

Take $r = 5$. Then

$$\begin{aligned} l_{r-2}l_{r-4} - 8k_{r-2}k_{r-4} &= l_3 l_1 - 8k_3 k_1 \\ &= 11(1) - 8(4)(1) \\ &= -21 \end{aligned}$$

Thus, Equation (3) holds for $r = 5$.

Now assume that the Equation (3) holds for $5 < r \leq t$. In particular, it holds for $r = t-1$. Then $l_{t-3}l_{t-5} - 8k_{t-3}k_{t-5} = -21$. Let $r = t+1$. Then

$$\begin{aligned} l_{r-2}l_{r-4} - 8k_{r-2}k_{r-4} &= l_{t-1}l_{t-3} - 8k_{t-1}k_{t-3} \\ &= (6l_{t-3} - l_{t-5})\, l_{t-3} - 8\,(6k_{t-3} - k_{t-5})\, k_{t-3} \\ &= -42 - (l_{t-3}l_{t-5} - 8k_{t-3}k_{t-5}) \\ &= -42 + 21 \\ &= -21 \end{aligned}$$

Hence, Equation (3) is true for all $r \geq 5$.

The result follows once Equation (3) is substituted in Equation (2).

Lemma 2.5. For $n \in N$ and $n > 1$, $n^4+2n^3+3n^2+6n+4$ is not a perfect square.

Proof. Suppose that $n^4+2n^3+3n^2+6n+4 = \left(n^2+an+b\right)^2$. Then comparing the coefficient of n^3 and the constant term, we get $b^2 = 4,\ 2a = 2$. These choices imply $b = 2,\ a = 1$. This is a possible choice of a and b. Also, we need to equate the terms $3n^2+6n$ and $\left(a^2+2b\right)n^2+2abn$. If we do so, we obtain $n = \frac{2(ab-3)}{3-a^2-2b}$. Since $n \in \mathbb{N}$, we have two possibilities:

i) $ab > 3$ and $a^2+2b < 3$

ii) $ab < 3$ and $a^2+2b > 3$

For the case (i), we could not have any suitable a and b. But for case (ii), we have two suitable a and b, as $a = 1, b = 2$(which was the earlier one) and $a = 2, b = 1$(an impossible one). Thus, the only choice of a and b is $a = 1$ and $b = 2$. This leads to the value of n as 1. This completes the proof.

Lemma 2.6. For $n \in \mathbb{N}$ and $n > 2$, $n^4+4n^3+11n^2+2n+4$ is not a perfect square.

Proof. The proof is similar to that of Lemma 2.5.

Definition 2.7. For $n \in \mathbb{N}$, a triangular number t_n is defined as $t_n = \frac{n(n+1)}{2}$.

Lemma 2.8. The sum of two consecutive triangular numbers is a perfect square. In particular, $t_n + t_{n+1} = (n+1)^2$.

These are the methods or results we employ in this paper to solve the exponential Diophantine equation.

3 Results and Discussion

This section is split into two subsections 3.1 and 3.2. In subsection 3.1, it is discussed and proved that the ED triangles over t_n exist for the following sides:

- $1, n+2$ and nz
- $n+1,\ 2$ and nz
- $n+1,\ n+2$ and nz
- $x+1,\ 3,$ and $z(x>1)$
- $nx+1, ny+2,$ and $nz(x,y>1)$

In subsection 3.2, it is discussed and proved that the ED triangles over t_n does not exist for the following sides:

- $1,\ 2,$ and nz
- $1,\ ny+2$ and $nz(y>1)$
- $nx+1,\ 2$ and $nz(x>1)$
- $nx+1,\ n+2$ and $nz(x>1)$
- $n+1, ny+2$ and $nz(y>1)$

3.1 Existence of ED Triangles over t_n

In this subsection, the existence of ED triangles over t_n with some particular sides are discussed. Also, for the choice $x>1, y>1$, a python code is displayed to determine the existence.

Theorem 3.1.1. There exist infinitely many ED triangles over t_n with sides $1, n+2$ and nz for some $n \in N$ and $z \in N \cup \{0\}$.

Proof. Comparing the sides $1, n+2$, and nz with the sides $nx+1, ny+2$, and nz, we see that $x = 0$ and $y = 1$. Substituting these values of x and y in $t_n^x + t_{n+1}^y = z^2$, we obtain $1 + t_{n+1} = z^2$. Replacing t_{n+1} by $\frac{(n+1)(n+2)}{2}$, we receive that $z = \sqrt{\frac{n^2+3n+4}{2}}$. Since z is an integer, we must have that $n^2+3n+4 = 2k^2$ for some $k \in \mathbb{Z}$. Considering this as a quadratic equation in n, the possible value of n is obtained as $n = \frac{-3+\sqrt{8k^2-7}}{2}$. For this n to be a natural number, one must have that $8k^2-7 = l^2$ for some $l \in \mathbb{Z}$. By Lemma 2.4, there are infinitely many such l and $k's$ exist. So we found infinitely many n (say n_r) which is calculated as $\frac{l_r-3}{2}(r \geq 1)$.

Theorem 3.1.2. There exist infinitely many ED triangles over t_n with sides $n+1,\ 2$ and nz for some $n \in \mathbb{N}$, and $z \in \mathbb{N} \cup \{0\}$.

Proof. Comparing the sides $n+1,\ 2$, and nz with the sides $nx+1, ny+2$, and nz, we see that $x = 1$ and $y = 0$. Substituting these values of x and y in $t_n^x + t_{n+1}^y = z^2$, we obtain $t_n + 1 = z^2$. Doing the same process as in Theorem 3.1.1, it is seen that $n_r = \frac{l_r-1}{2}\left(r \geq 1\right)$.

Theorem 3.1.3. There exist infinitely many ED triangles over t_n with sides $n+1,\ n+2$ and nz for some $n \in \mathbb{N}$, and $z \in \mathbb{N} \cup \{0\}$.

Proof. Here $x = y = 1$. So the obtained equation is $t_n + t_{n+1} = z^2$. By Lemma 2.8, the result follows.

Corollary 3.1.4. For any natural number n, one can construct an ED triangle over t_n.

Theorem 3.1.5. There exist infinitely many ED triangles over t_n with sides $x+1,\ 3,$ and z for some $x(>1),\ z \in \mathbb{N} \cup \{0\}$.

Proof. Here $n = 1, y = 1$. So the obtained equation is $1^x + 3 = z^2$. For any choice of $x > 1$, this equation is satisfied.

Theorem 3.1.6. There exist ED triangles over t_n with sides $nx+1, ny+2$, and nz for some $x\,(>1)\,, y\,(>1)\,,\ z \in N \cup \{0\}$.

The existence of this case is given by the following Python coding (Figure 1) and its output (Figure 2).

```
import math
import sympy
def square(k):
 print('\ttn\ttm\ttx\ty\tz')
 for n in range(1,k+1):
  for x in range(2,k+1):
   for y in range(2,k+1):
    for z in range(2,k+1):
     tn=(n*(n+1))//2
     tm=((n+1)*(n+2))//2
     if tn**x+tm**y==z**2:
        print('\t',tn,'\t',tm,'\t',x,'\t',y,'\t',z)
k=int(input("k:"))
square(k)
```

Fig 1. Python Coding for the existence of ED triangles over t_n for $x, y > 1$

```
k:50
        tn      tm      tx      y       z
         3       6       2       3       15
         3       6       6       4       45
         10      15      3       2       35
         21      28      2       2       35
>>>
```

Fig 2. Output for the coding in Figure 1

3.2 Non - Existence of ED Triangles over t_n

In this section, the non-existence of ED triangles over t_n with some particular sides discussed using Catalan's conjecture and some properties of congruence[13].

Theorem 3.2.1. One cannot find an ED triangle over t_n with sides $1,\ 2$, and nz for some $n \in \mathbb{N},$ and $z \in \mathbb{N} \cup \{0\}$.

Proof. Comparing the sides $1, 2$ and nz with the sides $nx+1, ny+2$, and nz, we see that $x = 0$ and $y = 0$. Incorporating these values of x and y in $t_n^x + t_{n+1}^y = z^2$, the latter equation becomes $z^2 = 2$, which is an impossible one.

Theorem 3.2.2 . One cannot find an ED triangle over t_n with sides $1,\ ny+2$ and nz for some $n \in \mathbb{N},$ and $z \in \mathbb{N} \cup \{0\}$.

Proof. Comparing the sides $1, ny+2$, and nz with the sides $nx+1, ny+2$, and nz, we see that $x = 0$. Incorporating this value of x in $t_n^x + t_{n+1}^y = z^2$, this exponential Diophantine equation becomes $z^2 - t_{n+1}^y = 1$. By Catalan's conjecture, $t_{n+1} = 2$. But 2 is not a triangular number.

Theorem 3.2.3. There are no ED triangles over t_n exists with sides $nx+1,\ 2$ and nz for some $n \in \mathbb{N},$ and $z \in \mathbb{N} \cup \{0\}$.

Proof. By comparing the sides as usual, here it is noted that $y = 0$. If so, we get the exponential Diophantine equation as $z^2 - t_n^x = 1$. If $n > 1$, then by Catalan's conjecture, $t_n = 2$, which is not possible. If $n = 1$, then $z^2 = 2$. This is also not possible.

Theorem 3.2.4. There are no ED triangles over t_n exists with sides $nx+1,\ n+2$ and nz for some $n \in \mathbb{N}\backslash\{1\}$ and $x(>1), z \in \mathbb{N} \cup \{0\}$.

Proof. Here $y = 1$. Since $n > 1$, we must have that $t_n > 1$ and so $t_n^x > t_n$. This leads to the fact that $z^2 > t_n^x + t_{n+1} > t_n + t_{n+1}$. Applying Lemma 2.8, it is clear that $z^2 > (n+1)^2$. So take $z = n+k$ for some $k = 2, 3, \cdots$. Thus, the equation $t_n^x + t_{n+1} = z^2$ becomes

$$\frac{n^x(n+1)^x}{2^x} + \frac{(n+1)(n+2)}{2} = (n+k)^2$$

$$n^x + \binom{x}{1} n^{x+1} + \binom{x}{2} n^{2+x} + \cdots + \binom{x}{x} n^{2x} + 2^{x-1}\left(n^2+3n+2\right) - 2^x\left(n^2+k^2+2nk\right) = 0 \tag{4}$$

If $x = 2$, then Equation (4) becomes $4k^2 + 8nk + \left(-n^4 - 2n^3 + n^2 - 6n - 4\right) = 0$. Solving this for k, we arrive at $k = \frac{-8n \pm 4\sqrt{n^4+2n^3+3n^2+6n+4}}{8}$. Using Lemma 2.5, we conclude that k is not an integer.

Suppose $x > 2$. Then comparing the coefficients of n^2 in Equation (4) gives $2^{x-1} - 2^x = 0$. This leads to an impossible situation that $-1 = 0$.

Theorem 3.2.5. There are no ED triangles over t_n exists with sides $n+1, ny+2$ and nz for some $n \in \mathbb{N}$ and $y(>1), z \in \mathbb{N} \cup \{0\}$.

Proof. Here $x = 1$. If $n = 1$, then the considered exponential Diophantine equation becomes $1 + 3^y = z^2$. This has no solution byCatalan's conjecture. If $n = 2$, then we've $3 + 6^y = z^2$. Since $y > 1$, $6^y \equiv 0\ (mod\ 4)$. This leads to $z^2 \equiv 3\ (mod\ 4)$. This is not possible. Assume that $n > 2$. Then as in Theorem 3.2.4, one can take $z = n + k$ for some $k = 2, 3, \cdots$ and by the same process we arrive at the relation

$$2^{y-1}\left(n^2+n\right) - 2^y\left(n^2+k^2+2nk\right) + \left[1 + \binom{y}{1} n + \binom{y}{2} n^2 + \cdots + \binom{y}{y} n^y\right]\left[1 + \binom{y}{1} n + \binom{y}{2} n^2 + \cdots + \binom{y}{y} n^y\right]\left[n^y + 2\binom{y}{1} n^{y-1} + \cdots +\right.$$

If $y = 2$, then Equation (5) becomes $4k^2 + 8nk - n^4 - 4n^3 - 7n^2 - 2n - 4 = 0$. By Lemma 2.6, we conclude that k is not an integer.

Suppose $y > 2$. Then equating the coefficients of n^2 in Equation (5) gives

$$2^{y-1} - 2^y + 2^{y-2}\binom{y}{y-2} + \binom{y}{y-1} 2^{y-1}\binom{y}{1} + \binom{y}{2} 2^y = 0$$

$$y = \frac{5}{9} \notin Z$$

This completes the proof.

4 Conclusion

In this article, we successfully discussed the existence and non-existence of Exponential Diophantine triangles over t_n. Apart from just existence, the exact sides of the triangles are mentioned. Also, Python coding to verify the existence case for the choice $x, y > 1$ is provided. This article stands in a new path in the sense of employing an exponential Diophantine equation to construct a newfangled set of triangles. Also, with a deep point of view, one can understand that all the possible choices of sides are discussed thoroughly. This research can be extended, modified or generalized in many ways. Some of these include replacing triangular numbers with some other special numbers or changing the form of exponential Diophantine equations.

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