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A study of κ- Contraction and the Triangular α Orbital Admissibility Condition in Quasi-metric Space

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Abstract

Objective: To establish the presence of fixed point under a novel contraction condition and a freshly defined distance function, we harness the concept of triangular α orbital admissible mappings. **Method**: Consider two mapping in quasi-metric space. These two mapping satisfy a new contraction condition and also the triangular αorbital admissible condition. Define the sequences for two mappings. Consider two cases for odd and even sequences. Show that a fixed point is common for two mappings and then demonstrate its uniqueness. **Findings**: Unique common fixed point exists. **Novelty**: A new technique is used, so the length of proof become very short as compare to theorems available in literature.

Keywords: Quasi metric; Fixed point; α Orbital admissible; Function; κ Contraction

1 Introduction

The foundational principle of Banach Contraction, known for its precision and efficacy, forms the bedrock of metric fixed point theory. It offers a systematic approach for addressing a wide array of problems across mathematical sciences, economics and engineering. Numerous authors have explored applications, extensions and advancements of this principle in diverse directions. In various spaces such as b-metric space, Cone metric space, Menger space, Fuzzy metric space, G-metric space, Quasi metric space, researchers have established fixed point theorems.

Our focus lies on quasi-metric spaces, chosen due to their fewer constraints compared to metric spaces, leading to distinct results. Recent work of authors[(1)] and[(2)] have delved into quasi- metric spaces. The term "generalized contraction" was introduced by[(3)], while[(4)] presented the concept of "almost z-contraction" within metric spaces. In dislocated quasi-metric space,[(5)] has demonstrated a fixed point theorem for Geraghty quasi-contraction type mappings, employing classical methods for proof.[(6)] concentrated on quasi-partial b-metric space and established non-unique fixed points.[(7)] employed the concept of quasi-triangular orbital admissible mapping in a complete metric space, whereas we applied this concept in context of quasi-metric spaces.

The proposed study is distinguished from these findings because we need not require to prove Cauchy sequence as we have added a concept of triangular α orbital admissible "κ–distance function", along with a unique contraction condition and a novel technique for proving the existences of a unique fixed point. Furthermore, our approach reduces the length of the proof for the theorem, enhancing the efficiency of method outlined in this paper.

Definition [8]: The mapping $q : I \times I \to [0,\infty)$ termed as Quasi Metric, if it fulfills:

(i) $q(i,j) = 0 \Leftrightarrow i = j$

(ii) $q(i,j) \leq q(i,l) + q(l,j), \forall i,j,l \in I$

The couple (I,q) termed as quasi-metric space.

Definition [9]: Consider a non- empty set I and a function $\alpha : I \times I \to [0,\infty)$. Then the mapping $F : I \to I$ is triangular α orbital admissible, if it is α orbital admissible plus.

$\alpha(i,j) \geq 1$ and $\alpha(j,Fj) \geq 1 \Rightarrow \alpha(i,Fj) \geq 1$

Remark 1: In this context, it's important to note that a metric space encompasses quasi-metric space as a subset, but opposite relationship does not always hold.

Remark 2: The conclusions that hold true in quasi-metric space do not necessarily hold true in metric space.

Example: Let $I = (0,\infty)$ Define $q(i,j) = \left(\begin{array}{l} j, i \geq j \\ \frac{j-i}{4}, i < j \end{array} \right.$

Here, we observe $q(i,j) \neq q(j,i)$, indeed it pertains the quasi-metric space rather than a metric space.

The objective of the current study is to define a new function, new contraction condition and also apply fixed point theory to quasi-metric space.

2 Result and Discussion

We have defined κ- function as here under:

Definition: A function $\kappa : (0,\infty) \to (0,\infty)$ is named as κ-distance function when κ satisfies $\kappa(x^*) = 0 \Leftrightarrow x^* = 0$

Now, using new contraction condition, definition of κ -distance function and utilizing the concept of triangular α orbital admissible, we prove the theorem, as stated below:

Theorem 2.1: Assume (I,q) metric space, $F : I \to I, G : I \to I$ be triangular α orbital admissible mappings. The pair (F, G) fulfilling the succeeding contraction for each $i,j \in I$.

(i) $\alpha(i,j)\kappa(q(Gi,Fj) \leq \kappa\{a_1 q(i,j) + a_2 q(i,Gi) + a_3 q(j,Fj) + a_4 q(i,Fj) + a_5 q(Gi,j)\}$

(ii) $\alpha(i,j)\kappa(q(Fi,Gj) \leq \kappa\{a_1 q(i,j) + a_2 q(i,Fi) + a_3 q(j,Gj) + a_4 q(i,Gj) + a_5 q(Fi,j)\}$

Also assume $q(i_n, i_{n+1}) = 0$ or $q(i_{n+1}, i_n) = 0$, for some $n \in NU\{0\}, \sum_{i=1}^{5} a_i \leq 1$. Then i_n is unique common fp of F,G.

Proof: Consider $i_0 \in I$, the sequence $\{i_n\}$ in I be defined by

$F(i_{2n}) = i_{2n+1}$ and $G(i_{2n+1}) = i_{2n+2}$ for all $n \in NU\{0\}$. (2.1.1)

Consider the following cases

Case 1: $q(i_n, i_{n+1}) = 0$. If n is even, the $n = 2k$ for some $k \in NU\{0\}$.$\therefore q(i_{2k}, i_{2k+1}) = 0$ (2.1.2)

Now

$$\kappa(q(i_{2k+1}, i_{2k+2}) \leq \alpha(i_{2k}, i_{2k+1})\kappa\{q(Gi_{2k}, Fi_{2k+1})\} \leq \kappa\{a_1 q(i_{2k}, i_{2k+1}) + a_2 q(i_{2k}, Gi_{2k}) + a_3 q(i_{2k+1}, Fi_{2k+1}) + a_4 q(i_{2k}, Fi_{2k+1}) + a_5 q(Gi_{2k}, i_{2k+1})\}$$

$$= \kappa\{a_1 q(i_{2k}, i_{2k+1}) + a_2 q(i_{2k}, i_{2k+1}) + a_3 q(i_{2k+1}, i_{2k+2}) + a_4 q(i_{2k}, i_{2k+2}) + a_5 q(i_{2k+1}, i_{2k+1})\}, \text{ using } (2.1.1)$$

$$= \kappa\{(a_3 + a_4) q(i_{2k+1}, i_{2k+2})\}, \text{ using } (2.1.2)$$

$\therefore q(i_{2k+1}, i_{2k+2}) = 0$

Now

$$\kappa(q(i_{2k+2}, i_{2k+1}) \leq \alpha(i_{2k+1}, i_{2k})\kappa\{q(Fi_{2k+1}, Gi_{2k})\}$$

$$\leq \kappa\{a_1 q(i_{2k+1}, i_{2k}) + a_2 q(i_{2k+1}, Fi_{2k+1}) + a_3 q(i_{2k}, Gi_{2k}) + a_4 q(i_{2k+1}, Gi_{2k}) + a_5 q(i_{2k}, Fi_{2k+1})\}$$

$$= \kappa\{a_1 q(i_{2k+1}, i_{2k}) + a_2 q(i_{2k}, i_{2k+2}) + a_3 q(i_{2k}, i_{2k+1}) + a_4 q(i_{2k+1}, i_{2k+1}) + a_5 q(i_{2k+2}, i_{2k})\}$$

$$= \kappa\{(a_2 + a_5) q(i_{2k+2}, i_{2k+1})\}$$

using (2.1.2) and (ii) of definition [8]

$\therefore q(i_{2k+2}, i_{2k+1}) = 0$

$\therefore q(i_{2k+2}, i_{2k+1}) = 0 = q(i_{2k}, i_{2k+1})$

$\Rightarrow q(i_{2k+2}, i_{2k+1}) = 0 = q(i_{2k}, i_{2k+1}) \Rightarrow i_{2k} = i_{2k+1} = i_{2k+2}$, by (i) definition[8]
$\Rightarrow i_n = i_{n+1} = i_{n+2} \Rightarrow i_n = Gi_n = Fi_{n+1}$
Hence i_n is common fixed point of F and G for even value of n.
Case 2: $q(i_n, i_{n+1}) = 0$. If n is odd, then $n = 2k+1$ for some $k \in NU\{0\}$
$\therefore q(i_{2k+1}, i_{2k+2}) = 0$
Now

$$\kappa(q(i_{2k+2}, i_{2k+3}) \leq \alpha(i_{2k+1}, i_{2k+2})\, \kappa\{q(Fi_{2k+1}, Gi_{2k+2})\}$$

$$\leq \kappa\{a_1 q(i_{2k+1}, i_{2k+2}) + a_2 q(Fi_{2k+1}, i_{2k+1}) + a_3 q(i_{2k+2}, Gi_{2k+2}) + a_4 q(i_{2k+1}, Gi_{2k+2}) + a_5 q(i_{2k+2}, Fi_{2k+1})\}$$

$$= \kappa\{a_1 q(i_{2k+1}, i_{2k}) + a_2 q(i_{2k+2}, i_{2k+1}) + a_3 q(i_{2k+2}, i_{2k+3}) + a_4 q(i_{2k+1}, i_{2k+3}) + a_5 q(i_{2k+2}, i_{2k+2})\}$$

$$= \kappa\{(a_3 + a_4)\, q(i_{2k+2}, i_{2k+3})\}$$

$$\therefore q(i_{2k+2}, i_{2k+3}) = 0$$

Now

$$\kappa(q(i_{2k+3}, i_{2k+2})) \leq \alpha(i_{2k+1}, i_{2k+2})\, \kappa\{q(Gi_{2k+2}, Fi_{2k+1})\}$$

$$\leq \kappa\{a_1 q(i_{2k+2}, i_{2k+1}) + a_2 q(i_{2k+2}, Gi_{2k+2}) + a_3 q(i_{2k+1}, Fi_{2k+1}) + a_4 q(i_{2k+2}, Fi_{2k+1}) + a_5 q(Gi_{2k+2}, i_{2k+1})\}$$

$$= \kappa\{a_1 q(i_{2k+2}, i_{2k+1}) + a_2 q(i_{2k+2}, i_{2k+3}) + a_3 q(i_{2k+1}, i_{2k+2}) + a_4 q(i_{2k+2}, i_{2k+2}) + a_5 q(i_{2k+3}, i_{2k+1})\}$$

$$= \kappa\{a_5 q(i_{2k+3}, i_{2k+2})\}$$

$$\therefore q(i_{2k+3}, i_{2k+2}) = 0$$

$$\Rightarrow q(i_{2k+3}, i_{2k+2}) = 0 = q(i_{2k+2}, i_{2k+1}) \Rightarrow i_{2k+1} = i_{2k+2} = i_{2k+3}$$

$\Rightarrow i_n = i_{n+1} = i_{n+2} \Rightarrow i_n = Gi_n = Fi_{n+1}$.
Hence i_n is common fixed point of F and G for odd value of n.

3 Uniqueness of fixed point

$\kappa(q(p_1, p_2)) = \kappa\{q(Gp_1, Fp_2)\}$, where p_1 and p_2 are distinct fixed points.

$$\leq \kappa\{a_1 q(p_1, p_2) + a_2 q(p_1, Gp_2) + a_3 q(p_2, Fp_2) + a_4 q(p_1, Fp_1) + a_5 q(Gp_1, p_2)\}$$

$$= \kappa\{(a_1 + a_2 + a_5)\, q(p_1, p_2)\}$$

$\therefore q(p_1, p_2) = 0 \Rightarrow p_1 = p_2$
F and G therefore share a single fixed point.
Example: Let $I = R, q : I \times I \to (0, \infty), \alpha : I \times I \to (0, \infty)$ defined by

$$\alpha(i,j) = \begin{cases} 1, \text{ for } i,j \geq 0, i \geq j \\ \frac{1}{20}, \text{ for } i,j < 0, i \geq j \\ 0, \text{ otherwise} \end{cases}, q(i,j) = 2(i-j)$$

$k(x^*) = \frac{x^*}{1+\frac{x^*}{2}}$
$F : I \to I$, defined by $F(i) = \frac{i}{1+2i}$ and $G : I \to I$ defined by $G(i) = \frac{i}{8}$.
Let $a_1 = 0.2, a_2 = 0.3, a_3 = 0.1, a_4 = 0.1, a_5 = 0.2 \therefore \sum_{i=1}^{i=5} a_i = 0.9 < 1$.
Case 1: $i,j \geq 0, i \geq j$. Consider $i = 2, j = 1$. Then in condition (i) of theorem 2.1
L.H.S. = -0.1818 and R.H.S. = 0.89.
Then in condition (ii) of theorem 2.1
L.H.S. = 0.43 and R.H.S. =0.89
Case 2: $i,j \geq 0, i \geq j$. Consier i=-1, j=-2. Then in condition (i) of theorem 2.1
L.H.S. = -1.0246 and R.H.S. = 0.9333
Then in condition (ii) of theorem 2.1
L.H.S. = 0.055 R.H.S. = 0.0952.

Therefore all conditions specified in theorem 2.1 are met and $F(0) = G(0) = 0$. Therefore 0 is unique common fixed point for both F and G.

4 Conclusion

A new function and a new condition in quasi metric space has given singular common fixed point statement. The idea of triangular α orbital admissible function is useful to prove the result in very few steps. The theorem can be extended for four mappings.

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