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RESEARCH ARTICLE

A study on the sum of ‘n+1’ Consecutive Coral Numbers

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Abstract

Objectives: To find new formulas for Sums of n+1 Coral Numbers and its matrix form. **Methods:** Here an attempt made to communicate the formula for Recursive Matrix form and some of its applications. If applications are also provided. Moreover results are obtained by employing mathematical calculations and algebraic simplifications. Results are established by main theorems and their corollary and matrix representations. **Findings:** A formula for the Sum of n+1 consecutive Coral numbers is obtained by employing a lemma. Matrix form for the sum and Recursive forms are attained here. The matrix form of sums of n+1 consecutive Cullen Numbers is also gained. In the application part some interesting associations between Special Numbers, Cullen Numbers and Carol Numbers are given. **Novelty:** In the analysis, entirely new formulae are procured. Matrix representation and its recursive forms are new finding in the area of research. Also, different types of correlations between Carol Numbers and other special numbers are provided.

Keywords: Carol Numbers; Sum of Squares; Cullan Numbers; Woodall numbers; Kynea numbers

1 Introduction

The formula $\zeta(\mu) = 4^{\mu} - 2^{\mu+1} - 1$ is the general form of Carol number. For the amenity of the reader, we shall exhibit obviously define these sequence. The first five terms of this sequence are $-1,\ 7,\ 47,\ 223,\ 959\ldots$ [1]. In this sequence there is sub-sequence which has only prime numbers. For example, some of those elements are $7,\ 47,\ 223,\ 3967,\ 16127\ldots$

The generalized Cullen numbers in linear recurrence sequences of higher order has been discussed[2]. Numerous writers have looked for unique Cullen number characteristics and their generalisations. We consider[3–5] as the outcomes of primality for these integers, and [6] as their greatest common divisor. In the past two decades, there has been a lot of interest in the issue of discovering Cullen numbers that belong to other recognized sequences.

A study has been carried out to communicate the formula for the sums of squares of ‘m’ Woodall numbers and its matrix form. And some correlations between Woodall numbers and other special numbers also expressed[7]. Generalized Woodall sequences in terms of modified Woodall, modified Cullen, Woodall and Cullen sequences has been investigated [8] in 2022 with identities and matrices related to these sequences.

Closed forms of the sum formulas for generalized Hexanacci numbers with the special case of summation formulas of Hexanacci, Hexanacci-Lucas, and other sixth-order recurrence sequences has been studied by Soykan et al. in [9].

The generalized Mersenne number has been introduced and dealt with two special cases, namely, Mersenne and Mersenne-Lucas sequences [10]. They discussed the Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Soykan et al. studied the closed forms of the sum formulas for the squares of generalized Tribonacci numbers. As special cases, the summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin numbers and the other third order recurrence relations were investigated [11].

The Carol number is first studied by Cletus Emmanuel. The sum of squares of '$m+1'$ consecutive Carol numbers and its Matrix representation has been investigated in 2023 by Shanmuganandham et. al [12]. And recursive form of sum of squares of n consecutive Carol numbers has also been established. Not much research has yet been done on coral numbers. This article derives the indistinguishable identities for Pell, Lucas, Fibonacci, Jacobsthal and Polygonal numbers using the properties of divisibility. Also, an integer solutions and the sum of the recursive Matrix of n consecutive carol Numbers are demonstrated.

Notations:

Carol number- $\zeta(\mu)$

Cullan number-$\kappa(\mu)$

Woodall number -$W(\mu)$

Thabitibnkurrah number-$T(\mu)$

Kynea number- $ky(\mu)$

Hexogonal Number - Hex_μ

Gnomonic Number - Gno_μ

2 Methodology

Definition: 3.1 The **Carol number** is defined by the formula $\zeta(\mu) = 4^\mu - 2^{\mu+1} - 1$

Definition: 3.2 The general form of **Cullan number** is defined as $\kappa(\mu) = \mu(2^\mu) + 1$

Definition: 3.3 The general form of **Woodall number** is defined as $W(\mu) = \mu(2^\mu) - 1$

Definition: 3.4 Thabit ibn kurrah number is defined by the formula $T(\mu) = 3(2^\mu) - 1$

Definition: 3.5 Kynea number is defined by the formula $ky(\mu) = 4^\mu + 2^{\mu+1} - 1$

3 Results and Discussion

Theorem: 4.1 For all $\mu \geq 1$, $l = 0, 1, 2, \ldots n$the formula for sum of consecutive '$n+1$' Carol number is $\sum_{l=0}^{n} \zeta(\mu+l) = \frac{1}{6}\sum_{l=0}^{n}(8E(E-3) - 2F(F-6) - 3l(Tri)_l]$ where $E = 2^{\mu+l}$ and $F = 2^\mu$. The following lemma is necessary to achieve the deducible steps in the Proof of the Main theorem.

Lemma: For all $\mu \geq 1$, $l = 0, 1, 2, \ldots n$, the following equality hold,

$$\sum_{l=0}^{n} \zeta(\mu+l) = \frac{4^\mu}{3}\left(4^{n+1} - 1\right) - 2^{\mu+l}\left(2^{n+l} - 1\right) - (n+1)$$

Proof of Lemma: By the definition of Carol number,

$\zeta(\mu) = 4^\mu - 2^{\mu+1} - 1$

replacing μ by $\mu+1$in equation (1) successively, the n^{th} Carol number is given by

$$\zeta(\mu+n) = 4^{\mu+n} - 2^{\mu+n+1} - 1 \tag{1}$$

Adding all $(n+1)$ Carol numbers, we have

$$\zeta(\mu) + \zeta(\mu+1) + \zeta(\mu+2) + \cdots + \zeta(\mu+n)$$

$$= \left(4^\mu + 4^{\mu+1} + 4^{\mu+2} + \ldots 4^{\mu+n}\right) - \left(2^{\mu+1} + 2^{\mu+2} + 2^{\mu+3} + \cdots + 2^{\mu+(n+1)}\right) - (n+1) \tag{2}$$

The first and second terms of RHS of equation (2) forms a geometric series with common ratio 4 and 2 respectively whose sum is $\frac{4^{n+1}-1}{3}$ and $(2^{n+1} - 1)$

Therefore, equation (2) becomes,

$$\sum_{l=0}^{n} \zeta(\mu+l) = \frac{4^{\mu}}{3}\left(4^{l+1}-1\right) - 2^{\mu+1}\left(2^{l+1}-1\right) - (l+1)$$

Hence, the proof of the lemma.

Proof of main theorem:

By the lemma,

$$\sum_{l=0}^{n} \zeta(\mu+l) = \frac{4}{3}\left(4^{l+1}\right) - \frac{4^{\mu}}{3} - 4\left(2^{\mu+l}\right) + 2(2^{\mu}) - (l+1)$$

$\sum_{l=0}^{n} \zeta(\mu+l) = \frac{1}{6}\sum_{l=0}^{n}(8E(E-3) - 2F(F-6) - 3l(Tri)_l]$ where $E = 2^{\mu+l}$ and $F = 2^{\mu}$.

Matrix form of Sum of 'n' consecutive Carol Number

Theorem:4.2 For $\mu > 0$, the matrix form of 'n' consecutive carol number is

$$\begin{bmatrix} \zeta(\eta+n-2) \\ \zeta(\eta+n-1) \\ \zeta(\eta+n) \end{bmatrix} = \begin{bmatrix} G^2S^4 & -GS & -1 \\ G^2S^2 & -G & -1 \\ G^2 & -2G & -1 \end{bmatrix} \begin{bmatrix} X^2 \\ X \\ 1 \end{bmatrix}$$

where $G = 2^n$ and $S = \frac{1}{2}$

Proof: From the definition of Carol number, we have $\zeta(\mu+1) = 4(2^{\mu})^2 - 2^2(2^{\mu}) - 1$

$\zeta(\mu+2) = 4^2(2^{\mu})^2 - 2^3(2^{\mu}) - 1$and

$$\zeta(\mu+3) = 4^3(2^{\mu})^2 - 2^4(2^{\mu}) - 1$$

The above equations has the matrix form as, $C = AY$ where

$C = \begin{bmatrix} \zeta(\eta+1) \\ \zeta(\eta+2) \\ \zeta(\eta+3) \end{bmatrix}, A = \begin{bmatrix} 4 & -2^2 & -1 \\ 4^2 & -2^3 & -1 \\ 4^3 & -2^4 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} (2^{\mu})^2 \\ 2^{\mu} \\ 1 \end{bmatrix}$

Consider the initial matrix

$$A = A(\zeta_1) = \begin{bmatrix} 4 & -2^2 & -1 \\ 4^2 & -2^3 & -1 \\ 4^3 & -2^4 & -1 \end{bmatrix}$$

The coefficient matrix representation of next set of equations for,

$\zeta(\mu+2),\ \zeta(\mu+3),\ \zeta(\mu+4)$ is of the form,

$$A(\zeta_2) = \begin{bmatrix} 4^2 & -2^3 & -1 \\ 4^3 & -2^4 & -1 \\ 4^4 & -2^5 & -1 \end{bmatrix}$$

Proceeding in this manner, we obtain the coefficient matrix representation of the final set of equation for, $\zeta(\mu+n-2), \zeta(\mu+n-1),\ \zeta(\mu+n)$ is of the form,

$$A(\zeta_n) = \begin{bmatrix} 4^{n-2} & -2^{n-1} & -1 \\ 4^{n-1} & -2^{n} & -1 \\ 4^{n} & -2^{n+1} & -1 \end{bmatrix}$$

This also be written as

$$A(\zeta_n) = \begin{bmatrix} \frac{2^{2n}}{16} & -\frac{2^n}{2} & -1 \\ \frac{2^{2n}}{4} & -2^{n} & -1 \\ 4^{n} & -2^{n+1} & -1 \end{bmatrix}$$

Therefore, by suitable substitution, the matrix representation of the required system is

$$\begin{bmatrix} \zeta(\eta+n-2) \\ \zeta(\eta+n-1) \\ \zeta(\eta+n) \end{bmatrix} = \begin{bmatrix} G^2S^4 & -GS & -1 \\ G^2S^2 & -G & -1 \\ G^2 & -2G & -1 \end{bmatrix} \begin{bmatrix} X^2 \\ X \\ 1 \end{bmatrix}$$

where $G = 2^n$ and $S = \frac{1}{2}$

Hence the proof of the theorem.

The recursive Matrix form of Sum of 'n' consecutive Carol Numbers

Theorem: 4.3 For $\mu > 0$, the recursive matrix form of 'n' consecutive carol number is

$$A(\zeta_n) = \begin{bmatrix} H^2 a_{11} & H a_{12} & a_{13} \\ H^2 a_{21} & H a_{22} & a_{23} \\ H^2 a_{31} & H a_{32} & a_{33} \end{bmatrix}$$

where $H = 2^{(n-1)}$

Proof: Take the initial matrix

$$A(\zeta_1) = \begin{bmatrix} 4 & -2^2 & -1 \\ 4^2 & -2^3 & -1 \\ 4^3 & -2^4 & -1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We can write the elements of the next order matrix $A(\zeta_2)$ using the elements of previous one as follows,

$$A(\zeta_2) = \begin{bmatrix} 4^2 & -2^3 & -1 \\ 4^3 & -2^4 & -1 \\ 4^4 & -2^5 & -1 \end{bmatrix} = \begin{bmatrix} 4a_{11} & 2a_{12} & a_{13} \\ 4a_{21} & 2a_{22} & a_{23} \\ 4a_{31} & 2a_{32} & a_{33} \end{bmatrix}$$

Like the same, the elements of third order matrix has the form,

$$A(\zeta_3) = \begin{bmatrix} 4^2 a_{11} & 2^2 a_{12} & a_{13} \\ 4^2 a_{21} & 2^2 a_{22} & a_{23} \\ 4^2 a_{31} & 2^2 a_{32} & a_{33} \end{bmatrix}$$

Therefore, the recursive matrix of n^{th} order written as

$$A(\zeta_n) = \begin{bmatrix} 2^{2(n-1)} a_{11} & 2^{n-1} a_{12} & a_{13} \\ 2^{2(n-1)} a_{21} & 2^{n-1} a_{22} & a_{23} \\ 2^{2(n-1)} a_{31} & 2^{n-1} a_{32} & a_{33} \end{bmatrix}$$

Using the substitution, $H = 2^{(n-1)}$, we obtain the required recursive matrix as in the form,

$$A(\zeta_n) = \begin{bmatrix} H^2 a_{11} & H a_{12} & a_{13} \\ H^2 a_{21} & H a_{22} & a_{23} \\ H^2 a_{31} & H a_{32} & a_{33} \end{bmatrix}$$

where $H = 2^{(n-1)}$

Hence the Proof.

Relation of Carol numbers and Special numbers

Theorem: 4.4 $3(\zeta(\mu) + ky(\mu) + \kappa(\mu) - W(\mu)]$ is a nasty number.

Proof: By the definitions of special numbers, $\zeta(\mu) + ky(\mu) = 2^{2\mu+1} - (\kappa(\mu) - W(\mu)]$

Therefore, $(\zeta(\mu) + ky(\mu) + \kappa(\mu) - W(\mu)] = 2(2^{\mu})^2$.

This leads to, $3(\zeta(\mu) + ky(\mu) + \kappa(\mu) - W(\mu)]$ is a nasty number

Theorem: 4.5 $3\zeta(\mu)ky(\mu) = Hex_{\mu} - 10Gno_{\mu} - (m+18)$

Proof: Again, by the definition of special numbers, $\zeta(\mu)ky(\mu) = (4^{\mu} - 1)^2 - 2^{2(\mu+1)}$

ie, $\zeta(\mu)ky(\mu) = 2^{4\mu} - 6\left(2^{2\mu}\right) + 1$

$\zeta(\mu)ky(\mu) = m^2 - 6m + 1$ where $m = 2^{2\mu}$

$$\zeta(\mu)ky(\mu) = \frac{1}{3}\left(\left(3m^2 + 3m + 1\right) - 21m + 2\right\}$$

Therefore, $3\zeta(\mu)ky(\mu) = Hex_{\mu} - 10\left(Gno_{\mu}\right) - (m+18)$.

Corollary:4.6 $(Hex)2^{2\mu} - 10(Gno)2^{2\mu} - 3\zeta(\mu)ky(\mu) - 2^{2\mu} \equiv 0\ (mod\ 6)$

Theorem: 4.7 $2\mu^2(T(\mu) + 1] + 3\left(\mu^2\zeta(\mu) - \kappa(\mu)W(\mu)\right] = 0$

Proof: By the definitions, $\kappa(\mu)W(\mu) = \mu^2 4^{\mu} - 1$

ie, $\kappa(\mu)W(\mu) = \mu^2\zeta(\mu) + \mu^2\left(2^{\mu+1}\right)$

Therefore, $\mu^2\zeta(\mu) = \kappa(\mu)W(\mu) - 2\mu^2(2^{\mu})$

$$\mu^2\zeta(\mu) = \kappa(\mu)W(\mu) - \frac{2\mu^2}{3}(T(\mu) + 1]$$

Hence $2\mu^2(T(\mu) + 1] + 3\left(\mu^2\zeta(\mu) - \kappa(\mu)W(\mu)\right] = 0$

Corollary: 4.89 $\left[\kappa(\mu)W(\mu) - \mu^2\zeta(\mu)\right] - 6\mu^2 T(\mu)$is a nasty number.

4 Conclusion

A formula for the Sum of n+1 consecutive Coral numbers is obtained by lemma. Matrix form for the sum and Recursive forms are attained. In the application form interesting association and Special Numbers are given. This study can be extended to more special numbers.

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