INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

A Study on the Sum of 'm+1' Consecutive Woodall Numbers

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OPEN ACCESS
Received: 16-08-2023
Accepted: 14-10-2023
Published: 20-11-2023

Citation: Shanmudanantham P, Deepika T (2023) A Study on the Sum of 'm+1' Consecutive Woodall Numbers. Indian Journal of Science and Technology 16(44): 3982-3986. https://doi.org/10.17485/IJST/v16i44.2084

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Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objectives: The Objective of this article is to find new formulas for Sums of m+1 Woodall Numbers and its matrix form. Here an attempt made to communicate the formula for Recursive Matrix form and some of its applications. **Methods:** Theorems are proved using the definitions of Woodall numbers. Some applications are also provided. Moreover, results are obtained by employing mathematical calculations and algebraic simplifications. Results are established by main theorems and their corollary and matrix representations. **Findings:** A formula for the Sum of m+1 consecutive Woodall numbers is obtained by utilizing a lemma. Matrix form for the sum and Recursive forms are attained here. The matrix form of sums of m+1 consecutive Cullen Numbers is also gained. In the application part some interesting associations between Special Numbers, Cullen Numbers and Carol Numbers are given. **Novelty:** In the analysis, entirely new formulae are procured. Matrix representation and its recursive forms are new finding in the area of research. Also, different types of correlations between Woodall Numbers and other special numbers are provided.

Keywords: Cullen Numbers; Carol Numbers; Woodall Numbers; Hex number; Fermat number

1 Introduction

The Woodall numbers $\{\omega(\tau)\}$ are numbers with the pattern $\omega(\tau) = \tau 2^{\tau} - 1$ also known as Riesel numbers and Cullen numbers of the second sort [1]. The initial terms of Woodall numbers are:

1,7,23,63,159,383,895,2047,4607,10239,22527,49151,106495,229375,491519,1048575,...

In 1917, Allan J. C. Cunningham and H. J. Woodall conducted the first research of the Woodall numbers, drawing inspiration from James Cullen's previous investigation of the similarly defined Cullen numbers [2]. The numbers with the formula $Cu(\tau) = \tau 2^{\tau} + 1$ are known as Cullen numbers. The initial Cullen digits are:

1,3,9,25,65,161,385,897,2049,4609,10241,22529,49153,106497,229377,491521,...

Numerous writers have looked for unique characteristics of Cullen and Woodall numbers as well as their generalizations. We use [3–5] as the results of optimality for these integers, and as the value of their greatest common divisor. In the past two decades, there has been a lot of interest in the issue of identifying Cullen and Woodall numbers that belong to other recognized sequences. We mention [6] for the pseudo-prime Cullen and Woodall numbers.

A research has been done to explain the matrix shape and the formula for the sums of squares of 'm' Woodall numbers [6,7]. Additionally, there are some relationships between Woodall numbers and other unique numbers. In 2022, generalized Woodall sequences were studied in terms of modified Woodall, modified Cullen, Woodall, and Cullen sequences, together with matrices and identities associated to these sequences. Soykan et al [8] explored closed forms of the sum formulas for generalized Hexanacci numbers with the particular case of summation formulae of Hexanacci, Hexanacci-Lucas, and other sixth-order recurrence sequences.

The Mersenne and Mersenne-Lucas sequences have been discussed as two particular examples of the generalized Mersenne number [9]. They spoke about the generating functions, Simson formulae, Binet formulas, and summation formulas for these sequences. The sum formulae for the squares of generalized Tribonacci numbers' closed forms were researched by Soykan et al. The Padovan, Perrin, Tribonacci, and Tribonacci-Lucas summation formulae as well as additional third order recurrence relations were examined as special examples [10,11].

In 2023, Shanmuganandham et al. studied the sum of squares of'm+1' consecutive Woodall numbers and its matrix representation· Additionally, the sum of squares of n successive Woodall numbers in recursive form has been developed. Woodall numbers have not yet been the subject of considerable study. The goal of this article is to determine the sum of successive Woodall numbers up to m+1. This article uses the principles of divisibility to obtain the indistinguishable identities for Pell, Lucas, Fibonacci, Jacobsthal, and Polygonal numbers. The total of the recursive matrix of n successive Woodall Numbers is also shown, along with an integer answer. The topic of graph labeling has a considerable applicability for this investigation.

2 Methodology

Definition 2.1:

For $\tau \geq 1$, the Woodall number is defined by the formula $\omega(\tau) = \tau 2^{\tau} - 1$.

Definition 2.2: For $\tau \geq 1$, the Fermat number is defined as $(Fer)_{\tau} = 2^{\tau} + 1$.

Definition 2.3: For $\tau \geq 1$, the general term of the Carol number is given by $(2^{\tau} - 1)^2 - 2$.

Definition 2.4: For $\tau \geq 1$, the Hex number is defined by the formula $(Hex)_{\tau} = 3\tau(\tau+1) + 1$.

Definition 2.5: For $\tau \geq 1$, the Hexagonal number is defined as $H(6,\tau) = \tau(2\tau - 1)$.

Definition 2.6: For all $\tau \geq 1$, the Triangular number is given by the formula $(Tri)_{\tau} = \frac{\tau(\tau+1)}{2}$.

Definition 2.7: For all $\tau \geq 1$, the Cullen number is defined by the formula $Cu(\tau) = \tau 2^{\tau} + 1$.

3 Result and discussion

Theorem 3.1: For all $\tau \geq 1$, the following equality holds,

$\sum_{k=0}^{m}(\tau+k) = \sum_{k=0}^{m}[2(K-1)L - (\tau-2)]X - \frac{2}{k}(Tri)_m$, where $X = 2^{\tau}; K = \tau + k$ and $L = 2^k$.

The following lemma is necessitating attaining the derivable pace in the proof of the main theorem.

Lemma: For all $\tau \geq 1, \sum_{k=0}^{m}\omega(\tau+k) = \tau(2^{\tau}) + \tau(2^{\tau})\sum_{k=1}^{m}2^k + 2^{\tau}\sum_{k=1}^{m}2^k - \sum_{k=0}^{m}1$.

Proof of the lemma: This lemma can be solved in two different ways. One is direct form the definition of Woodall number and second one is the induction method on $k = 0,1,2,\ldots m$.

Proof (1): From the definition, $\omega(\tau) = \tau(2^{\tau}) - 1$

Therefore, $\omega(\tau+1) = (\tau+1)(2^{\tau+1}) - 1 = 2\tau(2^{\tau}) + 2(2^{\tau}) - 1$

Similarly, $\omega(\tau+2) = (\tau+2)(2^{\tau+2}) - 1 = 4\tau(2^{\tau}) + 2(2^2)(2^{\tau}) - 1$

We extend this result up to $(m+1)^{th}$ term, at last, arrive the result as follows,

$\omega(\tau+m) = (\tau+m)(2^{\tau+m}) - 1 = 2^m\tau(2^{\tau}) + m(2^m)(2^{\tau}) - 1$. Adding all the above terms we have

$$\sum_{k=0}^{m}\omega(\tau+k) = \tau(2^{\tau}) + \tau(2^{\tau})\sum_{k=1}^{m}2^k + 2^{\tau}\sum_{k=1}^{m}2^k - \sum_{k=0}^{m}1 \tag{1}$$

Proof (2): Consider the equation (1), by the induction on k, we can give the alternate proof of the lemma.

When k=1, the equation (1) becomes, $\omega(\tau+1) = (\tau+1)(2^{\tau+1}) - 1 = 2\tau(2^{\tau}) + 2(2^{\tau}) - 1$

Therefore, the lemma is true when k=1. By induction, assume that the lemma is true for $k = m-1$.i.e.

$$\begin{aligned}\sum_{k=1}^{m-1}\omega(\tau+k-1) = \tau\left(2^{\tau}\right)\left(2+2^2+2^3+\cdots+2^{k-1}\right) + \\ \left(1.2+2\cdot 2^2+3\cdot 2^3+\ldots+(k-1)\cdot 2^{k-1}\right)\left(2^{\tau}\right) - k\end{aligned} \tag{2}$$

By replacing k by $k+1$ in equation (2) and by using induction hypothesis, we proved lemma.

Hence, $\sum_{k=0}^{m}\omega(\tau+k)=\tau(2^{\tau})+\tau(2^{\tau})\sum_{k=1}^{m}2^{k}+2^{\tau}\sum_{k=1}^{m}2^{k}-\sum_{k=0}^{m}1$

Proof of main theorem:

In the RHS of equation (2), first term consists of a geometric series whose common ratio is 2. Therefore its sum is equal to 2^{k} -1. Then we can easily prove the sum of the series

$\left(1.2+2\cdot 2^{2}+3\cdot 2^{3}+\ldots+(k-1)\cdot 2^{k-1}\right)$ is equal to $\left[(k-1)2^{k+1}+2\right]$.

Equation (2) modified as follows,

$$\sum_{k=0}^{m}\omega(\tau+k)=\tau(2^{\tau})2^{k+1}-\tau(2^{\tau})+[(k-1)2^{k+1}](2^{\tau})+2^{\tau+1}-(k+1)$$

Rearranging the terms, $\sum_{k=0}^{m}\omega(\tau+k)=\sum_{k=0}^{m}\{[(\tau+k-1)2^{k+1}-\tau+2](2^{\tau})-(k+1)\}$

By suitable substitution, we have $\sum_{k=0}^{m}(\tau+k)=\sum_{k=0}^{m}(2(K-1)L-(\tau-2)]X-\frac{2}{k}(Tri)_{m}$where $X=2^{\tau};\ K=\tau+k$ and$L=2^{k}$. Hence the theorem.

Corollary3.2: The sum of consecutive m+1 Cullen numbers is

$$\sum_{k=0}^{m}(\tau+k)=\sum_{k=0}^{m}(2(K-1)L-(\tau-2)]X-\frac{2}{k}(Tri)_{m}$$

Where$X=2^{\tau};\ K=\tau+k$ and $L=2^{k}$

Matrix form of sum of consecutive Woodall numbers

Theorem 3.3: For all$\tau\geq 1$, and $k=1,2,3,\ldots m$,

$$W(\tau)=\begin{bmatrix} P & (k-2)P & -1 \\ Q & (k-1)Q & -1 \\ R & kB & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ 1 \end{bmatrix}$$

Where$P=2^{k-2}$, $Q=2^{k-1}$, and $R=2^{k}$, $X_1=\tau(2^{\tau}), X_2=2^{\tau}$and

$$W(\tau)=\begin{bmatrix} \omega(\tau+k-2) \\ \omega(\tau+k-1) \\ \omega(\tau+k) \end{bmatrix}$$

Proof: By the definition, $\omega(\tau)=\tau 2^{\tau}-1$

Also, the next Woodall number is $\omega(\tau+1)=\tau(2^{\tau+1})+(2^{\tau+1})-1=2\tau(2^{\tau})+2(2^{\tau})-1$

In the similar way we have, $\omega(\tau+2)=\tau(2^{\tau+2})+(2^{\tau+2})-1=4\tau(2^{\tau})+2.2^{2}(2^{\tau})-1$

The above three equations can revise in matrix form as $W_{1,1}(\tau)=D_{1,1}(\tau)Y_{1,1}(\tau)$, where

$W_{1,1}(\tau)=\begin{bmatrix} \omega(\tau) \\ \omega(\tau+1) \\ \omega(\tau+2) \end{bmatrix} D_{1,1}(\tau)=\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -1 \\ 4 & 8 & -1 \end{bmatrix}$ and $Y_{1,1}(\tau)=\begin{bmatrix} \tau(2^{\tau}) \\ (2^{\tau}) \\ 1 \end{bmatrix}$

Indistinguishable $W_{1,2}(\tau)$ and$D_{1,2}(\tau)$can be acquired by put back (τ) by $(\tau+1)$in the respective $W_{1,2}(\tau)$matrix, keeping $Y_{1,1}(\tau)$is fixed and $D_{1,2}(\tau)=\begin{bmatrix} 2 & 2 & -1 \\ 4 & 8 & -1 \\ 8 & 24 & -1 \end{bmatrix}$.

In this fashion $W_{1,k-1}(\tau)$ and $D_{1,k-1}(\tau)$ gets the following matrix depiction as $W_{1,k-1}(\tau)=D_{1,k-1}(\tau)Y_{1,1}(\tau)$ where

$W_{1,k-1}(\tau)=\begin{bmatrix} \omega(\tau+k-2) \\ \omega(\tau+k-1) \\ \omega(\tau+k) \end{bmatrix} D_{1,k-1}(\tau)=\begin{bmatrix} 2^{k-2} & (k-2)2^{k-2} & -1 \\ 2^{k-1} & (k-1)2^{k-1} & -1 \\ 2^{k} & k2^{k} & -1 \end{bmatrix}$ and $Y_{1,1}(\tau)=\begin{bmatrix} \tau(2^{\tau}) \\ (2^{\tau}) \\ 1 \end{bmatrix}$

By the appropriate substitution for $P=2^{k-2}$, $Q=2^{k-1}$and $R=2^{k}$the matrix representation reduced to

$W(\tau)=\begin{bmatrix} P & (k-2)P & -1 \\ Q & (k-1)Q & -1 \\ R & kB & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ 1 \end{bmatrix}$

Where

$W(\tau)=\begin{bmatrix} \omega(\tau+k-2) \\ \omega(\tau+k-1) \\ \omega(\tau+k) \end{bmatrix} Y_{1,1}(\tau)=\begin{bmatrix} X_1 \\ X_2 \\ 1 \end{bmatrix}$

And $X_1 = \tau(2^\tau), X_2 = 2^\tau$. Hence the theorem.

Corollary 3.4: The matrix form of sums of m+1 consecutive Cullen number is,

$$Cu(\tau) = \begin{bmatrix} P & (k-2)P & 1 \\ Q & (k-1)Q & 1 \\ R & kB & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ 1 \end{bmatrix}$$

Where

$$Cu(\tau) = \begin{bmatrix} cu(\tau+k-2) \\ cu(\tau+k-1) \\ cu(\tau+k) \end{bmatrix} Y_{1,1}(\tau) = \begin{bmatrix} X_1 \\ X_2 \\ 1 \end{bmatrix} X_1 = \tau\left(2^\tau\right), X_2 = 2^\tau$$

Recursive Matrix form:

Theorem 3.5: For all $n \geq 0$the recursive coefficient matrix has the form

$$W(n) = \begin{bmatrix} {a_{11}}^n & n{a_{12}}^n & a_{13} \\ {a_{21}}^{n+1} & (n+1){a_{22}}^{n+1} & a_{23} \\ {a_{31}}^{n+2} & (n+2){a_{32}}^{n+2} & a_{33} \end{bmatrix}$$

where$a_{13} = a_{23} = a_{33} = -1$.

Proof: Consider the initial coefficient matrix, from $D_1(\tau)$of theorem 3.3 as given below,

$$W(n=1) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -1 \\ 4 & 8 & -1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The elements of next order matrix $W(n=2)$depend on the earlier order elements in $W(n=1)$ Excluding the elements of last column.

$$W(n=2) = \begin{bmatrix} 4 & 8 & -1 \\ 8 & 24 & -1 \\ 16 & 64 & -1 \end{bmatrix} = \begin{bmatrix} {a_{11}}^2 & 2{a_{12}}^2 & a_{13} \\ {a_{21}}^2 & 3{a_{22}}^3 & a_{23} \\ {a_{31}}^4 & 4{a_{32}}^4 & a_{33} \end{bmatrix}$$

In this way the elements of nth order matrix $W(n)$ can be rewritten as follows,

$$W(n) = \begin{bmatrix} {a_{11}}^n & n{a_{12}}^n & a_{13} \\ {a_{21}}^{n+1} & (n+1){a_{22}}^{n+1} & a_{23} \\ {a_{31}}^{n+2} & (n+2){a_{32}}^{n+2} & a_{33} \end{bmatrix}$$

where $a_{13} = a_{23} = a_{33} = -1$, then the theorem concludes.

Corollary 3.6: The recursive matrix form of sum of Cullen number is,

$$Cu(n) = \begin{bmatrix} {a_{11}}^n & n{a_{12}}^n & -a_{13} \\ {a_{21}}^{n+1} & (n+1){a_{22}}^{n+1} & -a_{23} \\ {a_{31}}^{n+2} & (n+2){a_{32}}^{n+2} & -a_{33} \end{bmatrix}$$

Applications

Theorem 3.7 : For all $\tau \geq 0$, $w(\tau+3) - 2w(\tau+1) - w(\tau+2) - 6 = 4(thakur)_\tau$

Proof: By the definition of Woodall number, we have

$$2w(\tau+1) + w(\tau+2) = 8\tau(2^\tau) + 12(2^\tau) - 3$$

Therefore, $2w(\tau+1) + w(\tau+2) - w(\tau+3) = -12(2^\tau) - 2$

This becomes, $w(\tau+3) - 2w(\tau+1) - w(\tau+2) = 4[3(2^\tau) - 1] + 6$

Hence $w(\tau+3) - 2w(\tau+1) - w(\tau+2) - 6 = 4(thakur)_\tau$

Corollary 3.8:$4[w(\tau+3) - 2w(\tau+1) - w(\tau+2) - 4(thakur)_\tau]$ is a nasty number.

Corollary 3.9:$[w(\tau+3) - 2w(\tau+1) - w(\tau+2) - 4(thakur)_\tau] \equiv 0 (mod 6)$

Product of two consecutive Woodall numbers in terms of some special numbers

Theorem 3.10 : For all$\tau \geq 0$,

$$w(\tau+1)w(\tau+2) = [C(16,\tau) + 8(Gno)_\tau + 23](2^{2\tau}) - [3(Gno)_\tau + 13](2^\tau) + 1$$

Proof: By the product of two consecutive Woodall number,

$$w(\tau+1)w(\tau+2) = 8(\tau^2+3\tau+2)(2^{2\tau}) - (6\tau+10)(2^{\tau}) + 1$$

$$= C(16,\tau)(2^{2\tau}) + (6\tau+15)(2^{2\tau}) - (6\tau+10)(2^{\tau}) + 1$$

$$= C(16,\tau)(2^{2\tau}) + 8(Gno)_{\tau}(2^{2\tau}) + 23(2^{2\tau}) - 3(Gno)_{\tau}(2^{\tau}) - 13(2^{\tau}) + 1$$

Hence, $w(\tau+1)w(\tau+2) = [C(16,\tau) + 8(Gno)_{\tau} + 23](2^{2\tau}) - [3(Gno)_{\tau} + 13](2^{\tau}) + 1$.

Theorem 3.11 : Woodall number, Cullen number and Carol number satisfies the Diophantine Equation of the form$x^2+y^2 = 2z^2 + w^2 - T$where$x =$Woodall number, $y =$Cullen number, $T =$Carol number, $z = \tau(2^{\tau})$ and $w = 2^{\tau} - 1$.

Proof: By adding the square of Woodall number and Cullen number we have,

$$x^2+y^2 = 2\tau^2(2^{2\tau}) + 2$$

By substituting the definition of carol number, equation (3) becomes,

$$x^2+y^2 = 2(\tau 2^{\tau})^2 + (2^{\tau}-1)^2 - (Carol)_{\tau}$$

Hence, $x^2+y^2 = 2z^2+w^2-T$where$x =$Woodall number, $y =$Cullen number, $T =$Carol number, $z = \tau(2^{\tau})$ *and* $w = 2^{\tau}-1$.

4 Conclusion

A formula for the Sum of m+1 consecutive Woodall numbers is obtained by utilizing a lemma. Matrix form for the sum and Recursive forms are attained here. The matrix form of sums of m+1 consecutive Cullen Numbers is also gained. In the application part some interesting associations between Special Numbers, Cullen Numbers and Carol Numbers are given. This study can be extended to more special numbers.

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