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* **Corresponding author**.

santhivaiyapuri2019@gmail.com

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Connectedness in Projection Graphs of Commutative Rings

S Teresa Arockiamary[1], C Meera[2], V Santhi[3]*

1 Assistant Professor, Department of Mathematics, Stella Maris College, University of Madras, Tamil Nadu, India
2 Assistant Professor, Department of Mathematics, Bharathi Women's College, University of Madras, Tamil Nadu, India
3 Assistant Professor, Department of Mathematics, Presidency College, University of Madras, Tamil Nadu, India

Abstract

Background/Objectives: Connectedness plays an essential role in applications of graph theory. Connected graphs are used to represent social networks, transportation networks, and communication networks. In this study, various classes of commutative rings are explored to identify connected projection graphs. **Methods:** An investigation is carried out to exclude classes of rings whose projection graphs possess isolated vertices and connectedness is shown by establishing spanning subgraphs. **Findings :** Unipotent units and zero-divisors are found non-isolated, in which zero-divisors include idempotents and nilpotents. The element 2 is isolated if it is invertible. Necessary condition for the rings to have connected projection graphs is derived. Projection graphs of finite Boolean rings are connected and Hamiltonian. The local rings of integers modulo n with even characteristics are connected and contain spanning bistars. A criterion for certain class of nonlocal rings to have connected projection graphs is described. **Novelty:** Study on projection graphs in the perception of unipotent units is carried out, which is not done earlier in any other algebraic graph.

Keywords: Unipotent; Von Neumann regular ring; Boolean ring; Spanning subgraph; Bistar

1 Introduction

The structure of ring is studied by using graph theoretic tools. Graphs from rings are defined with the help of algebraic relations between elements or substructures of rings. During last thirty years, many authors published research articles on zero-divisor graphs in different forms. To mention some of them, Abdulaziz et al. characterized rings using generalized zero-divisor graph[(1)]. In 2021 Vijay Kumar Bhat considered matrix ring of order n over Z_p and associated zero-divisor graph[(2)]; Pradeep Singh et al. investigated connectivity[(3)]; Sriparna Chattopadhyay et al. studied Laplacian eigenvalues[(4)].

In this line, projection graph $P_1(R)$ of ring R with unital element $1 \neq 0$ is introduced as graph with nontrivial elements of R as vertices and $x \sqcap y(x, y\backslash$ are adjacent) iff xy equals either x or y. Connectedness of $P_1(R)$ is investigated in this paper.

Several kinds of rings are considered for investigation and by method of exhaustion results are obtained. For rings of size greater than four, it is proved that $P_1(R)$ is totally disconnected if R is domain. Zero-divisors and unipotent units are proved as non-isolated. Local rings of characteristic 2 are characterized via connectivity of $P_1(R)$. Connectedness of projection graphs of some local and nonlocal rings are established by showing the presence of spanning stars and bistars. Projection graphs of Boolean rings are also proved to be connected.

One can refer [5] for basic concepts, properties and terminologies used in this article. "$x \in R\backslash\{0,1\}$ is called nontrivial". "Any x in R is Von Neumann regular if there is a with $x = x^2\ a$ and R is Von Neumann regular ring (VNR) if every x in R is Von Neumann regular [5]". "A unit x is unipotent if there is nilpotent z such that $x = 1+z$ and ring whose units are all unipotents is called UU ring [6]". "Bistar $B_{k,k}$ is graph obtained from K_2 by joining centers of $2K_{1,n}$ by an edge [7]".

2 Results and Discussion

In this section a criterion for $P_1(R)$ to have no isolated vertices is determined. Subclasses of different kinds of rings with this criterion are analyzed for identifying rings with connected projection graphs.

Proposition 2.1 If R is of cardinality four, then $P_1(R)$ is either totally disconnected or complete.

Proof: If R is with cardinality four, then R is one of non-isomorphic rings $Z_2 \times Z_2$, F_4, Z_4 and R', where R' is ring of matrices $\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}$, $x,y \in Z_2$. Now $P_1(Z_2 \times Z_2) \cong P_1(F_4) \cong \overline{K_2}$, which is totally disconnected and, $P_1(Z_4) \cong P_1\left(R'\right) \cong K_2$, which is complete.

As a consequence of the above proposition R is considered as ring with five or more elements throughout this section.

Proposition 2.2 In $P_1(R)$, (i) zero-divisors are non-isolated.

(ii) Let $x(\neq 2)$ be nonzero-divisor. Then x is non-isolated if $1-x$ is zero-divisor (equivalently, $Ann(1-x)$ is nontrivial).

Proof: (i) Suppose $x \in ZD(R)\backslash\{0\}$ and $xy = 0$. If y equals x, then $x^2 = 0$ and $x(1-x) = x$, where $1-x$ is nontrivial and $x \sqcap (1-x)$.

If y is different from x, then $x(1-y) = x$, where $1-y$ is nontrivial and $x \sqcap (1-y)$ if $1-y \neq x$. Suppose $1-y = x$. Then x is idempotent. If $xR \neq \{0,x\}$, there is nontrivial element z different from x in xR and $z = xr$ for nontrivial element r of R, which implies $xz = x(xr) = xr = z$. Thus, $x \sqcap z$. If $xR = \{0,x\}$, consider nontrivial element z in $(1-x)R$ different from $1-x$. Now $z = (1-x)r$ for nontrivial r, which implies $xz = 0$ and $x \sqcap (1-z)$, which completes proof of (i).

(ii) If x is non-isolated, there is nontrivial y with $xy = x$ or $xy = y$, which implies $x(1-y) = 0$ or $y(1-x) = 0$. Also, $1-x$ and $1-y$ are nontrivial and $x(1-y) = 0$ does not hold. Therefore, $y(1-x) = 0$ must be true and hence $1-x$ is zero-divisor, equivalently $Ann(1-x)$ is nontrivial.

Suppose $1-x \in ZD(R)$, in other words, $Ann(1-x)$ is nontrivial. Then $(1-x)z = 0$ for nontrivial z. Therefore, $xz = z$, where z is different from x. Hence, x is non-isolated.

UU rings guarantee the existence of projection graphs without isolated units.

Proposition 2.3 (i) $P_1(R)$ is totally disconnected if R is domain.

(ii) Nilpotents (respectively, idempotents) are non-isolated.

(iii) If R is Boolean ring, *$P_1(R)$* has no isolated vertex.

(iv) Unipotents are non-isolated.

(v) If R is UU, $P_1(R)$ has no isolated units.

(vi) If R is $F_2\ [x]/(x^k)$ (respectively, Z_{2^k}, for $k \geq 2$), $P_1\ (R)$ has no isolated units.

(vii) Let *char R* be 2^k, $k \geq 1$. If *$U(R)$* is 2-group, *$P_1(R)$* has no isolated unit.

Proof: (i) Suppose R is domain and $x \in R\backslash\{0,1\}$. If $xy = x$ (respectively, $xy = y$), then $x(1-y) = 0$ (respectively, $y(1-x) = 0$), which implies $y = 1$ (respectively, $y = 0$) since R has no zero-divisors. Hence, there is no $y \in Z_n\backslash\{0,1\}$ adjacent to x. Thus, $P_1(R)$ is totally disconnected.

Suppose R is not domain and $x_0y_0 = 0$ for $x_0, y_0 \in R\backslash\{0,1\}$. If $x_0 = y_0$, then $x_0^2 = 0$ and $x_0(1+x_0) = x_0$, where $1+x_0 \in R\backslash\{0,1,x_0\}$. If $x_0 \neq y_0$, then $x_0(1+y_0) = x_0$, where $1+y_0 \in R\backslash\{0,1,x_0\}$.

Thus, $x_0 \sqcap (1+y_0)$.

(ii) As nilpotents and idempotents are zero-divisors, (ii) follows from Proposition 2.2 (i).

(iii) If R is Boolean, then every nontrivial element is zero-divisor and hence $P_1\ (R)$ has no isolated vertices.

(iv) Let x be unipotent. Then $x = 1+z$ for nilpotent z. Suppose s is nilpotency index of z. Then $z^s = 0$. If x_0 denotes x^{s-1}, then $xx_0 = (1+z)x_0 = x_0$ and $x \sqcap x_0$.

(v) If R is UU, every unit is unipotent and thus no unit is isolated.

(vi) As $F_2[x]/(x^k)$, Z_{2^k} are UU [page 450, 6], their projection graphs have no isolated units.

(vii) Suppose $U(R)$ is $2-$group. Then R is UU ring by Theorem A [6], which shows $P_1(R)$ has no isolated unit.

In class of rings with *char* $R \neq 2$, element 2 plays an important role in determining connected projection graphs.

Proposition 2.4 If *char* R is $m(\neq 2)$, following are equivalent:

(i) 2 is non-isolated (ii) 2 is zero-divisor (iii) $Ann(2)$ is nontrivial (iv) m is even (v) -1 is non-isolated.

Proof: 2 is nontrivial by hypothesis.

(i) $\Leftrightarrow$(ii)

Suppose 2 is non-isolated. Then there is nontrivial x different from 2 with $2x = x$ or $2x = 2$.

If $2x = x$, then $x = 0$, which contradicts the choice of x. Therefore, $2x = 2$ holds, which implies $2(1-x) = 0$, where $1-x$ is nontrivial, as desired.

Suppose $2 \in ZD(R)$. Then there is nontrivial y with $2y = 0$, which gives $2 = 2(1-y)$, where $1-y$ is nontrivial and $1-y \neq 2$. Hence, $2\sqcap(1-y)$, which shows that 2 is non-isolated.

(ii) $\Leftrightarrow$ (iii)

It is known that 2 is zero-divisor if its annihilator is nontrivial.

(ii) $\Leftrightarrow$ (iv)

Suppose m is even.

Then $1+\cdots+1(m \text{ times}) = 0$ which implies $2+\cdots+2$ $(m/2 \text{ times}) = 0$.

Let $1+\cdots+1(m/2 \text{ times}) = x$. Then $2x = 0$, where x is nontrivial, which shows (iv) implies (ii).

If 2 is zero-divisor, there is nontrivial x with $2x = 0$, which shows additive order of x is 2. Hence, 2 divides m and thus (ii) implies (iv).

(ii) $\Leftrightarrow$ (v)

Suppose 2 is zero-divisor and $2x = 0$. Then $(-1)x = x$, showing $x \sqcap (-1)$.

Suppose -1 is non-isolated and $(-1)x = -1$ or $(-1)x = x$. If $(-1)x = -1$, then $x = 1$. Therefore, $(-1)x = x$ must hold, which implies $2x = 0$. Hence, (v) is equivalent to (ii).

Proposition 2.5 If $P_1(R)$ has no isolated vertex, then either *char* R is two or 2 is zero-divisor (equivalently, $Ann(2)$ is not trivial).

Proof: Suppose *char* R is not 2. If 2 is not zero-divisor, then by Proposition 2.4, 2 is isolated, which proves the result.

Illustration 2.6 (i) In $P_1(Z_{12}), Ann(2)$is not trivial and $2\sqcap 7$ and $11\sqcap 6$. (ii) In $P_1(Z_{21}), Ann(2)$ is trivial and 2, 20 are isolated.

Theorem 2.7 Let *char* R be two. Then $P_1(R)$ has no isolated vertex if either x is zero-divisor (equivalently, $Ann(x) = \{0\}$) or $1-x$ is zero-divisor (equivalently, $Ann(1-x) = \{0\}$) for every nontrivial element x.

Proof: Suppose $P_1(R)$ has no isolated vertex.

Let x be nontrivial and $xy = x$ or $xy = y$. Then either x or $1-x$ is zero-divisor.

Suppose $P_1(R)$ has isolated vertex x and, x is not zero-divisor. Then $1-x$ is zero-divisor by

Proposition 2.2 (ii) since $x \neq 2$ by hypothesis.

Proposition 2.8 (i) Let R be local ring. If $P_1(R)$ has no isolated vertex, *char* $R = 2^k, k \geq 1$.

(ii) Let *char* R be 2. $P_1(R)$ has no isolated vertex if $R = ZD(R) \cup (1+ZD(R))$.

Proof: (i) If *char* $R = m$,m is p^k, $k \geq 1$ by hypothesis.

Suppose $P_1(R)$ has no isolated vertex. If m is not two, then 2 is nontrivial element of R. Therefore, 2 is non-isolated and by Proposition 2.4 m must be even, which shows that *char* R is 2^k.

(ii) By hypothesis, $x+x = 0$ for every x. Hence, by Proposition 2.7, $P_1(R)$ has no isolated vertex if either x is zero-divisor or $1+x$ is zero-divisor for every nontrivial element x, which completes the proof.

Proposition 2.9 (i) Let R be local Artinian ring with *char* $R = 2^k, k \geq 2$. If $R = 2R \cup (1+2R)$, $P_1(R)$ contains spanning bistar $B_{|2R|-2,|2R|-2}$.

(ii) If R is $Z_{2^{k+1}}$, $k \geq 2$, $P_1(R)$contains spanning bistar $B_{2^{k-2},2^{k-2}}$ with center $(2^k, 1+2^k)$.

Proof: (i) By hypothesis 2 is nilpotent. Suppose $2^s = 0$. If x_0 denotes 2^{s-1}, then x_0 and $1+x_0$ are nontrivial elements of $2R$ and $(1+2R)$, respectively. Now $x_0 \sqcap y$, for $y \in (1+2R\backslash\{0\})$ and, $(1+x_0)\sqcap z$) for z in $2R\backslash\{0\}$. Also, $x_0 \sqcap (1+x_0)$. Thus induced subgraph on $(2R \cup (1+2R)) \setminus (0,1\}$ contains spanning bistar with center $(x_0, 1+x_0)$, as desired.

(ii) By hypothesis, 2 is nilpotent and $2^{(k+1)} = 0$. Hence, (ii) follows from (i).

Illustration 2.10 If R is Z_{16}, then 2 is nilpotent with degree of nilpotency 4 and $P_1(R)$ contains spanning bistar $B_{6,6}$ with center $\left(x_0 = 2^3, 1+x_0 = 1+2^3\right)$ as in Figure 1.

Proposition 2.11 (i) If R is $\frac{Z_2(x)}{(x^4)}$, $P_1(R)$ contains $B_{2,2}$ as spanning subgraph.

If R is $Z_4\,[x]/(x^2)$, then $P_1(R)$ contains $B_{6,6}$ as spanning subgraph.

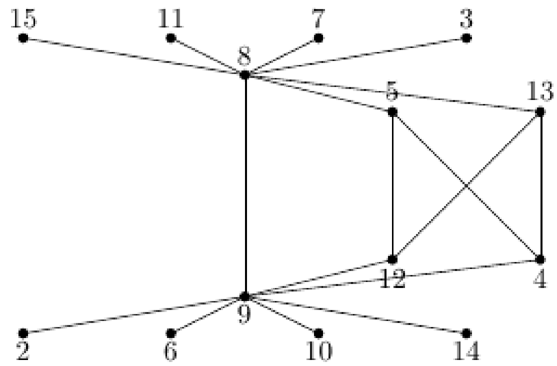

Fig 1. $P_1(Z_{16})$

Proof: (i) $R\setminus(0,1)=\{x, x^2, x^3, 1+x, 1+x^2, 1+x^3\}$; $x\sqcap(1+x^3)$, $x^2\sqcap\left(1+x^3\right)$, $x^2\sqcap\left(1+x^2\right)$, $x^3\sqcap\left(1+x^3\right)$, $x^3\sqcap\left(1+x^2\right)$, $x^3\sqcap(1+x)$.

Now let $S=\{x, ,x^2, x^3\}$; $T=\{1+x,\ 1+x^2,\ 1+x^3\}$. Then, $R\setminus\{0,1\}=S\cup T$ and $P_1(R)$ contains $B_{(S|-1,(T|-1}$ as its spanning subgraph.

(ii) $R\setminus\{0,1\}=\{2, x, 2x, 3x, 2+x, 2+2x, 2+3x, 3, 1+x, 1+2x, 1+3x, 3+x, 3+2x, 3+3x\}$, in which $2x$ is nilpotent element with nilpotency 2.

Now $Ann(2x)\setminus\{0\}=\{2, x, 2x, 3x, 2+x, 2+2x, 2+3x\}$;

$1+Ann(2x)\setminus\{0\}=\{3, 1+x, 1+2x, 1+3x, 3+x, 3+2x, 3+3x\}$;

Let $S=Ann(2x)\setminus(0\}$; $T=1+Ann(2x)\setminus(0\}$. Then $R\setminus\{0,1\}=S\cup T$; $(2x)\neg t$ for $t\in T$; $(1+2x\neg s)$ for $s\in S$. Hence, $P_1(R)$ contains $B_{(s|-1,(T|-1}$ with center $(2x, 1+2x)$ as its spanning subgraph.

Proposition 2.12 (i) Let R be VNR and 2 be zero-divisor. Then (i) $P_1(R)$ has no isolated vertices if R has finite unit group.

(ii) If R is finite reduced zero-dimensional ring with even characteristics, $P_1(R)$ has no isolated vertex.

Proof: (i) Suppose $U(R)$ is finite and $x^t=1$ for every $x\in U(R)$. Then $(1-x)z=0$,

where $z=1+x+x^2+\cdots+x^{t-1}$ is nontrivial, showing that $1-x\in ZD(R)$ and hence x is non-isolated.

Now by hypothesis, for every nontrivial x, there is y with $x^2y=x$, in which xy is nonzero. If $xy\neq x$, $x\sqcap(xy)$.

If $xy=1$, x is unit and so is non-isolated by above discussion.

If $xy=x$ and $y\neq x$, $x\sqcap y$.

If $xy=x$ and $y=x$, x is idempotent and hence is non-isolated.

(ii) By assumption, R is VNR and so (ii) follows from (i)

Proposition 2.13 Let F be infinite field.

(i) If $R=Z_2\times F$, $P_1(R)$ has no isolated vertices.

(ii) If $R=F[x]\setminus(x^2)$, $P_1(R)$ contains $K_{|ZD(R)|-1,|ZD(R)|-1}$ together with infinite isolated vertices.

(iii) If $R=Z[x]\setminus(x^2,mx)$, $m\geq 2$, $P_1(R)$ contains $K_{m-1,m-1}$ together with infinite isolated vertices.

Proof: (i) $R\setminus\{0,1\}$ has partition $(\{0\}\times F)\cup(\{1\}\times F)$ and $(0,x)\sqcap(0,1)$ for $(0,x)\in\{0\}\times F\setminus\{0\}$ (respectively, $(0,1)\sqcap(1,x)$ for $(1,x)\in\{1\}\times F\setminus\{1\}$). Hence, $P_1(R)$ contains no isolated vertices.

(ii) $R=\{a+bx \mid a,b\in F\}$; $ZD(R)\setminus\{0\}=\{ax \mid a\in F\setminus\{0\}\}$;

$1+ZD(R)\setminus\{0\}=\{1+ax \mid a\in F\setminus\{0\}\}$ and $ax(1+ax)=ax$ for every $a\in F\setminus\{0\}$. Thus induced subgraph on $X=(ZD(R)\setminus\{0\})\cup(1+(ZD(R)\setminus\{0\}))$ is $K_{|ZD(R)|-1,|ZD(R)|-1}$. Also, if set complement of X in $R\setminus\{0,1\}$ is denoted by W, then W is infinite and it contains no edges, which completes proof of (ii).

(iii) $R\setminus\{0,1\}$ contains infinite number of elements and $ZD(R)\setminus\{0\}=\{x, 2x, \cdots, (m-1)x\}$;

$1+ZD(R)\setminus\{0\}=\{1+x, 1+2x, \cdots, 1+(m-1)x\}$.

Also, $u\sqcap v$, for $u\in ZD(R)\setminus\{0\}$; $v\in 1+ZD(R)\setminus\{0\}$. Thus, $P_1(R)$ contains $K_{|ZD(R)|-1,|ZD(R)|-1}$ and there are no other edges.

The Proposition below proves necessary condition for connectedness of $P_1(R)$.

Proposition 2. 14 Let *char* R be m. If $P_1(R)$ is connected, m is even.

Proof: If $P_1(R)$ connected, it has no isolated vertices and therefore the proof follows from Propostion 2.4.

Proposition2.15 If $D_i\ncong Z_2$ is domain and $R=D_1\times D_2$,

(i) $P_1(R)$ contains two disjoint copies of complete tripartite graphs.

(ii) $x = (a,b)$ is isolated if $a \in D_1 \backslash \{0,1\}$ and $b \in D_2 \backslash \{0,1\}$.

Proof:$R \backslash (0,1\}$ has partition $B \cup S \cup T \cup W$, where $B = \{(0,1),(1,0)\}$;

$S = (\{0\} \times (D_2 \backslash \{0,1\})) \cup (D_1 \backslash \{0,1\}) \times \{1\}$

$T = ((D_1 \backslash \{0,1\}) \times \{0\}) \cup (\{1\} \times (D_2 \backslash \{0,1\}))$; $W = (D_1 \backslash \{0,1\}) \times (D_2 \backslash \{0,1\})$.

Now $N(u)$ is determined by $1 + Ann(u) \backslash \{0\}$, $Ann(1-u) \backslash \{0\}$ for every u as follows:

$ZD(R) = (\{0\} \times D_2) \cup (D_1 \times \{0\})$; $Ann((0,y)) = D_1 \times \{0\}$ for every $y \in D_2 \backslash \{0\}$;

$Ann((x,0)) = \{0\} \times D_2$ for every $x \in D_1 \backslash \{0\}$; Hence, $N((0,1)) = ((D_1 \backslash \{0\}) \times \{1\}) \cup (\{0\} \times (D_2 \backslash \{0\}))$;

$N((1,0)) = (\{1\} \times (D_2 \backslash \{0\})) \cup ((D_1 \backslash \{0\}) \times \{0\})$;

For $y \in D_2 \backslash \{0,1\}$, $N((0,y)) = (D_1 \backslash \{1\}) \times \{1\}$; For $x \in D_1 \backslash \{0,1\}$, $N((x,0)) = \{1\} \times (D_2 \backslash \{1\})$;

Therefore, from above discussion, for every $x \in D_1 \backslash (0,1\}$, $y \in D_2 \backslash (0,1\}$,

(a) $(0,1) \neg s$ for $s \in S$ (respectively, $(1,0) \neg t$ for $t \in T$);
(b) S contains $(x,1) \neg (0,y)$;
(c) T contains $(1,y) \neg (x,0)$;
(d) W contains no edges, completing proof of (i) and (ii).

Illustration 2.16 Let $R = Z_5 \times Z_7$. Then

$S = ((0\} \times (Z_7 \backslash \{0,1\})) \cup ((Z_5 \backslash \{0,1\}) \times \{1\})$; $T = ((Z_5 \backslash \{0,1\}) \times (0\}) \cup ((1\} \times (Z_7 \backslash \{0,1\}))$;

$W = (Z_5 \backslash \{0,1\}) \times (Z_7 \backslash \{0,1\})$.

If $x = (1,0)$, $N(x) = (\{1\} \times \{2,3,4,5,6\}) \cup (\{2,3,4\} \times \{0\})$.

If $x = (0,1)$, $N(x) = (\{2,3,4\} \times \{1\}) \cup (\{0\} \times \{2,3,4,5,6\})$.

Also W contains 15 isolated vertices. Hence, $P_1(R) \cong 2K_{1,5,3} \cup \overline{K_{15}}$.

Proposition 2.17 If $R = Z_2 \times D$, $D \ncong Z_2$ is finite,

$P_1(R) \cong 2K_{1,k}$, where $k = |D| - 2$. Also, following statements hold:

(i) minimum and maximum degree are 1 and $|D| - 2$, respectively.

(ii) domination number is 2.

(iii) Independence number is $2(|D| - 2)$.

(iv) number of connected components is 2.

(v) girth is ∞.

(vi) chromatic number is 2.

Proof: (i) $R \backslash \{0,1\} = B \cup S \cup T$; $B = ((0,1),(1,0)\}$; $T = -$; $S = \{(1,b) | b \in D \backslash \{0,1\}\}$.

If $x \in T$, $N(x) = \{(0,1)\}$ and $N((0,1)) = T$; If $x \in S$, $N(x) = \{(1,0)\}$ and $N((1,0)) = S$.

Hence, $P_1(R) \cong 2K_{1,m}$ on $(T \cup \{(0,1)\}) \cup (S \cup \{(1,0)\})$ with respective centers $(0,1)$, $(1,0)$, as desired. The degree of $(0,1)$ and that of $(1,0)$ are same and is equal to $|D| - 2$ and hence (i) is proved.

(ii) Every edge is incident either from $(0,1)$ or from $(1,0)$ and so $\{(0,1),(1,0)\}$ is minimal dominating set. Hence, domination number is 2.

(iii) $X = T \cup S$ contains no edge and so X is maximal independent set. Hence, independence number is $|X|$, which is equal to $2(|D| - 2)$.

(iv) There are two connected components $T \cup \{(0,1)\}$ and $S \cup \{(1,0)\}$, which completes proof of (iv).

(v) $P_1(R)$ is acyclic and hence girth of $P_1(R)$ is ∞.

(vi) Let c_1 and c_2 denote two distinct colors. Define $\lambda : R \backslash \{0,1\} \to \{c_1, c_2\}$ by

$-$. Then λ is bijection on set of vertices and hence chromatic number is 2.

Proposition 2.18 If R_1 has at least 3 elements and $R = Z_2 \times R_1$,

(i) $P_1(R)$ contains $2K_{1,|R_1|-2}$ as its spanning subgraph.

(ii) $P_1(R)$ is connected if R_1 has zero-divisors.

Proof: (i) As in previous proposition, if T denotes $\{0\} \times R_1 \backslash \{0,1\}$, S denotes $\{1\} \times R_1 \backslash \{0,1\}$,

$R \backslash \{0,1\} = (T \cup \{(0,1)\}) \cup (S \cup \{(1,0)\})$. Hence, $P_1(R)$ contains $2K_{1,|R_1|-2}$ as its spanning subgraph.

(ii) From (i), $T \cup \{(0,1)\}$ and $S \cup \{(1,0)\}$ are connected components. Suppose $zz' = 0$. Then $(0,z)$ is in T and $\left(1, 1 - z'\right)$ is in $S \cup \{(1,0)\}$.

Also, $(0,z)\left(1, 1 - z'\right) = (0,z)$. Therefore, $(0,z) \sqcap (1, 1 - z')$. Thus, $P_1(R)$ is connected.

Suppose that R_1 has no zero-divisors. Then $P_1(R) \cong 2K_{1,|R_1|-2}$, which is not connected.

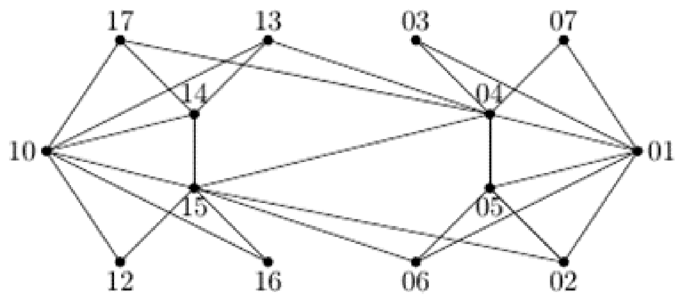

Fig 2. $P_1(\mathbb{Z}_2 \times \mathbb{Z}_8)$

Illustration 2.19 If $R = \mathbb{Z}_2 \times \mathbb{Z}_8, T = \{0\} \times \mathbb{Z}_8 \backslash \{0,1\}, S = \{1\} \times \mathbb{Z}_8 \backslash \{0,1\}$ Also, $z = 2$ is zero-divisor in $\mathbb{Z}_8$ with $2(4) = 0$ and $(0,2) \sqcap (1,5)$ as in Figure 2.

Proposition 2.20 Let $x(\neq -1)$ be nontrivial element with $2x \neq 0, 1, -1$. Then
(i) $x \sqcap (1-x)$ if either $x \in Nil(R)$ or $1-x \in Nil(R)$ having nilpotency 2.
(ii) $x \sqcap (1+x)$ if either $x \in Nil(R)$ with 2 as nilpotency or $x \in U(R)$ with order 2.
(iii) $x \sqcap (1-x)$, $x \sqcap (1+x)$ simultaneously if $x \in Nil(R)$ with nilpotency 2.
(iv)($1-x$) $\sqcap$ ($1+x$) if either x is idempotent or $x^2 = -x$.
(v) $(-x) \sqcap x$ if either x is idempotent or $x(1+x) = 0$.
Proof: By choice of x, ($1-x$), ($1+x$) and $-x$ are distinct.
(i) Suppose $x \sqcap (1-x)$. $If\ x(1-x) = x,\ x^2 = 0$; $If\ x(1-x) = 1-x,\ (1-x)^2 = 0$
If $x^2 = 0, x(1-x)=x$; If $(1-x)^2 = 0, 1-x = x - x^2 = x(1-x)$, showing $(1-x) \sqcap x$.
(ii) Suppose $x \sqcap (1+x)$. If $x(1+x)=x, x^2=0$; If $x(1+x) = 1+x, x^2 = 1$.
If $x^2 = 0, x(1+x) = x$; If $x^2 = 1, x(1+x) = 1+x$, showing$(1+x) \sqcap x$.
(iii) Proof follows from (i) and (ii)
(iv) Suppose $(1-x) \sqcap (1+x)$. If $(1-x)(1+x) = 1-x$, x is idempotent; If $(1-x)(1+x) = 1+x$, $x^2 = -x$.
Suppose $x^2 = x$. Then $(1-x)(1+x) = 1-x$ and, hence $(1-x) \sqcap (1+x)$.
Suppose $x^2 = -x$. Then $(1-x)(1+x) = 1+x$ and, hence ($1-x$)$\sqcap$($1+x$), as desired.
(v) Suppose *(-x)*$\sqcap$*x*. Then *x(-x) =x or x(-x) =-x*.
If $x(-x) = x$, $x(1+x) = 0$; If $x(-x) = -x$, x is idempotent. If x is idempotent, $x(-x) = -x$. Also, if $x(1+x) = 0$, $x(-x) = -x = x$, which completes proof.

Remark 2.21 In Boolean ring R, (i) if x is nontrivial element, x is adjacent to every nontrivial element in xR other than x itself.
(ii) nontrivial elements of R can be paired as (x,y), where $x+y = 1$ and $xy = 0$, called orthogonal complements. Also, orthogonal complements are not adjacent.

Proposition 2.22 Let R be Boolean and (x,y) be pair of nontrivial orthogonal complements. Then
(i) $z \in N(x)$ if ($1-z$) $\in N(y)$.
(ii) $N(x) \cap N(y) = \varnothing$.
(iii) Degree of x and that of y are same.
Proof: By hypothesis, $x+y=1$; $xy = 0$; $1-x$, $1-y$ are nontrivial.
(i) Note that $xz= x$ if $y(1-z)=1-z$. Also $xz=z$ if $y(1-z)=y$. Hence (i) follows.
(ii) Suppose $z \in N(x)$. If $xz=z, yz = 0$. If $xz = x, (1-y)z = x$. If $yz = z$ (respectively, $yz = y$), $x = 0$ (respectively, $z = 1$), which shows $z \notin N(y)$ and concludes $N(x) \cap N(y) = \varnothing$.
(iii) From (i), (ii), $|N(x)| = |N(y)|$.

Illustration 2.23 If $R = \mathbb{Z}_2^3$, then vertex set is
$\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)\}$. The pairs of orthogonal complements and their neighbors are listed below:

Proposition 2.24 Let $R = Z_2^k$. If k is 3, then $P_1(R)$ is

Table 1. Pairs of orthogonal complements in R and their neighbors in $P_1(R)$

x	$y = \mathbf{1} - x$	$N(x)$	$N(y)$	$deg(\mathbf{x}) = deg(y)$
$(1,1,0)$	$(0,0,1)$	$(0,1,0),(1,0,0)$	$(1,0,1),(0,0,1)$	2
$(1,0,1)$	$(0,1,0)$	$(1,0,0),(0,0,1)$	$(0,1,1)(1,1,0)$	2
$(0,1,1)$	$(1,0,0)$	$(0,1,0),(0,0,1)$	$(1,0,1),(1,1,0)$	2

(i) Hamiltonian.
(ii) Eulerian.
(iii) $P_1(R)$ is 2- regular.
(iv) degree sequence is $\left(2^{(6)}\right)$.
(v) bipartite graph.
(vi) planar.
If k is 4, then $P_1(R)$
(vii) is bipartite.
(viii) Contains Hamiltonian cycle.

Proof: Suppose k is 3. Then using Table 1, $P_1(R)$ is drawn in Figure 3, from which it is clear that $(0,0,1) \sqcap (1,0,1) \sqcap (1,0,0) \sqcap (1,1,0) \sqcap (0,1,0) \sqcap (0,1,1) \sqcap (0,0,1)$, which is both Hamiltonian and Eulerian cycle. Degree of each vertex is found to be 2. Hence, $P_1(R)$ is $2-$ regular and degree sequence is $(2^{(6)})$. Since $P_1(R)$ is $6-$cycle, it is connected bipartite graph, which is planar, proving (i) — (vi).

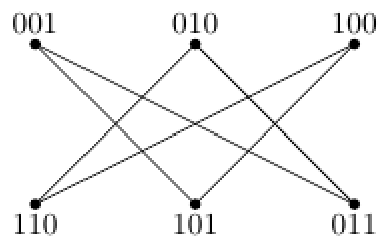

Fig 3. $P_1(Z_2^3)$

Suppose k is 4. Then
(vii) R\{0,1} = S_1 U S_2 U S_3 is a partition, where $S_1 = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$;
$S_2 = \{(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(0,0,1,1)\}$;
$S_3 = \{(1,1,1,0),(1,1,0,1),(1,0,1,1),(0,1,1,1)\}$.
Also, S_i contains no edge and every edge has one end in S_i and other in S_j for $i \neq j$ as shown in Figure 4. Hence, $P_1(R)$ is $3-$partite.
(viii) $P_1(R)$ contains Hamiltonian cycle as drawn in Figure 5.
Boolean rings R are associated with connected $P_1(R)$.

Proposition 2.25 Let $R = Z_2^k, k \geq 3$. Then
(i) $P_1(R)$ is connected.
(ii) diameter is either 3 or 4.
(iii) $P_1(R)$ is k-1 partite.

Proof: Let $R = Z_2^k, k \geq 3$.
If $k = 3$, $P_1(R) \cong C_6$ and hence it is connected and its diameter is 3.
Let $k \geq 4$ and x, y be distinct nontrivial elements and w=x+y+xy. Now following two cases are considered.
Case (a) Suppose w is trivial.
Note that $w \neq 0$. For, if $w = 0$, $(1-x)(1-y) = 1$, which is not possible since R has only one unit 1.
Suppose w=1.
If $xy \neq 0$, $x \sqcap (xy) \sqcap y$ is path between x and y.
If $xy = 0$, choose z, which is different from x, y with z=xr+yr'.

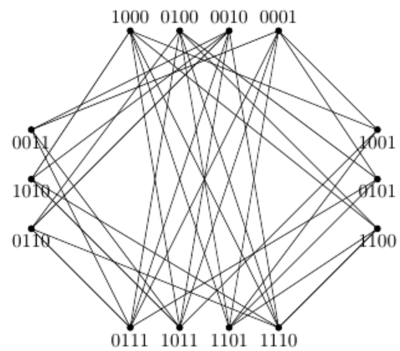

Fig 4. P_1 (Z_2^4)

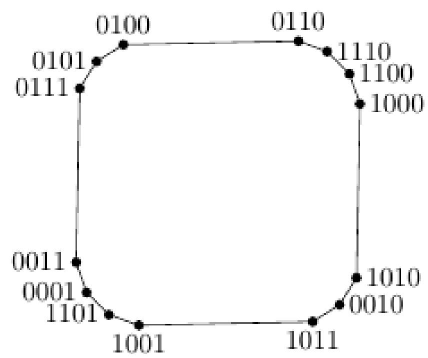

Fig 5. Hamiltonian cycle in P_1 (Z_2^4)

Now $xr\left(xr+yr'\right)=xr$, $yr'\left(xr+yr'\right)=yr'$ and, hence $x\sqcap xr\sqcap z\sqcap yr'\sqcap y$ is path.

Case (b) Suppose w is nontrivial.

If w is different from x, y, $x\sqcap w\sqcap y$ is path.

If w is nontrivial and is x or y, $x\sqcap y$.

Thus, $P_1(R)$ is connected and its diameter is 4, completing proof of (i) and (ii).

(iii) Obviously, nontrivial elements are zero-divisors, each of which is $k-$tuple with '0's and '1s'. Define $\sim$ on R by $x\sim y$ if number of '1's in x and number of '1's in y are same.

Then $\sim$ is equivalence relation and $R=\cup_{i=0}^{k}S_i$ and $S_i=[x]_{\sim}$, $x\in R$, are mutually disjoint subsets of R.

Note that $S_0=\{(0,0,\cdots,0)\}$ and $S_k=((1,1,\cdots,1)\}$. Therefore, S_i, $i\in\{1,\cdots k-1\}$ forms partition of vertices. If $x,y\in S_i$ are $k-$tuples, they differ by at least one place and therefore xy is neither x nor y, which shows x,y are not adjacent. Hence, S_i is independent for every i, which shows that there are $k-1$ maximal independent sets, as desired.

3 Conclusion

Necessary condition for $P_1(R)$ to be connected is obtained. Connected projection graphs are obtained from local rings Z_{2^k}, $\frac{F_2[x]}{(x^k)}$, $\frac{Z_4[x]}{(x^2)}$ and also from nonlocal rings $Z_2\times R_1$ with R_1 having zero-divisors and Boolean rings Z_2^k, $k\geq 3$. Connected projection graphs are so special that whenever $x\sqcap y$, $(1-x)\sqcap(1-y)$ and vice versa. Unipotent units are non-isolated; Denseness of $P_1(R)$ is assured by the unipotent property of units in R.

4 Declaration

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