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RESEARCH ARTICLE

Common Fixed Point Results for Generalized $(\varphi, \phi, \mathrm{L})_{\mathrm{f},\,\mathrm{g}}$- Weak Contraction Maps and their Applications to Best Approximation

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Abstract

Objective/Aim: To find the fixed point theorems and best approximation results for the pair of self maps in the setting of metric and normed spaces. **Methods:** We used the notion of generalized $(\varphi, \phi, L)_{f,\,g}$-weak contraction for pair of maps in metric spaces and property (N) in Normed spaces. **Findings:** Unique common fixed points for four maps satisfying generalized $(\varphi, \phi, L)_{f,\,g}$-weak contraction condition and some best approximation results have been established. **Novelty/Improvements:** Our results extend and improve the existing results of Dahiya et al., Arya et al., He and Zhao etc. Examples have also been added in support of our results.

Keywords: Metric space; Normed space; Common fixed point; Generalized $(\varphi, \phi, \mathrm{L})_{f,g}$ - weak contraction; Best approximation

1 Introduction

In 1997, Alber and Guerre-Delabriere [1] generalized the Banach contraction principle by introducing the concept of weakly contractive mappings and proving the existence of fixed points for weakly contractive mappings in Hilbert spaces. Thereafter, Rhoades [2] assumed ϕ -weakly contractive mappings and Zhang and Song [3] introduced the generalized ϕ - weak contraction and proved the fixed point results for these mappings. Further, using the control function defined by Khan et al. [4], the above results have been generalized by many authors.

On the other hand, Berinde [5] introduced the notion of (k, L)-weak contraction and proved that many well known contractive conditions do imply (k, L)-weak contraction. Afterward, many authors studied this new class of weak contractions and obtained some significant results. In 2014, Rathee et al. [6] introduced the notion of $(\varphi, \phi, L)_{f,\,g}$- weak contraction and proved a fixed point result for this contraction. In 2017, He et al. [7] proved the common fixed point theorem for two mappings satisfying a generalized (φ, ϕ) - weak contractive type condition in a complete metric space. Afterwards, Dahiya et al. [8] established common fixed point results for generalized (φ, ϕ)- weak contractive mappings with constants in complete metric spaces. In 2023, Arya et al[9] generalized the results of He et al.[7].

In the present paper, we introduce the notion of generalized $(\varphi, \phi, L)_{f,g}$- weak contraction for the pair of self mappings and obtain common fixed point results for $(\varphi, \phi, L)_{f,g}$- weak contraction, which generalize the existing results for generalized (φ, ϕ) - weak contractive type conditions. As an application, some best approximation results are also established which generalize and extend various known results existing in literature.

Preliminaries

We now give some known definitions and standard notations that will be needed in the sequel:

Definition 2.1: Let (X, $\|\cdot\|$) be a normed space and T, S: X $\to$ X. A point x $\in$ X is called:

(1) fixed point of T if Tx = x.

(2) coincidence point of the pair $\{S, T\}$ if Tx = Sx,

(3) common fixed point of the pair $\{S, T\}$ if x = Tx = Sx.

We shall denote by Fix(T) the set of all fixed points of T and by C(S, T) the set of all coincidence points of S and T.

Definition 2.2: Let (X, $\|\cdot\|$) be a normed space and T, S: X $\to$ X. The pair {S, T} is called weakly compatible if STx = TSx for all x $\in$ C(S, T).

Definition 2.3: A subset M of a normed space (X, $\|\cdot\|$) is said to be convex if the segment

[x, y] = $\{(1-k)x + ky : 0 \leq k \leq 1\}$, joining x to y is contained in M for all x, y $\in$ M and k $\in$ [0, 1].

Definition 2.4: The subset M of a normed space (X, $\|\cdot\|$) is said to be q-starshaped if there exists q $\in$ M such that the segment [q, x] = $\{(1-k)q + kx : 0 \leq k \leq 1\}$, joining q to x is contained in M for all x $\in$ M and k $\in$ [0, 1].

Remark 2.5: Clearly, q-starshaped subsets of X contain all convex subsets of X as a proper subclass.

Definition 2.6[10]: A subset M of a normed space (X, $\|\cdot\|$) is said to have property (N) with respect to T if

(i) T: M$\to$M,

(ii) (1 – k_n)q + k_nTx $\in$ M, for some q $\in$ M and a fixed sequence of real numbers k_n (0 < k_n < 1) converging to 1 and for each x $\in$ M.

Remark 2.7 [10]: It is to be noted that each T-invariant q-starshaped set has property (N) but converse does not hold in general. This is shown by the following example:

Example 2.8 [10]: Let X = R be the set of real numbers and M = {$\frac{1}{n}$, where n is a natural number} be endowed with the usual norm. Define Tx = 1 for each x $\in$ M.

Then clearly, M is not

q-starshaped, but has property (N) with respect to T, for q = 1, k_n = 1– $\frac{1}{n}$.

Definition 2.9: Let M be a nonempty subset of a normed space (X, $\|\cdot\|$) and x $\in$ X. If there exists an element y_0 in M such that d(x, y_0) = d(x, M), then y_0 is called best approximations to x out of M. We denote by P_M(x), the set of all best approximation to x out of M.

2 Main Results

First, we introduce the concept of generalized $(\varphi, \phi, L)_{f,g}$-weak contraction for the pair of self mappings in the following way:

Definition 3.1: Let S, T: X $\to$ X be two self maps on the metric space (X, d). Then the pair {S, T} is called a generalized $(\varphi, \phi, L)_{f,g}$-weak contraction if for each x, y $\in$ X

$\varphi(d(Sx, Ty)) \leq \varphi(m(x, y)) - \phi(m(x, y)) + L\varphi(n(x, y))$,

where f, g: X $\to$ X, L $\geq$ 0,

m(x, y)= max{d(fx, gy), d(f x, Sx), d(gy, Ty), $\frac{1}{2}$ [d(fx, Ty) + d(gy, Sx)]},

n(x, y) = min {[d(fx, Sx) + d(gy, Ty)] , d(gy, Sx) , d(fx, Ty)} ,

and φ: [0,∞) $\to$ [0,∞) is a continuous nondecreasing function with φ(t) = 0 if and only if t = 0 and ϕ: [0,∞) $\to$ [0,∞) is a lower semi continuous function from right such that ϕ is positive on (0, ∞) and ϕ(0) = 0.

Remark 3.2: (i) If we take S = T in the above definition, then T is a generalized $(\varphi, \phi, L)_{f,g}$ - weak contraction.

(ii) if f = g = I (identity map), then {S, T} is called generalized(φ, ϕ, L)-weak contraction and if S = T, then T is called a generalized(φ, ϕ, L)-weak contraction.

(iii) If L = 0 and g = f = identity map, that is, m(x, y) coincides with M(x, y), then {S, T} is called generalized (φ, ϕ) - weak contraction.

(iv) If S = T, L = 0 and φ(t) = t, then T is called a generalized

$\varphi_{f,g}$ -weak contraction, which is the same as the generalized (f, g) -weak contraction investigated by Akbar et al. (see[11]) and if φ(t) = (1 - k)t for a constant k with 0 < k < 1, then T is called generalized (f, g) -contraction.

We now prove our main result which extends and improves many existing results, including the results of Rathee et al [6] and Dahiya et al. [8]. The following lemma is needed to prove our main result:

Lemma 3.3 [6]: If $\lim_{n\to\infty} d(y_{n+1}, y_n) = 0$ and $\{y_{2n}\}$ is a Cauchy sequence and converges to z, then $\{y_{2n+1}\}$ also converges to z.

Theorem 3.4: Let M be a nonempty subset of a metric space (X, d) and f, g, S, T be self mappings of M such that cl(T(M)) $\subseteq$ f(M) and cl(S(M)) $\subseteq$ g(M). Assume that cl(T(M)) or cl(S(M)) is complete and the pair {S, T} is a generalized $(\varphi,\ \phi,\ L)_{f,\,g}$-weak contraction. If the pairs {f, S} and {g, T} are weakly compatible, then M $\cap$ Fix(f) $\cap$ Fix(g) $\cap$ Fix(S) $\cap$ Fix(T) is singleton, i.e., f, g, S, T have a unique common fixed point in M.

Proof: Let x_0 be an arbitrary point in M. Since cl(T(M)) $\subseteq$ f(M), we can find $x_1 \in$ M such that Tx_0 = fx_1, and also, as cl(S(M)) $\subseteq$ g(M), there exists $x_2 \in$ M such that Sx_1 = gx_2. Continuing this process, we obtain a sequence $\{y_n\}$ in M such that for every n $\geq$ 0,

$y_{2n} = Tx_{2n} = fx_{2n+1}$ and $y_{2n+1} = Sx_{2n+1} = gx_{2n+2}$.

If, for some n, $y_{2n} = y_{2n+1}$, then $\{y_n\}$ turns out to be a constant sequence, hence it is Cauchy. Now suppose that $y_{2n} \neq y_{2n+1}$.

Using the fact that {S, T} is generalized $(\varphi,\ \phi,\ L)_{f,\,g}$ - weak contraction, for each n $\geq$ 0, we have

$\varphi(d(y_{2n+1}, y_{2n})) = \varphi(d\,(Sx_{2n+1}, Tx_{2n}))$

$$\leq \varphi\left(m\left(x_{2n+1}, x_{2n}\right)\right) - \phi\left(m\left(x_{2n+1}, x_{2n}\right)\right) + L\varphi\left(n\left(x_{2n+1}, x_{2n}\right)\right) \quad (3.1)$$

where

$m(x_{2n+1}, x_{2n}) = \max\{d(fx_{2n+1}, gx_{2n}), d(fx_{2n+1}, Sx_{2n+1}), d(gx_{2n}, Tx_{2n}),$
$\frac{1}{2}[d(fx_{2n+1}, Tx_{2n}) + d(gx_{2n}, Sx_{2n+1})]\}$

$= \max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \frac{1}{2}[d(y_{2n-1}, y_{2n+1}]\}$

$\leq \max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\}$

$n(x_{2n+1}, x_{2n}) = \min\{[d(fx_{2n+1}, Sx_{2n+1}) + (gx_{2n}, Tx_{2n})], d(gx_{2n}, Sx_{2n+1}), d(fx_{2n+1}, Tx_{2n})\}$

$= \min\{[d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})], d(y_{2n-1}, y_{2n+1}), 0\} = 0.$

Since $\varphi(0) = 0$, and $n(x_{2n+1}, x_{2n}) = 0$, therefore eqn. (3.1) becomes

$$\varphi\left(d\left(y_{2n+1}, y_{2n}\right)\right) \leq \varphi\left(m\left(x_{2n+1}, x_{2n}\right)\right) - \phi\left(m\left(x_{2n+1}, x_{2n}\right)\right) \quad (3.2)$$

This implies that

$\varphi(d(y_{2n+1}, y_{2n})) \leq \varphi(m(x_{2n+1}, x_{2n}))$.

As φ is a nondecreasing function, therefore, for each n $\geq$ 0, we have

$d(y_{2n+1}, y_{2n}) \leq m(x_{2n+1}, x_{2n})$.

Now, we show that $\{d(y_{n+1}, y_n)\}$ is monotonically decreasing sequence.

If possible, let $d(y_{2n+1}, y_{2n}) > d(y_{2n}, y_{2n-1})$, then, by (3.1),

$m(x_{2n+1}, x_{2n}) \leq d(y_{2n+1}, y_{2n})$ which implies that

$$\varphi\left(d\left(y_{2n+1}, y_{2n}\right)\right) \leq \varphi\left(d\left(y_{2n+1}, y_{2n}\right)\right) - \phi\left(d\left(y_{2n+1}, y_{2n}\right)\right) \quad (3.3)$$

This is a contradiction since $\phi(t) > 0$. Therefore, we have

$d(y_{2n+1}, y_{2n}) \leq d(y_{2n}, y_{2n-1})$.

Similarly, it can be shown that $d(y_{2n}, y_{2n-1}) \leq d(y_{2n-1}, y_{2n-2})$. Therefore, for each n $\geq$ 0, we have $d(y_{n+1}, y_n) \leq d(y_n, y_{n-1})$.

Thus, the sequence $\{d(y_{n+1}, y_n)\}$is nonincreasing. Also, the sequence $\{d(y_{n+1}, y_n)\}$ is bounded below, therefore, there exists r $\geq$ 0 such that

$\lim_{n\to\infty} d(y_{n+1}, y_n) = r$.

After letting n $\to \infty$ in (3.3), we obtain $\varphi(r) \leq \varphi(r) - \phi(r)$, which is a contradiction unless r = 0. Hence,

$$\lim_{n\to\infty} d\left(y_{n+1}, y_n\right) = 0 \quad (3.4)$$

Now, we show that $\{y_{2n}\}$ is a Cauchy sequence in M. Let m, n$\in N$ and m > n. Then,

$d(y_{2n}, y_{2m}) \leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, d(y_{2m-1}, y_{2m})$

Now, taking the limit as m, n $\to \infty$ and using (3.4), we get

$\lim_{m,\,n\to\infty} d(y_{2n}, y_{2m}) = 0$. Thus, $\{y_{2n}\}$ is a Cauchy sequence in cl(T(M)). Therefore, by the completeness of cl(T(M)), there exists some z $\in$ cl(T(M)) such that

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} f x_{2n+1} = z \quad (3.5)$$

By Lemma 3.3, $\{y_{2n+1}\}$ also converges to z. Therefore,

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} gx_{2n+2} = z \tag{3.6}$$

Further, cl(T(M)) $\subseteq$ f(M), therefore there exists u $\in$ M such that fu = z. Now, we show that u is a coincidence point of S and f in M. Since {S, T} is a generalized $(\varphi,\ \phi,\ L)_{f,\,g}$-weak contraction, therefore, we have

$$\leq \varphi\left(\mathrm{m}\left(\mathrm{u},\mathrm{x}_{2\mathrm{n}}\right)\right)-\phi\left(\mathrm{m}\left(\mathrm{u},\mathrm{x}_{2\mathrm{n}}\right)\right)+\mathrm{L}\varphi\left(\mathrm{n}\left(\mathrm{u},\mathrm{x}_{2\mathrm{n}}\right)\right), \tag{3.7}$$

where

m(u, x_{2n}) = max {d(fu, gx_{2n}), d(fu, Su), d(gx_{2n}, Tx_{2n}), $\frac{1}{2}$[d(fu, Tx_{2n}) + d(gx_{2n}, Su)]}

n(u, x_{2n}) = min {[d(fu, Su) + d(gx_{2n} , Tx_{2n})] , d(gx_{2n}, Su), d(fu, Tx_{2n})}

Now using (3.5) and (3.6), we get m(u, x_{2n}) $\to$ d(Su, z) and n(u, x_{2n}) $\to$ 0 as n $\to\infty$.

Therefore, letting n $\to\infty$ in (3.7), we get

φ(d(Su, z)) $\leq$ φ(d(Su, z)) – ϕ(d(Su, z)).

This is true only if d(Su, z) = 0, that is, Su = z. But z = fu, therefore, Su = fu. Thus, u is coincidence point of S and f in M. Since the pair {f, S} is weakly compatible, therefore,

Sz = Sfu = fSu = fz. Now, we show that Sz = z. Since {S, T} is generalized $(\varphi,\ \phi,\ L)_{f,\,g}$- weak contraction, we have

φ(d(Sz, fx_{2n+1})) = φ(d(Sz, Tx_{2n}))

$$\leq \varphi\left(\mathrm{m}\left(\mathrm{z},\mathrm{x}_{2\mathrm{n}}\right)\right)-\phi\left(\mathrm{m}\left(\mathrm{z},\mathrm{x}_{2\mathrm{n}}\right)\right)+\mathrm{L}\varphi\left(\mathrm{n}\left(\mathrm{z},\mathrm{x}_{2\mathrm{n}}\right)\right) \tag{3.8}$$

where

m(z, x_{2n}) = max {d(fz, gx_{2n}), d(fz, Sz), d(gx_{2n} , Tx_{2n}), $\frac{1}{2}$ [d(fz, Tx_{2n}) + d(gx_{2n}, Sz)]}

= max {d(Sz, gx_{2n}), d(Sz, Sz), d(gx_{2n} , Tx_{2n}), $\frac{1}{2}$ [d(Sz, Tx_{2n}) + d(gx_{2n}, Sz)]}

n(z, x_{2n}) = min {[d(fz, Sz) + d(gx_{2n} , Tx_{2n})] , d(gx_{2n}, Sz), d(fz, Tx_{2n})}

Now using (3.5) and (3.6), we can write

m(z, x_{2n}) $\to$ d(Sz, z) and n(z, x_{2n}) $\to$ 0 as n $\to\infty$.

Therefore, letting n $\to\infty$ in (3.8), we get

φ(d(Sz, z)) $\leq$ φ(d(Sz, z)) – ϕ(d(Sz, z)). This is true only if d(Sz, z) = 0; that is, Sz = z = fz.

The next step is to show that z is also a common fixed point for mappings g and T. Since cl(S(M)) $\subseteq$ g(M), there exists a point y in M such that Sz = gy. Now, we show that gy = Ty,

φ(d(gy, Ty)) = φ(d(Sz, Ty))

$\leq$ φ(m(z, y)) $-$ ϕ(m(z, y)) + L φ(n(z, y)),

where

m(z, y) = max {d(fz, gy), d(fz, Sz), d(gy , Ty), $\frac{1}{2}$ [d(fz, Ty) + d(gy, Sz)]}

= max {0, 0, d(gy, Ty), $\frac{1}{2}$ [d(gy, Ty) + 0]} = d(gy , Ty)

n(z, y) = min {[d(fz, Sz) + d(gy, Ty)] , d(gy, Sz), d(fz, Ty)} = 0.

Thus, we get

φ(d(gy, Ty)) $\leq$ φ(d(gy, Ty)) $-$ ϕ(d(gy, Ty)).

This is true only if d(gy, Ty) = 0; that is, gy = Ty. Now since {g, T} is weakly compatible, therefore Tz = TSz = Tgy = gTy = gz. Finally, we show that Tz = z.

φ(d(z, Tz)) = φ(d(Sz, Tz))

$\leq$ φ(m(z, z)) – ϕ(m(z, z)) + Lφ(n(z, z)), where

m(z, z) = max {d(fz, gz), d(fz, Sz), d(gz , Tz), $\frac{1}{2}$ [d(fz, Tz) + d(gz, Sz)]}

= max {d(z, Tz), 0, 0, $\frac{1}{2}$ [d(z, Tz) + d(z, Tz)]} = d(z, Tz)

n(z, y) = min {[d(fz, Sz) + d(gy, Ty)] , d(gy, Sz), d(fz, Ty)} = 0.

Thus, we get

φ(d(z, Tz)) $\leq$ φ(d(z, Tz)) $-$ ϕ(d(z, Tz)), which implies z = Tz and hence Sz = fz = Tz = gz = z. Thus, z is common fixed point of f, g, S and T in M.

The proof is similar if cl(S(M)) is complete. Moreover, it can be easily shown that this z is unique and hence, M $\cap$ Fix(f) $\cap$ Fix(g) $\cap$ Fix(S) $\cap$ Fix(T) is singleton.

Now, we give an example in support of Theorem 3.4:

Example 3.5: Let X = R and M = [0, 1) be equipped with the usual metric d(x, y) = $|x-y|$ for x, y $\in$ X. Let f, g, S, T: M $\to$ M be defined by

S(x) = log(1+ x), T(x) = log(1 + $\frac{x}{4}$), f(x) = e^{4x} - 1 and g(x) = e^{x} – 1.

Define φ, ϕ: [0,∞) → [0,∞) by φ(t) = 2t and ϕ(t) = $\frac{3}{2}$t for all t ≥ 0.

Now we will show that {S, T} is a generalized $(\varphi,\ \phi,\ L)_{f,\,g}$-weak contraction. Let x, y ∈ M, then using the mean value theorem, we have

$\varphi(\mathrm{d}(\mathrm{Sx},\mathrm{Ty})) = 2\,\mathrm{d}(\mathrm{Sx},\mathrm{Ty}) = 2\quad|\mathrm{Sx}-\mathrm{Ty}\quad| = 2\quad\left|\log(1+\mathrm{x})-\log\left(1+\frac{\mathrm{y}}{4}\right)\quad\right| \leq 2\cdot\frac{1}{4}\quad|4\mathrm{x}-\mathrm{y}\quad|$

$$\leq \frac{1}{2}\left|e^{4x}-e^{y}\right|$$

$$= \frac{1}{2}\quad|fx-gy|$$

$$\leq \frac{1}{2}\max\left\{|fx-gy|,|fx-Sx|,|gy-Ty|,\frac{1}{2}[|fx-Ty|+|gy-Sx|]\right\},$$

$$= \frac{1}{2}\ \mathrm{m}(x,y) = 2\ \mathrm{m}(x,y) - \frac{3}{2}\ \mathrm{m}(x,y)$$

$$= \varphi(\mathrm{m}(\mathrm{x},\mathrm{y})) - \phi(\mathrm{m}(\mathrm{x},\mathrm{y})).$$

This yields that

φ(d(Sx, Ty) ≤φ(m(x, y)) – ϕ(m(x, y)) + L φ(n(x, y)),

holds for all L ≥ 0. Therefore, all the conditions of Theorem 3.4 are satisfied and hence S, T, f, g have a common fixed point. Here it is clear that 0 is the unique common fixed point of S, T, f and g.

Remark 3.6: If we take S = T in Theorem 3.4, then we have the following result of Rathee et al.:

Corollary 3.7 [6]: Let M be a nonempty subset of a metric space (X, d) and f, g and T be self-mappings of M such that cl(T(M)) ⊆ f(M) ∩ g(M). Assume that cl(T(M)) is complete and T is a generalized $(\varphi,\ \phi,\ L)_{f,\,g}$- weak contraction. Then the pair (T, f) and (T, g) have a unique point of coincidence in M. Also, if the pairs (T, f) and (T, g) are weakly compatible, then M ∩ Fix(f) ∩ Fix(g) ∩ Fix(T) is singleton.

Corollary 3.8: Let (X, d) be a complete metric space and let T be a self-mapping of X. If T is a generalized (φ, ϕ)-weak contraction, then T has a unique fixed point.

Remark 3.9: If we take f = g = identity mappings in Theorem 3.4, then we have the following result:

Corollary 3.10: Let M be a nonempty subset of a metric space (X, d) and let S and T be self-mappings of M such that cl(T(M)) ⊆ M and cl(S(M)) ⊆ M. Assume that cl(T(M)) or cl(S(M)) is complete and the pair {S, T} is generalized (φ, ϕ, L) -weak contraction. Then, M ∩ Fix(S) ∩ Fix(T) is singleton.

Now, we prove another fixed point result using corollary 3.10:

Theorem 3.11: Let M be a nonempty subset of a metric space (X, d) and f, g, S, T be self mappings of M such that cl(T(M)) or cl(S(M)) is complete and the pair {S, T} is a generalized $(\varphi,\ \phi,\ L)_{f,\,g}$- weak contraction. Further assume that Fix(f) ∩ Fix(g) is nonempty, cl(S(Fix(f) ∩ Fix(g))) ⊆ Fix(f) ∩ Fix(g) and cl(T(Fix(f) ∩ Fix(g))) ⊆ Fix(f) ∩ Fix(g). Then M ∩ Fix(S) ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) is singleton.

Proof: Firstly, suppose that cl(T(M)) is complete. Then cl(T(Fix(f) ∩ Fix(g))) is complete by the completeness of cl(T(M)). Now, for any x, y ∈ Fix(f) ∩ Fix(g), we have:

φ(d(Sx, Ty) ≤φ(m(x, y)) – ϕ(m(x, y)) + Lφ(n(x, y)),

= φ(M(x, y)) – ϕ(M(x, y)) + Lφ(N(x, y)),

where

M(x, y) = max{d(x, y), d(x, Sx), d(y, Ty), $\frac{1}{2}$ [d(x, Ty) + d(y, Sx)]},

N(x, y) = min {[d(x, Sx) + d(y, Ty)], d(y, Sx) , d(x, Ty)}.

Hence {S, T} is generalized (φ, ϕ, L)-weak contraction mapping on Fix(f) ∩ Fix(g). Also, cl(S(Fix(f) ∩ Fix(g))) ⊆ Fix(f) ∩ Fix(g) and cl(T(Fix(f) ∩ Fix(g))) ⊆ Fix(f) ∩ Fix(g), therefore by corollary 3.10, S and T have a common fixed point in Fix(f) ∩ Fix(g) and consequently, M ∩ Fix(S) ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) is singleton.

The proof is similar if cl(S(M)) is complete.

Corollary 3.12 [6]: Let M be a nonempty subset of a metric space (X, d) and f, g and T be self-mappings of M such that cl(T(M)) is complete, T is generalized $(\varphi,\ \phi,\ L)_{f,\,g}$- weak contraction, Fix(f) ∩ Fix(g) is nonempty and cl(T(Fix(f) ∩ Fix(g))) ⊆ Fix(f) ∩ Fix(g). Then, M ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) is singleton.

In Theorem 3.11, if we take S = T and L = 0, then we easily obtain the following results which properly contains Theorem 3.3 of Akbar et al.[11].

Corollary 3.13: Let M be a nonempty subset of a metric space (X, d) and f, g, T be self mappings of M such that cl(T(M)) is complete, T is a generalized $(\varphi,\ \phi)_{f,\,g}$- weak contraction, Fix(f) ∩ Fix(g) is nonempty and cl(T(Fix(f) ∩ Fix(g))) ⊆ Fix(f) ∩ Fix(g). Then M ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) is singleton.

Common Fixed Point Theorems for Generalized $(\varphi,\ \phi,\ L)_{f,\,g}$- Weak Contraction in Normed Spaces:

Now, we prove common fixed point results for generalized $(\varphi,\ \phi,\ L)_{f,\,g}$- weak contraction in the setting of normed spaces, which contain similar results of Akbar et al.[11] and Rathee et al.[6]:

Theorem 4.1: Let M be a nonempty subset of a normed space X and S, T, f and g be self-maps of M. If Fix(f) ∩ Fix(g) has the property (N) with respect to S and T, cl(T(Fix(f) ∩ Fix(g))) ⊆ Fix(f) ∩ Fix(g) and cl(S(Fix(f)∩Fix(g))) ⊆ Fix(f) ∩Fix(g) and S, T, f, g satisfy

$$\varphi(\mathrm{k}\|\mathrm{Sx}-\mathrm{Ty}\|)\leq\varphi\left(m_1(\mathrm{x},\mathrm{y})\right)-\mathrm{k}\phi\left(m_1(\mathrm{x},\mathrm{y})\right)+\mathrm{kL}\varphi\left(n_1(\mathrm{x},\mathrm{y})\right) \tag{4.1}$$

for all k ∈ (0, 1) and x, y ∈ M, where

(a)φ: [0,∞) → [0,∞) is a continuous monotone nondecreasing function with φ(t) = 0 if and only if t = 0,

(b)ϕ: [0,∞) → [0,∞) is a lower semcontinuous function from right such that ϕ(t) > 0 if t > 0 and ϕ(0) = 0,

(c)m_1(x, y) = max{‖fx - gy‖, dist(fx, [q, Sx]), dist(gy, [q, Ty]), $\frac{1}{2}$[dist(gy, [q, Sx]) + dist(fx, [q, Ty])]},

(d)n_1(x, y) = min{[dist(fx, [q, Sx]) + dist(gy, [q, Ty])], dist(gy, [q, Sx]), dist(fx, [q, Ty])

Then, M ∩ Fix(S) ∩ Fix(T) ∩ Fix(f) ∩Fix(g) ≠∅, provided that cl(S(M)) or cl(T(M)) is compact, S and T are continuous.

Proof: As T(Fix(f)∩Fix(g)) ⊆ Fix(f)∩Fix(g), S(Fix(f)∩Fix(g)) ⊆ Fix(f)∩Fix(g) and Fix(f)∩Fix(g) has the property (N) with respect to S and T, therefore, for each n ∈ N, we can define T_n: Fix(f) ∩ Fix(g) →Fix(f) ∩Fix(g) by T_nx = (1− k_n)q+k_nTx and

S_nx = (1− k_n)q+k_nSx, for all x ∈ Fix(f) ∩Fix(g) and a fixed sequence of real numbers k_n (0 < k_n < 1) converging to 1. Since Fix(f) ∩ Fix(g) has the property (N) with respect to S, T and cl(T(Fix(f) ∩ Fix(g)))⊆Fix(f) ∩ Fix(g), cl(S(Fix(f) ∩ Fix(g))) ⊆ Fix(f) ∩ Fix(g), we have cl(T_n(Fix(f)∩Fix(g))) ⊆ Fix(f)∩Fix(g) and cl(S_n(Fix(f)∩Fix(g))) ⊆ Fix(f)∩Fix(g) for each n ∈N. Also, by the inequality (4.1)

φ(‖ S_nx – T_ny ‖) = φ(k_n ‖Sx – Ty‖)

≤φ(m_1(x, y)) – $k_n\phi$(m_1 (x, y)) + k_nLφ(n_1(x, y))

= φ(m_1(x, y)) – ϕ_n(m_1 (x, y)) + $L_n\varphi$(n_1(x, y)),

where ϕ_n = $k_n\phi$, L_n= k_nL,

m_1(x, y) = max{‖fx - gy‖, dist(fx, [q, Sx]), dist(gy, [q, Ty]), $\frac{1}{2}$[dist(gy, [q, Sx]) + dist(fx, [q, Ty])]},

≤ max{‖fx - gy‖, ‖fx - S_nx‖, ‖gy - T_ny‖, $\frac{1}{2}$[‖gy - S_nx‖ + ‖fx - T_ny‖]},

and

n_1(x, y) = min{[dist(fx, [q, Sx]) + dist(gy, [q, Ty])], dist(gy, [q, Sx]), dist(fx, [q, Ty]),

≤ min{[‖fx - S_nx‖ + ‖gy - T_ny‖], ‖gy - S_nx‖, ‖fx - T_ny‖].

for all x, y ∈ Fix(f) ∩ Fix(g) and 0 < k_n < 1. Clearly, ϕ_n: [0,∞) → [0,∞) is a lower semicontinuous function from right such that ϕ_n is positive on (0,∞) and ϕ_n(0) = 0 and L_n ≥ 0. Thus, for each n ∈N, {S_n, T_n} is generalized $(\varphi,\ \phi_n,\ L_n)_{f,\,g}$ - weak contraction.

Now, suppose that cl(T(M)) is compact. Therefore, for each n ∈ N, cl(T_n(M)) is compact and hence complete for each n ∈ N. By Theorem 3.11, for each n ≥ 1, there exists {x_n} in M such that x_n = f(x_n) = g(x_n) = S_n(x_n) = T_n(x_n).

Again, the compactness of cl(T(M)) implies that there exists a subsequence {Tx_m} of {Tx_n} such that Tx_m → z ∈ cl(T(M)). Since {Tx_m} is a sequence in T(Fix(f) ∩ Fix(g)) and cl(T(Fix(f) ∩ Fix(g))) ⊆ Fix(f) ∩ Fix(g), therefore z ∈ Fix(f) ∩ Fix(g).

Moreover

x_m = T_m (x_m) = (1 - k_m)q + k_mTx_m→ z,

As S and T are continuous on M, we have Sz = Tz = z. Similarly, we can prove the result if cl(S(M)) is compact. Thus, M ∩ Fix(S) ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) ≠∅.

In theorem 4.1, if we take S = T, then we obtain the following result, which generalize the result of Rathee et al.[6]:

Corollary 4.2 : Let M be a nonempty subset of a normed space X and T, f, g be self-maps of M. If Fix(f) ∩ Fix(g) has the property (N) with respect to T, cl(T(Fix(f)∩Fix(g))) ⊆ Fix(f) ∩Fix(g) and T, f, g satisfy

φ(k‖Tx – Ty ‖) ≤φ(m_1(x, y)) – kϕ(m_1 (x, y)) + kLφ(n_1(x, y)),

for all k ∈ (0, 1) and x, y ∈ M, where

(a)φ: [0,∞) → [0,∞) is a continuous monotone nondecreasing function with φ(t) = 0 if and only if t = 0,

(b)ϕ: [0,∞) → [0,∞) is a lower semicontinuous function from right such that ϕ is positive on (0,∞) and ϕ(0) = 0,

(c)m_1(x, y) = max{‖fx - gy‖, dist(fx, [q, Tx]), dist(gy, [q, Ty]), $\frac{1}{2}$[dist(gy, [q, Tx]) + dist(fx, [q, Ty])]},

(d)n_1(x, y) = min{[dist(fx, [q, Tx]) + dist(gy, [q, Ty])], dist(gy, [q, Tx]), dist(fx, [q, Ty])

Then, M ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) ≠∅, provided that cl(T(M)) is compact and T is continuous.

In Theorem 4.1, if we take S = T, φ(t) = t and ϕ(t) = ($\frac{1}{k}$ - 1)t for a constant k with 0 < k < 1, then we easily obtain the following result:

Corollary 4.3: Let M be a nonempty subset of a normed space X and T, f and g be self-maps of M. If Fix(f) ∩ Fix(g) has the property (N) with respect to T, cl(T(Fix(f)∩Fix(g))) ⊆Fix(f)∩Fix(g) and T, f, g satisfy

$\|Tx - Ty\| \leq m_1(x, y) + Ln_1(x, y)$, for all x, y ∈ M, where

$m_1(x, y) = \max\{\|fx - gy\|, dist(fx, [q, Tx]), dist(gy, [q, Ty]), \frac{1}{2}[dist(gy, [q, Tx])$

$+ dist(fx, [q, Ty])]\}$,

$n_1(x, y) = \min\{[dist(fx, [q, Tx]) + dist(gy, [q, Ty])], dist(gy, [q, Tx]), dist(fx, [q, Ty])$

Then, M ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) $\neq \varnothing$, provided that cl(T(M)) is compact and T is continuous.

Applications to Best Approximation:

Theorem 5.1: Let M be a nonempty subset of a normed space X and S, T, f, g be self mappings of X. If u ∈ X and D = P_M(u), D_0 = D ∩ Fix(f) ∩ Fix(g) is q-starshaped, cl(S(D_0)) ⊆D_0 and cl(T(D_0)) ⊆D_0, cl(S(D)) or cl(T(D)) is compact, S and T are continuous on D and (4.1) holds for all x, y ∈ D. Then, P_M(u) ∩ Fix(S) ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) $\neq \varnothing$.

Proof: Since D_0 is q - starshaped set and it is S and T- invariant, therefore, D_0 satisfies property (N) with respect to S and T. Also, all the conditions of theorem 4.1 are satisfied; therefore by theorem 4.1, P_M(u) ∩ Fix(S) ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) $\neq \varnothing$.

We now give another best approximation result:

Theorem 5.2: Let M be a nonempty subset of a normed space (X, ‖.‖) and S, T, f and g be continuous self mappings of X such that S(∂M ∩M) ⊆ M and T(∂M ∩M) ⊆ M, where ∂M denotes the boundary of M. Let u be common fixed point of f, g, S, T for some u ∈ X\M. Further suppose that D = P_M(u) ∩ Fix(f) ∩ Fix(g) has the property (N) with respect to S and T and q ∈ Fix(f) ∩ Fix(g) and f(D) = g(D) = D. If the conditions ‖Sx - Tu‖ ≤ ‖fx - gu‖ and ‖Su - Ty‖ ≤ ‖fu - gy ‖ and (4.1) holds for all x, y ∈ D and one of the cl(S(D)) or cl(T(D)) is compact, then P_M(u) ∩ Fix(S) ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) $\neq \varnothing$.

Proof. Let x ∈ D. Then, for any h ∈ (0, 1), ‖hu + (1- h)x - u‖ = (1- h) ‖x - u‖< dist(u, M). It follows that the line segment {hu + (1 - h)x : 0 < h < 1} and the set M are disjoint. Thus x is not in the interior of M and so x ∈ ∂M ∩M. Since S(∂M ∩M) ⊆ M and T(∂M ∩M) ⊆ M, therefore, Sx and Tx must be in M. Also, since f(D) = D, fx ∈ D and u is common fixed point of f, g, S, T, therefore, from the given contractive condition, we obtain

‖Sx - u‖ = ‖Sx - Tu‖≤ ‖fx - gu‖ = ‖fx - u‖ = d(u, M),

and ‖u - Tx‖ = ‖Su - Tx‖ ≤‖fu - gx‖ = ‖u - gx‖ = d(u, M). Thus, D is both S and T- invariant. Hence, S(D) ⊂ D = g(D) and T(D) ⊂ D = f(D) Thus all the conditions of the theorem 4.1 are satisfied; therefore, there exists z ∈ D such that z is a common fixed point of S and T, i.e., P_M(u) ∩ Fix(S) ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) $\neq \varnothing$.

Corollary 5.3: Let M be a nonempty subset of a normed space (X, ‖.‖) and T, f and g be continuous self mappings of X such that T(∂M ∩M) ⊆ M, where ∂M denotes boundary of M. Let u be common fixed point of f, g and T for some u ∈ X\M. Further suppose that D = P_M(u) ∩ Fix(f) ∩ Fix(g) has the property (N) with respect to T and q ∈ Fix(f) ∩ Fix(g) and f(D) = g(D) = D. If the conditions ‖Tx - Tu‖ ≤ ‖fx - gu‖ and (4.1) holds for all x, y ∈ D and cl(T(D)) is compact, then P_M(u) ∩ Fix(S) ∩ Fix(T) ∩ Fix(f) ∩ Fix(g) $\neq \varnothing$.

Remark 5.4: Corollary 5.1 and Theorem 5.2 extend and generalize the corresponding results of Akbar et al.[11] and Rathee et al.[6].

3 Conclusion

In this paper, we extend the notion of generalized $(\varphi, \phi, L)_{f,g}$-weak contraction mappings to the pair of mappings and prove the common fixed point theorems for such types of mappings in the setting of metric space and normed space. As an application, some best approximation results have also been established. The results proved in this paper generalize various known existing results in the literature. Examples are also given that verify our theorems.

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