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RESEARCH ARTICLE

2-Peble Triangles Over Figurate Numbers

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Abstract

Objectives: The main aim of this article is to discuss the existence or non-existence of 2-Peble triangles over some figurate numbers. **Methods:** A few quartic equations over integers are solved to complete the objective at hand. This is done with the aid of the transformation of variables. Additionally, fundamental concepts such as mathematical induction and parity of integers are used. **Findings:** Here, it is demonstrated that there are no 2-Peble triangles over triangular, hexagonal, and octagonal numbers. The same process is explained for particular special numbers as an exceptional instance. **Novelty:** This article defines a triangle, the d-Peble triangle over figurate numbers, which creates a link between the Pell equation and a common geometric shape. So many previous researchers, when examining a problem involving geometric shapes, attain their expected result using Diophantine equations. But this concept differs from those as this uses figurate numbers and a Pell equation to create a triangle.

Keywords: 2-Peble Triangle; Figurate Numbers; Pell Equations; Exponential Diophantine Triangle; Quartic Equation

1 Introduction

In this article, a triangle namely d-Peble triangle over a special number s_n is defined for a positive integer d, which is not a perfect square. This triangle is constructed by connecting the Pell equation and a special number. The non-existence of 2-Peble triangles over a specific variety of figurate numbers is the research problem chosen for investigation here.

When seen from an in-depth perspective, it is understood that the research question under attention rests primarily on resolving the Pell equation over the selected special numbers. In this article, the term "Peble" is defined, which bears the names of prominent mathematicians who made contributions to the quadratic Diophantine equation $x^2 - dy^2 = 1$. Such mathematicians are William Brouncker, an English mathematician, who discovered the solution method that Euler had incorrectly credited to Pell, and Lagrange who developed the entire theory that underlies the solution, which is based on the theory of continued fractions.

The articles that use Lucas numbers [1], x- coordinate of Pell's equation with powers of 2 [2], and Fibonacci numbers [3] in Pell equations served as inspirations for this article. New types of rectangles and triangles are defined and some of their properties are explored by using Diophantine equations in articles [4,5]. Likewise, this article connects both the Pell equation over special numbers and a well-known geometric shape (triangle)but innovatively and uniquely. There are a few quartic Diophantine equations that must be solved to arrive at the non-existence. To resolve those equations, the method of variable transformation is adopted as in [6,7]. Also, the basic number theoretic concepts are retrieved from "Fundamental Perceptions in Contemporary Number Theory" [8], contributed by J. Kannan and Manju Somanath.

This article can be categorized into the following sections and subsections, excluding the introduction, conclusion, and references: Methodology (section 2), 2-Peble triangles over T_n(subsection 3.1), 2-Peble triangles over h_n(subsection 3.2), 2-Peble triangles over O_n(subsection 3.3) and 2-Peble triangles over g_n and P_n(subsection 3.4).

2 Methodology

The definitions and examples required for the article's primary work are provided in this part. Also solutions of the Pell equation $x^2-2y^2=1$ are displayed.

Definition 2.1. (Figurate number). A number that may be expressed as a regular geometrical arrangement of equally spaced points is commonly known as a figurate number.

Example. Triangular, Hexagonal, and Octagonal numbers.

Definition 2.2. (Formula for T_n**).** For $n \in \mathbb{N}$, the triangular number is given by $T_n=\frac{n(n+1)}{2}$.

Example. $T_1=1, T_2=3, T_{60}=1830$

Definition 2.3. (Formula for h_n). If n is a natural number, then the hexagonal number is given by $h_n=2n^2-n$.

Example. $h_1=1, h_2=6, h_{17}=561$

Definition 2.4. (Formula for O_n). If n is an integer greater than 1, then the octagonal number is given by $O_n=3n^2-2n$.

Example. $O_1=1, O_2=6, O_{20}=1160$

Definition 2.5. (Formula for g_n). For $n \in \mathbb{N}$, the gnomonic number is given by $g_n=2n-1$.

Example. $g_1=1, g_2=3, g_5=11$

Definition 2.6. (Formula for P_n). For $n \in \mathbb{N}$, the pronic number is given by $P_n=n(n+1)$.

Example. $P_1=2,\ P_2=6, P_9=90$

Definition 2.7. [9] (Pell equation). If d is a positive integer that is not a perfect square, then the equation $x^2-dy^2=1$ is known as the Pell equation.

Lemma 2.8. [10] (Solutions of $x^2-2y^2=1$**).** The equation $x^2-2y^2=1$ has infinitely many solutions (x,y) in positive integers, and the general solution is given as

$$x_{n+1}=3x_n+4y_n$$

$$y_{n+1}=2x_n+3y_n$$

Note 2.9 . If x and y are integers such that $x^2-2y^2=1$, then x is odd and y is even.

Definition 2.10. (d-Peble triangle over special number s_n). Let d be a positive integer which is not a perfect square. A triangle is said to be a d-Peble triangle over special number s_n if it has the sides $m+n, mn, 2mn$ where m and n are distinct positive integers that satisfies the relation $s_m^2-ds_n^2=1$.

Example. A triangle with sides $3,2,4$ is an 8-Peble triangle over triangular number T_n, since $T_2^2-8T_1^2=3^2-8\left(1^2\right)=1$.

3 Results and Discussion

The non-existence of 2-Peble triangles over a few figurate numbers (triangular, hexagonal, and octagonal numbers) is looked at in this section. Also, in the end, the same concept for gnomonic and pronic numbers is observed.

3.1 2-Peble Triangles over T_n

This section deals with the 2-Peble triangles over triangular number T_n.

Lemma 3.1.1. $\frac{t^2-1}{8}$ is a triangular number for all odd integers $t>1$.

Proof. Let $t=2k+1$ where k is a positive integer. Then $\frac{t^2-1}{8}=\frac{k(k+1)}{2}$, a triangular number.

Lemma 3.1.2. $(\alpha,k)=(1,1)$ is the unique positive integer solution of the equation $\alpha^4-2\alpha^2-32k^2+33=0$ such that $k\leq\alpha$.

Proof. If $\alpha=1$, then $k=1$ is trivial to view. Also one can easily see that if $\alpha=2,3,4,5,6,7$, then k is not an integer. So in all these cases, there is no solution in N for the considered equation. Let us prove by induction that the equation

$$\alpha^4-2\alpha^2-32k^2+33=0 \tag{1}$$

has no solution for all $\alpha\geq 8$.

If $\alpha=8$, then $32k^2=4001$. This doesn't give a suitable integer k. So in this case there is no solution to the equation (1). Assume that the equation (1) has no solution for $\alpha-1$. Then

$$(\alpha-1)^4-2(\alpha-1)^2-32k^2+33=\beta(say)$$

where $\beta\neq 0$. On simplification, it is clear that

$$\beta=\alpha^4+4\alpha^2-4\alpha^3-32k^2+32 \tag{2}$$

Now suppose that $\alpha^4-2\alpha^2-32k^2+33=0$. Then $\beta=t\alpha^2-1$ where $t=-2(2\alpha-3)$. From equation (2), it is found that $t=\alpha^2+4-4\alpha-\frac{32k^2}{\alpha^2}+\frac{33}{\alpha^2}$.

Since $t<0$, it follows that

$$\alpha^4-4\alpha^3+4\alpha^2<32k^2-33$$

$$\alpha^4-4\alpha^3-28\alpha^2\leq -33$$

This is because of the fact that $k\leq\alpha$. But for $\alpha>8$, the term $\alpha^4-4\alpha^3-28\alpha^2\geq 0$. Thus, we get a contradiction. Hence the result follows.

Lemma 3.1.3. There are no positive integers r and $k(>1)$ such that $r^4+2r^3+r^2+2-2k^2=0$.

Proof. Replacing r with $u-\frac{1}{2}$ in the equation $r^4+2r^3+r^2+2-2k^2=0$, we get $u^4-\frac{1}{2}u^2+2-2k^2+\frac{1}{16}=0$. Again replacing u by $\sqrt{t}$, we get $16t^2-8t+33-32k^2=0$. The suitable zero of this equation is found as $t=\frac{1+4\sqrt{2k^2-2}}{4}$ (neglect the impossible other zero).

Thus we obtain $u=\sqrt{\frac{1+4\sqrt{2k^2-2}}{4}}=\frac{\sqrt{1+4\sqrt{2k^2-2}}}{2}$.

Since r is an integer, u must be a rational number. So $1+4\sqrt{2k^2-2}=\alpha^2$ for some integer α. This leads to the equality $\alpha^4-2\alpha^2-32k^2+33=0$. By lemma 3.1.2, k must be equal to 1. This completes the proof.

Lemma 3.1.4. Any positive integer y in $x^2-2y^2=1$ cannot be a triangular number.

Proof. From the Pell equation $x^2-2y^2=1$, one can find y as $y=\sqrt{\frac{x^2-1}{2}}$. Suppose $y=\frac{r(r+1)}{2}$ for some positive integer r. Then

$$\sqrt{\frac{x^2-1}{2}}=\frac{r(r+1)}{2}$$

$$\frac{x^2-1}{2}=\frac{r^2(r+1)^2}{4}$$

$$r^4+2r^3+r^2-2x^2+2=0$$

By lemma 2.8, it is seen that $x>1$. Hence by lemma 3.1.3, this situation won't happen.

Theorem 3.1.5. It is impossible to construct a 2-Peble triangle over T_n.

Proof. Suppose there exists distinct positive integers m and n such that $T_m^2-2T_n^2=1$. Then

$$\left(\frac{m(m+1)}{2}\right)^2-2\left(\frac{n(n+1)}{2}\right)^2=1$$

$$M^2 - 2N^2 = 4$$

where $M = m(m+1)$, and $N = n(n+1)$. If $x^2 - 2y^2 = 1$, then $(2x)^2 - 2(2y)^2 = 4$. This leads to the fact that $M = 2x$ and $N = 2y$.

Hence $m = \frac{-1+\sqrt{1+8x}}{2}$ and $n = \frac{-1+\sqrt{1+8y}}{2}$. As m and n are integers, both $\sqrt{1+8x}$ and $\sqrt{1+8y}$ are integers. By lemma 3.1.1, both x and y are triangular numbers. But lemma 3.1.4 shows that y cannot be a triangular number. Here is the proof.

3.2 2-Peble Triangles over h_n

This section deals with the 2-Peble triangles over hexagonal number h_n.

Lemma 3.2.1. $(\alpha, y) = (5, 2)$ is the unique positive integer solution of the equation $\alpha^4 - 2\alpha^2 - 128y^2 - 63 = 0$.

Proof. If $\alpha = 5$, then $y = 2$ is trivial to view. One can easily see that if $\alpha = 1,2,3,4,6,7,8,9,10,11,12,13$, then y is not an integer. So in all these cases, there is no solution in N for the considered equation. Let us prove by induction that the equation

$$\alpha^4 - 2\alpha^2 - 128y^2 - 63 = 0 \tag{3}$$

has no solution for all $\alpha \geq 14$.

If $\alpha = 14$, then $128y^2 = 37961$. This doesn't give a suitable integer y. So, in this case, there is no solution to the equation (3). Assume that the equation (3) has no solution for $\alpha - 1$. Then

$$(\alpha-1)^4 - 2(\alpha-1)^2 - 128y^2 - 63 = \beta(say)$$

where $\beta \neq 0$. On simplification, it is clear that

$$\beta = \alpha^4 + 4\alpha^2 - 4\alpha^3 - 64 - 128y^2 \tag{4}$$

Now suppose that $\alpha^4 - 2\alpha^2 - 128y^2 - 63 = 0$. Then $\beta = t\alpha^2 - 1$ where $t = -2(2\alpha - 3)$. From equation (4), it is found that $t = \alpha^2 + 4 - 4\alpha - \frac{128y^2}{\alpha^2} - \frac{63}{\alpha^2}$. Since $t < 0$, it follows that

$$\alpha^4 - 4\alpha^3 + 4\alpha^2 < 63 + 128y^2$$

$$\alpha^4 - 4\alpha^3 - 124\alpha^2 < 63$$

But for $\alpha > 14$, the term $\alpha^4 - 4\alpha^3 - 124\alpha^2 > 63$. Thus we get a contradiction. Hence, the result follows.

Lemma 3.2.2. There do not exist positive integers m and y such that $4m^4 - 4m^3 + m^2 - 2y^2 - 1 = 0$.

Proof. The equation $4m^4 - 4m^3 + m^2 - 2y^2 - 1 = 0$ becomes $u^4 - \frac{1}{8}u^2 - \frac{63}{256} - \frac{1}{2}y^2 = 0$, if m is replaced by $u + \frac{1}{4}$. The latter equation becomes $256t^2 - 32t - 63 - 128y^2 = 0$, if u is replaced by $\sqrt{t}$. This gives $t = \frac{1+8\sqrt{1+2y^2}}{16}$ and so $u = \frac{\sqrt{1+8\sqrt{1+2y^2}}}{4}$. So $1 + 8\sqrt{1+2y^2} = \alpha^2$ for some integer α. This leads to the equality $\alpha^4 - 2\alpha^2 - 128y^2 - 63 = 0$. By lemma 3.2.1, one can see that this equality does not hold.

Theorem 3.2.3. It is impossible to construct a 2-Peble triangle over h_n.

Proof. Suppose there exists distinct positive integers m and n such that $h_m^2 - 2h_n^2 = 1$. Then $\left(2m^2 - m\right)^2 - 2\left(2n^2 - n\right)^2 = 1$. Take $x = 2m^2 - m$. Substituting x in $x^2 - 2y^2 = 1$ gives $4m^4 - 4m^3 + m^2 - 2y^2 - 1 = 0$. By lemma 3.2.2, we see that this equation has no positive integer solution.

3.3 2-Peble Triangles over o_n

This section deals with the 2-Peble triangles over octagonal number o_n .

Lemma 3.3.1. The equation $\alpha^4 - 2\alpha^2 - 18y^2 - 8 = 0$ has no positive integer solution (α, y).

Proof. One can easily see that if $\alpha = 1,2,3,4,5$, and 6, then y is not an integer. So in all these cases, there is no solution in N for the considered equation. Let us prove by induction that the equation

$$\alpha^4 - 2\alpha^2 - 18y^2 - 8 = 0 \tag{5}$$

has no solution for all $\alpha \geq 7$.

If $\alpha = 7$, then $18y^2 = 2303$. This doesn't give a suitable integer y. So in this case there is no solution to the equation (5). Assume that the equation (5) has no solution for $\alpha - 1$. Then

$$(\alpha-1)^4 - 2(\alpha-1)^2 - 18y^2 - 8 = \beta(say)$$

where $\beta \neq 0$. On simplification, it is clear that

$$\beta = \alpha^4 + 4\alpha^2 - 4\alpha^3 - 9 - 18y^2 \tag{6}$$

Now suppose that $\alpha^4 - 2\alpha^2 - 18y^2 - 8 = 0$. Then $\beta = t\alpha^2 - 1$ where $t = -2(2\alpha - 3)$. From equation (6), it is found that $t = \alpha^2 + 4 - 4\alpha - \frac{18y^2}{\alpha^2} - \frac{8}{\alpha^2}$. Since $t < 0$, it follows that

$$\alpha^4 - 4\alpha^3 + 4\alpha^2 < 8 + 18y^2$$

$$\alpha^4 - 4\alpha^3 - 12\alpha^2 < 8$$

But for $\alpha \geq 8$, the term $\alpha^4 - 4\alpha^3 - 12\alpha^2 > 8$. Thus we get a contradiction. Hence the result follows.

Lemma 3.3.2. One cannot find positive integers m and y such that $9m^4 - 12m^3 + 4m^2 - 1 - 2y^2 = 0$.

Proof. Replacing m with $u + \frac{1}{3}$ in the equation $9m^4 - 12m^3 + 4m^2 - 1 - 2y^2 = 0$, we get $81u^4 - 18u^2 - 8 - 18y^2 = 0$. Again replacing u by $\sqrt{t}$, we get $81t^2 - 18t - 8 - 18y^2 = 0$. The suitable zero of this equation is found as $t = \frac{1+3\sqrt{1+2y^2}}{9}$ (neglect the impossible other zero). Thus we obtain $u = \sqrt{\frac{1+3\sqrt{1+2y^2}}{9}} = \frac{\sqrt{1+3\sqrt{1+2y^2}}}{3}$. Here $1 + 3\sqrt{1+2y^2} = \alpha^2$ for some integer α. This leads to the equality $\alpha^4 - 2\alpha^2 - 18y^2 - 8 = 0$. By lemma 3.3.1, one can see that this equality does not hold. This completes the proof.

Theorem 3.3.3. It is impossible to construct a 2-Peble triangle over O_n.

Proof. Suppose there exists distinct positive integers m and n such that $O_m^2 - 2O_n^2 = 1$. Then $\left(3m^2 - 2m\right)^2 - 2\left(3n^2 - 2n\right)^2 = 1$. Take $x = 3m^2 - 2m$. Substituting x in $x^2 - 2y^2 = 1$ gives $9m^4 - 12m^3 + 4m^2 - 2y^2 - 1 = 0$. By lemma 3.3.2, we see that this equation has no positive integer solution.

3.4 2-Peble Triangles over g_n and P_n

It is clear that g_n is odd for all $n \in N$ and P_n is even for all $n \in N$. So by note 2.9, it is apparent that there is no solution for the equations $g_m^2 - 2g_n^2 = 1$ and $P_m^2 - 2P_n^2 = 1$. Thus there are no 2-Peble triangles over g_n and P_n.

4 Conclusion

The non-existence of 2-Peble triangles over some figurate numbers, such as triangular (T_n), hexagonal(h_n), and octagonal(O_n) numbers, has been effectively explored in this article. Furthermore, it is demonstrated that no such triangles can be formed for gnomonic or pronic numbers. This leads to the fact that, if s_n is a special number such that s_m and s_n are of the same parity for distinct positive integers m and n, one can see that there is no 2-Peble triangle over s_n. In terms of resolving the Pell equation over particular integers, this article represents a new direction. This study can be expanded by looking at different special numbers or by taking into account different options for d.

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