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Balakrishnan Alpha Skew Generalized *t* Distribution: Properties and Applications

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Abstract

Objective: In this study, a new class of Balakrishnan distribution by extending the alpha skew generalized *t* distribution is proposed. **Methods:** Statistical properties of the Balakrishnan alpha skew generalized *t* distribution are studied in detail. In particular, specific expressions of the density and distribution function, moments, skewness, kurtosis and mode of the particular distribution are derived. The parameters of the new family of distributions are estimated by the method of maximum likelihood. Lastly, the flexibility, suitability and usefulness of the proposed distribution are illustrated by analyzing real-life data set. Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are used for model selection. **Findings:** The proposed distribution performs well in terms of AIC and BIC in real-life data modelling than the other symmetric and asymmetric distributions considered here. **Novelty:** The main uniqueness of the proposed distribution is to modal bimodal and heavy/light-tailed datasets.

Keywords: Skewed Distribution; Alpha Skew Generalized t Distribution; AIC; BIC; Optimization; Generalized t

1 Introduction

An alternative of the traditional normal distribution is proposed by McDonald & Newey [(1)], and they named it as generalized *t* distribution (GT). The probability function of this distribution is given by

$$f_{GT}(z;p,q) = \frac{p}{2q^{1/p}B\left(\frac{1}{p},q\right)}\left(1+\frac{|z|^p}{q}\right)^{-(q+1/p)} ; -\infty < z < \infty \tag{1}$$

This distribution function is more flexible to handle thicker as well as thinner tails for particular values of p and q. The GT distribution known as another form of normal distribution when $p = 2$ and $q \to \infty$

But this distribution function is not flexible enough to model asymmetric form of data. Recently, skewed distribution played a very prominent role in statistical literature to overcome the difficulty to handle real-life skewed data sets. Using the novel idea of[2], many researchers exclusively studied the non-normal asymmetric distributions; see for example[3–8].

Discussing the idea of[9], Balakrishnan[10] introduced a generalization of skew normal distribution, known as Balakrishnan skew normal distribution. The density function of the given distribution is

$$\phi_n(z;\lambda) = \frac{[\Phi(\lambda z)]^n \phi(z)}{L_n(\lambda)}; z \in R \tag{2}$$

Where $\phi(.)$ *and* $\Phi(.)$ represents the density and distribution function of the standard normal distribution respectively. After that, many researchers considered BSN distribution in the literature (for detail see[11,12]).

Actious et al.[8] introduced alpha skew generalized t distribution using the concept of[13]. The density of the distribution is given by

$$f(z) = \frac{(1-\alpha z)^2 + 1}{2+\alpha^2}\phi(z), \alpha \in R, z \in R \tag{3}$$

He mentioned the limiting forms of this distribution and identified the value of α, the transit point from unimodal distribution to bimodal distribution using numerical methods. Later,[14] demonstrated the alpha skew hyperbolic secant distribution using the ideal of alpha skew mechanism for modelling both unimodal and bimodal data with various levels of kurtosis. Kumar and Manju[15] introduced the gamma generalized logistic distribution, and this model is a generalization of logistic distribution of type I and type II. The exponentiated half logistic skew-t distribution studied by Adubisi *et. al.*[16] Guan *et al.*[17] coined a new *GT* distribution applicable for modelling the data with high kurtosis and a heavy tail by adding two shape parameters in the model. A new form of alpha skew normal distribution was employed by Hazarika el. al[18] by considering Balakrishnan[9] technique to create Balakrsihana alpha skew normal distribution defined by the *pdf.*

$$f(z) = \frac{\left((1-\alpha z)^2 + 1\right)^2}{C(\alpha)}\phi(z), \alpha \in R, z \in R \tag{4}$$

Where, $\phi(z)$ is the density of the standard normal distribution and $C(\alpha) = 3 - \frac{4}{1+\alpha^2}$ is the normalizing constant. Through a particular data set seen to be unimodal often it exhibits more than one mode. An attempt has been made to unify a new generalization of alpha skew generalized t distribution, which is not applied by any researcher. The primary motivation for developing the new model is to create a more flexible density with asymmetric, thicker, thinner tails and bimodal features for real-life data analysis. The main aim of this paper is to introduce the new form of skew-normal distribution considering the idea of Balakrishnan distribution and propose some of the properties of the same.

The article is summarized as follows: In Section 2, we define the proposed distribution and identify its particular case and provide valuable results regarding the same. In this same Section, we study some more of its essential distributional properties. A few extensions of this distribution is discussed in Section 3. The parameter estimation and the real-life data modelling of the proposed distribution are provided in Section 4. Lastly, conclusions are given in Section 5.

2 New Alpha Skew Generalized

Here, we formalize a new family of distribution known as Balakrishnan alpha skew generalized t (BASGT) distribution and give some of its primitive properties.

Definition 1: If a continuous random variable (r.v.) Z has a density function

$$f(z;\alpha,p,q) = \frac{\left((1-\alpha z)^2 + 1\right]^2}{C(\alpha,p,q)} f_{GT}(z;p,q), pq > 4, z \in R \tag{5}$$

where, $C(\alpha,p,q) = 4 + \frac{8\alpha^2 q^{2/p}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right) + \alpha^4 q^{4/p}\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)}$, then it is said to be the Balakrishnan alpha skew generalized t distribution with parameters $\alpha \in R, p > 0$ and $q > 0$ and $f_{GT}(z;p,q)$ is the pdf of generalized t distribution. We denote it as $BASGT(\alpha,p,q)$. The normalizing constant $C(\alpha,p,q)$ is calculated as follows:

$C(\alpha,p,q) = \int_{-\infty}^{\infty}\left[(1-\alpha z)^2 + 1\right]^2 f_{GT}(z;p,q)dz$

$= \int_{-\infty}^{\infty} \left[\alpha^4 z^4 - 4\alpha^3 z^3 + 8\alpha^2 z^2 - 8\alpha z + 4\right] f_{GT}(z;p,q)dz$

$$= \left[\begin{array}{l} \alpha^4 \left(\frac{q^{4/p}\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)} \right) - 4\alpha^3 \left(\frac{q^{3/p}\Gamma\left(\frac{4}{p}\right)\Gamma\left(q-\frac{3}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)} \right) + 8\alpha^2 \left(\frac{q^{2/p}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)} \right) \\ -8\alpha \left(\frac{q^{1/p}\Gamma\left(\frac{2}{p}\right)\Gamma\left(q-\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)} \right) + 4 \end{array} \right]$$

On simplifying we get,

$$= 4 + \frac{8\alpha^2 q^{2/p}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right) + \alpha^4 q^{4/p}\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)}$$

Remark 1: The pdf of the proposed distribution is formulated using the formula in Equation (2), by taking $\Phi(.) = \frac{(1-\alpha z)^2+1}{2+\alpha^2}$ and $n = 2$

2.1 Properties

By that provision, we discussed some of the primitive properties of the proposed distribution.

- If $\alpha = 0$, then we get the generalized t (GT) distribution of McDonald and Newey[1] given by $f_{GT}(z;p,q) = \frac{p}{2q^{1/p}B(1/p,q)}\left(1+\frac{|z|^p}{q}\right)^{-(q+1/p)}$.
- If $p = 2$, then we get

$$f(z;\alpha,q) = \frac{\left((1-\alpha z)^2+1\right]^2}{\left((8-4q\left(3+4\alpha^2\right)+q^2\left(4+8\alpha^2+3\alpha^4\right)\right]\Gamma(q-2)/q\right)} f_t(z;2q)$$

where $f_t(z;2q)$ is the pdf of student's t distribution with $2q$ degrees of freedom and $q > 2$. And this this distribution is named as Balakrishnan alpha skew Student's t distribution.

- If $Z \sim BASGT(\alpha,p,q)$, then $-Z \sim BASGT(-\alpha,p,q)$.
- If $Z \sim BASGT(\alpha,p,q)$, then $aZ \sim BASGT(a\alpha,p,q)$.

2.2 Limiting Cases

In this section, some of the limiting cases of $BASGT(\alpha,p,q)$ distribution are given.

- If $p = 1$ and $q \to \pm\infty$, then we get the $BASLa(\alpha)$ distribution of Shah et al.[19] given by $f(z;\alpha) = \left((1-\alpha z)^2+1\right]^2 e^{-|z|}/\left(8\left(1+4\alpha^2+6\alpha^4\right)\right)$.
- If $p = 2$ and $q \to \infty$, then we get the $BASN(\alpha)$ distribution of Hazarika et al.[18] given by $f(z;\alpha) = \left((1-\alpha z)^2+1\right]^2 \phi(z)/\left(4+8\alpha^2+3\alpha^4\right)$.
- If $q \to \infty$, then we get

$$f(z;\alpha,p) = \frac{\left((1-\alpha z)^2+1\right]^2}{4+8\alpha^2\left(\frac{\Gamma(3/p)}{\Gamma(1/p)}\right)+\alpha^4\left(\frac{\Gamma(5/p)}{\Gamma(1/p)}\right)} \frac{pe^{-|z|^p}}{2\Gamma(1/p)}$$

This is referred to as the Balakrishnan Alpha Skew Power Exponential distribution.

- When $\alpha \to \pm\infty$, we get a Bimodal generalized t (BGT(4)) distribution (see[18]) given by

$$f(z;p,q) = \frac{z^4}{C(p,q)} f_{GT}(z;p,q) \; where, \; C(p,q) = \frac{q^{4/p}\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)}.$$

2.3 Plots of the density function

In this subsection, the pdf of $BASGT(\alpha,p,q)$ distribution for different choices of the parameters α, p and q is plotted in Figure 1.

2.4 Moments

In this subsection, the moments of $BASGT(\alpha,p,q)$ distribution are derived.

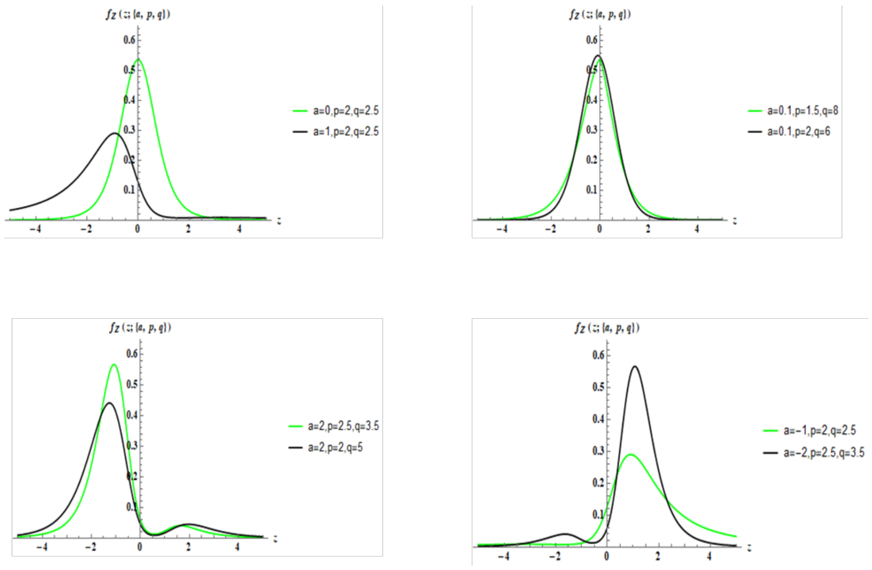

Fig 1. Plot of the density of the *BASGT*(α,p,q) distribution for difference choices of the parameters

Theorem 1: The n^{th} moment of $BASGT(\alpha, p, q)$ distribution for $n \in N$ is given by

$$E\left(Z^{2n}\right) = \frac{q^{\frac{2n}{p}}}{c(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)}\left[\begin{array}{c} 4\Gamma\left(\frac{2n+1}{p}\right)\Gamma\left(q-\frac{2n}{p}\right)+8\alpha^2 q^{\frac{2}{p}}\Gamma\left(\frac{2n+3}{p}\right)\Gamma\left(q-\frac{2(n+1)}{p}\right) \\ +\alpha^4 q^{\frac{4}{p}}\Gamma\left(\frac{2n+5}{p}\right)\Gamma\left(q-\frac{2(n+2)}{p}\right)\end{array}\right] \tag{6}$$

where $p > 0, q > 0$, and

$$E\left(Z^{2n+1}\right) = \frac{q^{2n/p}}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)}\left[-8\alpha q^{2/p}\Gamma\left(\frac{2n+3}{p}\right)\Gamma\left(q-\frac{2(n+1)}{p}\right)-4\alpha^3 q^{4/p}\Gamma\left(\frac{2n+5}{p}\right)\Gamma\left(q-\frac{2(n+2)}{p}\right)\right] \tag{7}$$

where $p > 0, q > 0$.

Proof: Lets starts with the even order moments of ...

$$E\left(Z^{2n}\right) = \frac{1}{C(\alpha,p,q)}\int_{-\infty}^{\infty} z^{2n}\left((1-\alpha z)^2+1\right]^2 f_{GT}(z;p,q)dz$$
$$= \frac{1}{C(\alpha,p,q)}\int_{-\infty}^{\infty}\left(4z^{2n}-8\alpha z^{2n+1}+8\alpha^2 z^{2n+2}-4\alpha^3 z^{2n+3}+\alpha^4 z^{2n+4}\right) f_{GT}(z;p,q)dz$$

$$= \frac{1}{C(\alpha,p,q)}\left(4E_{GT}\left(Z^{2n}\right)-8\alpha E_{GT}\left(Z^{2n+1}\right)+8\alpha^2 E_{GT}\left(Z^{2n+2}\right)-4\alpha^3 E_{GT}\left(Z^{2n+3}\right)+\alpha^4 E_{GT}\left(Z^{2n+4}\right)\right]$$
$$= \frac{1}{C(\alpha,p,q)}\left(4E_{GT}\left(Z^{2n}\right)+8\alpha^2 E_{GT}\left(Z^{2n+2}\right)+\alpha^4 E_{GT}\left(Z^{2n+4}\right)\right]\ (Odd\ order\ moments\ of\ GT\ are\ zero)$$

where, $E_{GT}\left(X^{2n}\right), E_{GT}\left(X^{2n+2}\right)$ and $E_{GT}\left(X^{2n+4}\right)$ are nothing but the even order moments of generalized t distribution. Therefore, the even order moments of $BASGT(\alpha, p, q)$ distribution becomes,

$$E\left(Z^{2n}\right) = \frac{q^{2n/p}}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)}\left[4\Gamma\left(\frac{2n+1}{p}\right)\Gamma\left(q-\frac{2n}{p}\right)+8\alpha^2 q^{2/p}\Gamma\left(\frac{2n+3}{p}\right)\Gamma\left(q-\frac{2(n+1)}{p}\right)+\alpha^4 q^{4/p}\Gamma\left(\frac{2n+5}{p}\right)\Gamma\left(q-\frac{2(n+2)}{p}\right)\right]$$

Similarly, the odd order moments of the studied distribution becomes

$$= \frac{q^{2n/p}}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)}\left[-8\alpha q^{2/p}\Gamma\left(\frac{2n+3}{p}\right)\Gamma\left(q-\frac{2(n+1)}{p}\right)-4\alpha^3 q^{4/p}\Gamma\left(\frac{2n+5}{p}\right)\Gamma\left(q-\frac{2(n+2)}{p}\right)\right]$$

Remarks 1: Thus, using Equation (6) and Equation (7), the first four moments of X can be obtained as

$$\mu_1{}' = E(Z) = -\frac{4\alpha q^{2/p}\left[q^{2/p}\alpha^2\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)+2\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)\right]}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)}$$

$$\mu_2{}' = E\left(Z^2\right) = \frac{q^{2/p}\left[q^{4/p}\alpha^4\Gamma\left(\frac{7}{p}\right)\Gamma\left(q-\frac{6}{p}\right)+8q^{2/p}\alpha^2\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)+4\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)\right]}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)}$$

$$\mu_3' = E\left(Z^3\right) = -\frac{4\alpha q^{4/p}\left[q^{2/p}\alpha^2\Gamma\left(\frac{7}{p}\right)\Gamma\left(q-\frac{6}{p}\right)+2\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)\right]}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)}$$

$$\mu_4{}' = E\left(Z^4\right) = \frac{q^{4/p}\left[q^{4/p}\alpha^4\Gamma\left(\frac{9}{p}\right)\Gamma\left(q-\frac{8}{p}\right)+8q^{2/p}\alpha^2\Gamma\left(\frac{7}{p}\right)\Gamma\left(q-\frac{6}{p}\right)+4\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)\right]}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)}$$

$$Var(Z) = \frac{\left[\begin{array}{c} C(\alpha,p,q)q^{2/p}\Gamma\left(\frac{1}{p}\right)\Gamma(q)\left(q^{4/p}\alpha^4\Gamma\left(\frac{7}{p}\right)\Gamma\left(q-\frac{6}{p}\right)+8q^{2/p}\alpha^2\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)+4\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)\right\} \\ -16q^{4/p}\left(q^{2/p}\alpha^3\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)+2\alpha\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)\right\}^2 \end{array}\right]}{\left(C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)\right\}^2}$$

2.5 Skewness and Kurtosis

In this subsection, the skewness $\left(\sqrt{\beta_1}\right)$ and the kurtosis (β_2) measures of $BASGT(\alpha,p,q)$ distribution are obtained based on the moments discussed in subsection 2.4 by using the following formulas define as follows:

$$\sqrt{\beta_1} = \frac{\mu_3{}' - 3\mu_2{}'\mu_1{}' + 2\mu_1{}^2}{\left(\mu_2{}' - \mu_1{}'^2\right)^{3/2}} \text{ and } \beta_2 = \frac{\mu_4{}' - 4\mu_3{}'\mu_1{}' + 6\mu_2{}'\mu_1{}^2 - 3\mu_1{}'^4}{\left(\mu_2{}' - \mu_1{}'^2\right)^2}.$$

where $\mu_i{}'$ denotes i^{th} the raw moments. Since the expression for the skewness and the kurtosis measures of $BASGT(\alpha,p,q)$ distribution are complicated, we give some numerical values of $\sqrt{\beta_1}$ and β_2 for different choices of the shape parameters (α,p,q).

Table 1. Possible values of the skewness and kurtosis coefficient

α	p	q	$\sqrt{\beta_1}$	β_2
0	2.5	3.5	0.000	1.294
0	2	5	0.000	0.800
0.5	1.5	5	-0.994	-9.475
1	1.5	10	1.491	-3.322
1.5	5	2	1.173	3.961
2	10	2	2.351	7.443
5	3.5	2.5	2.324	6.007
10	1.5	7	3.784	8.171

2.6 Distribution Function

In this subsection, we derive the cdf of $BASGT(\alpha,p,q)$ distribution using the same idea of[8,20]. For calculating the cdf of the studied distribution we are taking the help of incomplete beta function ratio, and it is defined as

$$I_z(a,b) = \frac{1}{B(a,b)}\int_0^z w^{a-1}(1-w)^{b-1}dw \tag{8}$$

For further detail of incomplete beta function ratio, see Chapter 1 of[21].

Theorem 2: The cdf of the random variable $Z \sim BASGT(\alpha, p, q)$ distribution is given by

$$G_{BASGT}(z) = \frac{1}{C(\alpha,p,q)} \left(4F_{GT}(z) - 8\alpha G_1(z) + 8\alpha^2 G_2(z) - 4\alpha^3 G_3(z) + \alpha^4 G_4(z)\right\} \tag{9}$$

Where, $F_{GT}(z) = \begin{cases} \frac{1}{2} - \frac{1}{2} I_{\left(1-\frac{1}{1+(-z)^{p/q}}\right)}\left(\frac{1}{p}, q\right), & z < 0 \\ \frac{1}{2} + \frac{1}{2} I_{\left(1-\frac{1}{1+z^{p/q}}\right)}\left(\frac{1}{p}, q\right), & z \geq 0 \end{cases}$

$$G_i(z) = \frac{q^{i/p}\Gamma\left(\frac{i+1}{p}\right)\Gamma\left(q-\frac{i}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)} \begin{cases} \frac{1}{2} - \frac{1}{2} I_{\left(1-\frac{1}{1+(-z)^{p/q}}\right)}\left(\frac{i+1}{p}, q-\frac{i}{p}\right), & z < 0 \\ \frac{1}{2} + \frac{1}{2} I_{\left(1-\frac{1}{1+z^{p/q}}\right)}\left(\frac{i+1}{p}, q-\frac{i}{p}\right), & z \geq 0 \end{cases}$$

and $I_y(a,b)$ is the incomplete beta function.

Proof:

$$G_{BASGT}(z) = \int_{-\infty}^{z} \frac{\left((1-\alpha x)^2 + 1\right]^2}{C(\alpha,p,q)} f_{GT}(x;p,q)dx$$

$$= \frac{1}{C(\alpha,p,q)} \int_{-\infty}^{2} \left(4 - 8\alpha x + 8\alpha^2 x^2 - 4\alpha^3 x^3 + \alpha^4 x^4\right) f_{GT}(x;p,q)dx$$

$$= \frac{1}{C(\alpha,p,q)} \begin{bmatrix} 4\int_{-\infty}^{2} f_{GT}(x;p,q)dx - 8\alpha\int_{-\infty}^{2} x f_{GT}(x;p,q)dx + 8\alpha^2\int_{-\infty}^{2} x^2 f_{GT}(x;p,q)dx \\ \left(-4\alpha^3\int_{-\infty}^{2} x^3 f_{GT}(x;p,q)dx + \alpha^4\int_{-\infty}^{2} x^4 f_{GT}(x;p,q)dx\right] \end{bmatrix}$$

Here first integral is nothing but the cdf of generalized t distribution and the remaining integrals are obtained from

$$G_i(z) = \int_{-\infty}^{z} x^i f_{GT}(x;p,q)dx$$

(for details see[20]). Therefore, the cdf of $BASGT(\alpha, p, q)$ is

$$G_{BASGT}(z) = \frac{1}{C(\alpha,p,q)} \left\{4F_{GT}(z) - 8\alpha G_1(z) + 8\alpha^2 G_2(z) - 4\alpha^3 G_3(z) + \alpha^4 G_4(z)\right\}$$

2.7 Location Scale Extension

In this subsection, we define the location-scale extension of $BASGT(\alpha, p, q)$ distribution.

If $Z \sim BASGT(\alpha, p, q)$ distribution then $Y = \mu + \sigma Z$ is considered as the location ($\mu \in R$) and scale ($\sigma > 0$) extension of Z. The pdf of the random variable Y is obtained as follows

$$f(y;\mu,\sigma,\alpha,p,q) = \frac{1}{\sigma} \frac{\left[\left\{1-\alpha\left(\frac{y-\mu}{\sigma}\right)\right\}^2 + 1\right]^2}{C(\alpha,p,q)} \frac{p}{2q^{1/p}B(1/p,q)} \left(1 + \frac{\left(\frac{y-\mu}{\sigma}\right|^p}{q}\right)^{-(q+1/p)} \tag{10}$$

where $\{y, \mu, \alpha\} \in R$, and $\{p, q, \sigma\} > 0$. We denote it as $Y \sim BASGT(\mu, \sigma, \alpha, p, q)$.

3 Maximum Likelihood Estimation

In this section, we derive the log-likelihood function and its partial derivatives for the estimation of the parameters of $BASGT(\alpha, p, q)$ distribution.

Let $y_1, y_2, \ldots, y_n$ be a random sample from $Y \sim BASGT(\mu,\sigma,\alpha,p,q)$ distribution, then the log-likelihood function for the shape parameters $\theta = (\mu,\sigma,\alpha,p,q)$ is given by

$$l(\theta) = 2\sum_{i=1}^{n} log\left(\left(1-\alpha\left(\frac{y_i-\mu}{\sigma}\right)\right\}^2+1\right] - nlogC(\alpha,p,q) - nlog(\sigma) + nlog(p) \quad -\frac{n}{p}log(q) - nlogB\left(\frac{1}{p},q\right) - \left(\frac{pq+1}{p}\right)\sum_{i=1}^{n} log\left(1+\frac{1}{q}\left(\frac{y_i-\mu}{\sigma}\right|^p\right) \tag{11}$$

$$\frac{\partial l}{\partial \mu} = \frac{4\alpha}{\sigma^2}\sum_{i=1}^{n}\frac{(\alpha\mu+\sigma-\alpha y_i)}{\left\{1-\alpha\left(\frac{y_i-\mu}{\sigma}\right)\right\}^2+1} - \frac{q\sigma(1+pq)}{pq\sigma}\sum_{i=1}^{n}\left\{-p\left|\frac{y_i-\mu}{\sigma}\right|^{p-1}\left(\frac{y_i-\mu}{\sigma}\right|^{l}\right\} \tag{12}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{4\alpha}{\sigma^3}\sum_{i=1}^{n}\frac{(\mu-y_i)(\alpha\mu+\sigma-\alpha y_i)}{\left\{1-\alpha\left(\frac{y_i-\mu}{\sigma}\right)\right\}^2+1} - \frac{(1+pq)}{pq\sigma^2}\sum_{i=1}^{n}\left\{p\mu\left|\frac{y_i-\mu}{\sigma}\right|^{p-1}\left|\frac{y_i-\mu}{\sigma}\right|^{l} - p\left|\frac{y_i-\mu}{\sigma}\right|^{p-1} y_i\left|\frac{y_i-\mu}{\sigma}\right|^{\prime}\right\} \tag{13}$$

$$\frac{\partial l}{\partial \alpha} = -\frac{n\left(4\alpha^3\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)+16\alpha q^{2/p}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)\right)}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)} \tag{14}$$

$$\frac{\partial l}{\partial p} = \frac{n}{p} + \frac{nlogq}{p^2} - \frac{\alpha^2 n}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)p^2}\left(\begin{array}{c}\alpha^2\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)\left(\psi\left(\frac{1}{p}\right)-5\psi\left(\frac{5}{p}\right)+4\psi\left(q-\frac{4}{p}\right)\right\} \\ \left(-8q^{2/p}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)\left(2logq-\psi\left(\frac{1}{p}\right)+3\psi\left(\frac{3}{p}\right)-2\psi\left(q-\frac{2}{p}\right)\right\}\right\}\end{array}\right. \\ -\frac{lognB\left(\frac{1}{p},q\right)\left(\psi\left(\frac{1}{p}\right)-\psi\left(\frac{1}{p}+q\right)\right\}}{p^2} + \frac{(1+pq)\sum_{i=1}^{n}\left(q+\left(\frac{y_i-(\mu|^p}{\sigma}\right|^p\right)}{p^2 q} + \frac{(1+pq)\sum_{i=1}^{n}\left|\frac{y_i-(\mu|^p}{\sigma}log\left(\left(\frac{y_i-\mu}{\sigma}\right|^p\right)\right.}{pq} \tag{15}$$

$$\frac{\partial l}{\partial q} = -\frac{n}{pq} - \frac{\alpha^2 n}{C(\alpha,p,q)\Gamma\left(\frac{1}{p}\right)\Gamma(q)pq}\left(\alpha^2 pq\Gamma\left(\frac{5}{p}\right)\Gamma\left(q-\frac{4}{p}\right)\left(\psi(q)-\psi\left(q-\frac{4}{p}\right)\right\}\right. \\ +8q^{2/p}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)\left(-2+pq\psi(q)-pq\psi\left(q-\frac{2}{p}\right)\right\} \\ -lognB\left(\frac{1}{p},q\right)\left(\psi(q)-\psi\left(p+\frac{1}{q}\right)\right\} - \frac{(1+pq)\sum_{i=1}^{n}\left(-\left(\frac{y_i-(\mu|^p}{\sigma}\right|^p\right)}{pq^2} \tag{16}$$

4 Applications

In this section, we consider one real life example to fit our proposed distribution with some other rival distributions.

For fitting our proposed $BASGT(\alpha,p,q)$ distribution, we consider a data set consists of 1150 heights measured at 1 micron intervals along the drum of a roller (i.e., parallel to the axis of the roller) which is available in the website http://lib.stat.cmu.edu/jasadata/laslett.

Using this data set, we compared the new distribution with some other distributions such as the alpha-skew-normal $ASN(\mu,\sigma,\alpha)$ distribution[13], the alpha-skew-logistic $ASLG(\mu,\sigma,\alpha)$ distribution[22], the alpha-skew-laplace $ASLa(\mu,\sigma,\alpha)$ distribution[23], the alpha-beta-skew-normal $ABSN(\mu,\sigma,\alpha,\beta)$ distribution[24], the alpha-beta-skew-Logistic

Table 2. MLE's, log-likelihood, AIC and BIC for 1150 Roller Data.

Distributions	μ	σ	α	β	p	q	log$L$	**AIC**	**BIC**
ASN	3.128	0.569	-1.17	–	–	–	-1065.3	2136.54	2151.68
ASLG	3.236	0.256	-0.732	–	–	–	-1076.8	2159.66	2174.81
ASLa	3.742	0.452	0.215	–	–	–	-1079.4	2164.78	2179.92
ABSN	3.095	0.688	-1.447	0.1607	–	–	-1059	2126.07	2146.26
ABSLG	3.706	0.311	0.059	0.0056	–	–	-1069.5	2146.93	2167.12
ABSGT	3.02	1.069	-1.588	0.004	13.119	0.46	-1095	2202.06	2232.35
ASGT	3.135	0.849	-1.691	–	2.635	5.456	-1061.4	2132.8	2158.04
BASGT	2.678	1.307	-1.293	–	4.009	5.631	**-1056.7**	**2123.32**	**2148.56**

$ABSLG(\mu,\sigma,\alpha,\beta)$ distribution [25], the alpha-beta-skew-generalized-t $ABSGT(\mu,\sigma,\alpha,\beta,p,q)$ distribution [26] and the alpha-skew generalized-t $ASGT(\mu,\sigma,\alpha,p,q)$ distribution [8].

With the help of 11^{th} version of Mathematica software, we obtain the MLE of the parameters. For comparing the models, the model selection criteria viz., AIC and BIC are considered. Table 2 shows the MLE's, log-likelihood, AIC and BIC of the above distributions for this data set.

It is observed from Table 2 that the proposed distribution provides best fit to the data set in terms of AIC and BIC.

5 Conclusion

Here, we introduced a new family of distribution which is called Balakrishnan alpha skew generalized t distribution. This distribution is flexible enough to support unimodal as well as bimodal and heavy/light-tailed datasets. Some of the distributional properties are investigated here. The parameters of the same distribution are estimated using the maximum likelihood method. The numerical optimization method shows that the proposed distribution provides better fits than the other rival distributions applied here.

Though the proposed distribution is flexible enough to model data in different circumstances, there is enough scope to study in different ways in the near future. For example, one can extend the work by considering truncated versions, bivariate and multivariate generalizations, bimodal and trimodal skewed extension, logarithmic forms, and inferential aspects of the distribution will be considered in follow-up work.

6 Declaration

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References

1) Mcdonald JB, Newey WK. Partially Adaptive Estimation of Regression Models via the Generalized T Distribution. *Econometric Theory*. 2010;4(3):428–457. Available from: https://doi.org/10.1017/S0266466600013384.
2) Azzalini A. A Class of Distributions Which Includes the Normal Ones. *Scandinavian Journal of Statistics*. 1985;12(2):171–178. Available from: http://www.jstor.org/stable/4615982.
3) Duarte J, Martínez-Flórez G, Gallardo DI, Venegas O, Gómez HW. A Bimodal Extension of the Epsilon-Skew-Normal Model. *Mathematics*. 2023;11(3):1–18. Available from: https://doi.org/10.3390/math11030507.
4) Shah S, Hazarika PJ, Pathak D, Chakraborty S, Ali MM. The Multimodal Extension of the Balakrishnan Alpha Skew Normal Distribution: Properties and Applications. *Pakistan Journal of Statistics and Operation Research*. 2023;19(1):203–217. Available from: https://doi.org/10.18187/pjsor.v19i1.4019.
5) Shah S, Hazarika PJ, Chakraborty S, Alizadeh M. The Balakrishnan-Alpha-Beta-Skew-Laplace Distribution: Properties and Applications. *Statistics, Optimization & Information Computing*. 2023;11(3):755–772. Available from: https://doi.org/10.19139/soic-2310-5070-1247.
6) Arslan O. The skew generalized t (SGT) distribution: properties and its applications in robust estimation. *Book of Abstracts*. 2011;p. 6. Available from: https://www.researchgate.net/publication/255647126_The_skew_generalized_t_SGT_distribution_properties_and_its_applications.
7) Hazarika PJ, Shah S, Chakraborty S. Balakrishnan Alpha Skew Normal Distribution: Properties and Applications. 2019. Available from: https://doi.org/10.48550/arXiv.1906.07424.
8) Acitas S, Senoglu B, Arslan O. Alpha-skew generalized t distribution. *Revista Colombiana de Estadística*. 2015;38(2):353–370. Available from: https://www.redalyc.org/pdf/899/89940050004.pdf.
9) Arnold BC, Beaver RJ, Azzalini A, Balakrishnan N, Bhaumik A, Dey DK, et al. Skewed multivariate models related to hidden truncation and/or selective reporting. *Test*. 2002;11:7–54. Available from: https://doi.org/10.1007/BF02595728.

10) Arnold BC, Beaver RJ, Azzalini A, Balakrishnan N, Bhaumik A, Dey DK, et al. Skewed multivariate models related to hidden truncation and/or selective reporting. *Test*. 2002;11:7–54. Available from: https://link.springer.com/article/10.1007/BF02595728.
11) Shah S, Hazarika PJ, Chakraborty S. The Balakrishnan alpha skew Laplace distribution: Properties and its applications. 2019. Available from: https://doi.org/10.48550/arXiv.1910.01084.
12) Shah S, Hazarika PJ. The Alpha-Beta-Skew-Logistic Distribution And Its Applications. 2019. Available from: https://doi.org/10.48550/arXiv.1910.01084.
13) Elal-Olivero D. Alpha-skew-normal distribution. *Proyecciones (Antofagasta, On line)*. 2011;29(3):224–240. Available from: http://dx.doi.org/10.4067/S0716-09172010000300006.
14) Korkmaz MÇ, Toibourani EF, Kioye JY, Chesneau C. The Alpha-Skew Hyperbolic Secant Distribution with Applications to an Astronomical Dataset. *Nicel Bilimler Dergisi*. 2022;4(1):70–84. Available from: https://doi.org/10.51541/nicel.1021116.
15) Kumar CS, Manju L. Gamma Generalized Logistic Distribution: Properties and Applications. *Journal of Statistical Theory and Applications*. 2022;21:155–174. Available from: https://doi.org/10.1007/s44199-022-00046-0.
16) Adubisi OD, Abdulkadir A, Farouk UA, Chiroma H. The exponentiated half logistic skew-t distribution with GARCH-type volatility models. *Scientific African*. 2022;16:1–14. Available from: https://doi.org/10.1016/j.sciaf.2022.e01253.
17) Guan R, Zhao X, Cheng W, Rong Y. A New Generalized t Distribution Based on a Distribution Construction Method. *Mathematics*. 2021;9(19):1–36. Available from: https://doi.org/10.3390/math9192413.
18) Hazarika PJ, Shah S, Chakraborty S. The Balakrishnan Alpha-Skew-Normal Distribution: Properties and Applications. *Malaysian Journal of Science*. 2020;39(2):71–91. Available from: https://doi.org/10.22452/mjs.vol39no2.5.
19) Shah S, Hazarika PJ, Chakraborty S. The alpha–beta skew normal distribution: properties and applications. 2019. Available from: https://doi.org/10.48550/arXiv.1910.01084.
20) Nadarajah S. On the generalized t(GT) distribution. *Statistics*. 2008;42(5):467–473. Available from: https://doi.org/10.1080/02331880701747660.
21) Gupta AK, Nadarajah S. Handbook of Beta Distribution and Its Applications. 1st ed. Boca Raton, Florida, USA. CRC Press. 2004. Available from: https://doi.org/10.1201/9781482276596.
22) Hazarika PJ, Chakraborty S. Alpha-Skew-Logistic Distribution. *IOSR Journal of Mathematics*. 2014;10(4):36–46. Available from: https://doi.org/10.9790/5728-10463646.
23) Harandi SS, Alamatsaz MH. Alpha-skew-Laplace distribution. *Statistics & Probability Letters*. 2013;83(3):774–782. Available from: https://doi.org/10.1016/j.spl.2012.11.024.
24) Shafiei S, Doostparast M, Jamalizadeh A. The alpha–beta skew normal distribution: properties and applications. *Statistics*. 2016;50(2):338–349. Available from: https://doi.org/10.1080/02331888.2015.1096938.
25) Esmaeili H, Lak F, Alizadeh M, Monfared MED. The Alpha-Beta Skew Logistic Distribution: Properties and Applications. *Statistics, Optimization & Information Computing*. 2020;8(1):304–317. Available from: https://doi.org/10.19139/soic-2310-5070-706.
26) Lak F, Alizadeh M, Monfared MED, Esmaeili H. The Alpha-Beta Skew Generalized t Distribution: Properties and Applications. *Pakistan Journal of Statistics and Operation Research*. 2019;15(3):605–616. Available from: https://doi.org/10.18187/pjsor.v15i3.2404.