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The Modular Chromatic Number of The Corona Product of a Generalized Jahangir Graph

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rat31323@gmail.com

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P Sumathi[1], S Tamilselvi[2]*

1 Associate Professor, Department of Mathematics, C. Kandaswami Naidu College for Men, Anna Nagar, 600 102, Chennai, India
2 Assistant Professor, Department of Mathematics, School of Arts And Science, AV Campus, Chennai, India

Abstract

Objectives: The objective of this study is to investigate modular coloring for product graphs such as the corona product of the family of Jahangir graphs. This paper minimizes the number of colors assigned to the vertex of the graph to achieve the modular chromatic number of $j_{n,m}{}^{\circ} j_{t,s}$. **Methods:** The modular k-coloring of a graph $G = (V(G),\ E(G))$ is an injective map $C:\ V(G) \rightarrow z_k,\ k \geq 2$, where the color sum $S(v) = \sum_{u \in N(v)} C(v)$ is different for adjacent vertices of G; that is, $S(u) \neq S(v)$, for $u, v \in E(G)$. The modular chromatic number $m_c(G)$ of G is the minimum k for which G has a modular k-coloring. **Findings:** In this paper, we verified the minimum k for which the corona product of a generalized Jahangir graph admits modular k-coloring and determined the modular chromatic number of $j_{n,m}{}^{\circ} j_{t,s}$ for all integers $m, n, s, t \geq 3$. **Novelty:** This paper presents the modular chromatic number of $j_{n,m}{}^{\circ} j_{t,s}$, and we obtain a new relation, $m_c(J_{n,m}) < m_c(J_{n,m}{}^{\circ} J_{t,s})$.

Keywords: Modular kColoring; Modular Chromatic Number; Vertex Color Sum; Corona Product; Generalized Jahangir Graph

1 Introduction

For a simple undirected and non - trivial connected graph $G(V,E)$, the modular coloring was initiated by F. Okamoto, E. Salehi, and P. Zhang [1]. The modular coloring for some standard graphs is discussed [1,2].

For a vertex v of a graph G, $N(v)$ denotes the neighborhood set of v (i.e., the set of vertices adjacent to v.

The color sum $S(v)$ of a vertex v of G is defined as the sum of the colors of the vertices in $N(v)$; that is, $\mathscr{S}(v) = \sum_{u \in N(v)} c(u)$. Let the coloring $C : V(G) \rightarrow z_k,\ (k \geq 2)$ be a vertex coloring of G where the adjacent vertices may be colored; the same is called a modular sum k-coloring, or simply, a modular k-coloring of G for z_k, while $S(u) \neq S(v)$ for $u, v \in E(G)$. A coloring C is a modular coloring if C is a modular k-coloring for some integer $k \geq 2$. The modular chromatic number $m_c(G)$ of G is the minimum k for which G has a modular k-coloring [1].

The generalized Jahangir graph $J_{n,m}$ for $n \geq 2$, $m \geq 3$ is a graph on $mn+1$ vertices, consisting of a cycle C_{nm}, with one additional vertex that is adjacent to m vertices of C_{nm} at a distance n to each other on C_{nm}.

N. Paramaguru determined modular colorings of the corona product of C_m with C_n and P_m with C_n [3,4]. P. Sumathi and S. Tamilselvi found the modular chromatic number for certain cyclic graphs like generalized Jahangir, Petersen, and uniform theta graphs [5] they also investigated the modular chromatic number of the inflated graphs of wheel, gear, fan, friendship, and flower graph [6].

The aim of this paper is to study the modular chromatic number of the corona product of a generalized Jahangir graph.

2 Preliminaries

2.1 Definition [7]

The generalized Jahangir graph $\mathrm{J_{n,m}}$ for $\mathrm{m} \geq 3$ for is a graph on $\mathrm{mn}+1$ vertices, consisting of a cycle $\mathrm{C_{nm}}$, with one additional vertex that is adjacent to $\mathrm{C_{nm}}$ vertices of $\mathrm{C_{nm}}$ at a distance n to each other on $\mathrm{C_{nm}}$.

Let, for a generalized Jahangir graph $J_{n,m}$, v_0 be a centre vertex; $v_i : i = 1,2,3,\ldots,m$ be the joining vertices; and $v_{ij} : i = 1,2,\ldots,m$ & $j = 1,2,\ldots,n-1$ be the petal vertices. The set of edges $E\left(J_{n,m}\right)= \{v_i v_{i+1} : i = 1,2,\ldots,m\,(n-1)\} \cup \{u_{mn}, v_1\} \cup \left\{v_0 v_{1+(m(i-1))} : i = 1,2,3,\ldots,m\right\}$.

2.2 Definition [3]

The corona of two graphs G and H is the graph $G\circ H$ formed from one copy of G and $|V(G)|$ copies of H, where the i_{th} vertex of G is adjacent to every vertex in the i_{th} copy of H. This type of graph products was introduced by Frucht and Harary in 1970.

2.3 Definition

Let $J_{n,m}$, $J_{t,s}$ be any two generalized Jahangir graphs. The corona of two generalized Jahangir graphs $J_{n,m}$ and $J_{t,s}$ is denoted by $\left(J_{n,m} \circ J_{t,s}\right)$, formed from one copy of $J_{n,m}$ and $|V(J_{n,m})|$ copies of $J_{t,s}$, where the i_{th} vertex of $J_{n,m}$ is adjacent to every vertex in the i_{th} copy of $J_{t,s}$ for $m,s \geq 3$.

The corona product of a generalized Jahangir graph is denoted by $\left(J_{n,m} \circ J_{t,s}\right)$, and let the vertices be $\mathrm{V}(J_{n,m} \circ J_{t,s}) = \left\{(v_{ij} : i = 0, 0 \leq j \leq m) \cup (v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n-1) \cup (V_h : 0 \leq h \leq mn\right\}$, where $V_h = \left\{(v_{ij}^h : i = 0, 0 \leq j \leq m\ , 0 \leq h \leq mn) \cup (v_{ij}^h : 1 \leq i \leq m, 1 \leq j \leq n-1, 0 \leq h \leq mn\right\}$, and let the edges be $\mathrm{E}\left\{\left(J^{\circ}_{n,m}\, J_{t,s}\right) = \left(v_{ij}v_{ij+1} : 1 \leq i \leq m, 0 \leq j \leq n-1\right\} \cup (v_{00}v_{i0} : 1 \leq i \leq m\} \cup (E_h : h = 0,1,2\ldots,mn\} \cup \{X_h : h = 0,1,2\ldots,mn\right\}$, where $\left\{E_h = \left\{v_{ij}^h v_{ij+1}^h : 1 \leq i \leq m, 0 \leq j \leq n-1, h = 0,1,2\ldots,mn\right\} \cup \{v_{00}^h v_{i0}^h : 1 \leq i \leq m, h = 0,1,2\ldots,mn\right\}$ and $X_h = \{v_{ij}v_{ij}^h : 0 \leq i \leq m, 0 \leq j \leq n-1, h = 0,1,2\ldots,mn\}$.

3 Results

3.1 Lemma

For any integer $m,n,s,t \geq 3$, where n and t are even and $\mathscr{S} \equiv 0\,(mod\ 3)$, then $J_{n,m} \circ J_{t,s}$ is modular 3- coloring.

Proof

Let G be the corona product of the generalized Jahangir graph; it is denoted by $\left(J_{n,m} \circ J_{t,s}\right)$; it has a center graph $J_{n,m}$ and h copies of outer graphs $J_{t,s}$, while each $J_{t,s}$ is adjacent to the h_{th} vertex of $J_{n,m}$, where the h_{th} copy of $J_{t,s}$ is denoted by $J^h_{t,s}$. Let the vertices be $\mathrm{V}(J_{n,m} \circ J_{t,s}) = \left\{(v_{ij} : i = 0, 0 \leq j \leq m) \cup (v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n-1) \cup (V_h : 0 \leq h \leq mn)\right\}$, where $V_h = \left\{(v_{ij}^h : i = 0, 0 \leq j \leq m\ , 0 \leq h \leq mn) \cup (v_{ij}^h : 1 \leq i \leq m, 1 \leq j \leq n-1, 0 \leq h \leq mn)\right\}$. We have three cases to prove the lemma. Let m be any positive integer; n and t are even, and $S \equiv 0\,(mod\ 3)$. The different t values lead the following cases.

Case 1 . $t \equiv 0\,(mod\ 3)$

Since t is even, by theorem 3.1, $m_c\left(J_{t,s}\right) = 2$[5]. By this $J^h_{t,s}$ of $\left(J_{n,m} \circ J_{t,s}\right)$ is receiving mod 2-coloring. Each $J^h_{t,s}$ is adjacent to the h_{th} vertex v_{ij} of $J_{n,m}$; without loss of generality, a minimum of 3 colors is required for coloring the graph $\left(J_{n,m} \circ J_{t,s}\right)$. That is, $m_c\left(J_{n,m} \circ J_{t,s}\right) \geq 3$.

Consider the coloring for $J_{n,m}$ is $C : V\left(J_{n,m}\right) \rightarrow \{0,1,2\}$, defined as follows:

$$C\left(v_{ij}\right)=\begin{cases}0 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ even\\ 1 & if\ i,j=0\\ 2 & 1\leq i\leq m, 1\leq j\leq n-1,\ where j\ is\ odd\end{cases},$$

and the coloring of $j^h_{t,s}$ is defined by the following two subcases.

Subcase 1.1. When $h=\{(i,j): i=0, 0\leq j\leq m\}$,

$$C\left(v^h_{ij}\right)=\begin{cases}0 & if\ 0\leq i\leq m,\ j=0\ \ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ even\\ 2 & 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ odd\end{cases}.$$

The modular coloring of $J^h_{t,s}$ depends on the vertex coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$. That vertex v_{ij} may receive any of the two colors 0 and 1. According to the given coloring, the modular coloring of $J^h_{t,s}$ is, respectively, as follows:

$$\mathscr{S}\left(v^h_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 1 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}$$

$$\mathscr{S}\left(v^h_{ij}\right)=\begin{cases}1 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

Subcase 1.2. When $h=\{(i,j): 1\leq i\leq m, 1\leq j\leq n-1\}$,

$$C\left(v^h_{ij}\right)=\begin{cases}0 & if\ 0\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ even\\ 1 & 1\leq i\leq m, 1\leq j\leq n-1,\ where j\ is\ odd\end{cases}.$$

The modular coloring of $J^h_{t,s}$ depends on the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$, while the vertex v_{ij} receives the colors 0 and 2. The modular coloring of $J^h_{t,s}$ is, respectively, as follows:

$$\mathscr{S}\left(v^h_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}$$

$$\mathscr{S}\left(v^h_{ij}\right)=\begin{cases}1 & if\ 1\leq i\leq m,\ j=0 and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\\ 2 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\end{cases}.$$

Applying the coloring as defined above, the modular coloring of $J_{n,m}$ is assigned as follows:

$$\mathscr{S}\left(v_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 1 & if\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\\ 2 & if\ 1\leq i\leq m,\ j=0\end{cases}.$$

This indicates that every adjacent vertex receives a distinct color sum.

Case 2. $t\equiv 1\,(mod\ 3)$

Case 3. $t\equiv 2\,(mod\ 3)$

The proofs for the above two cases are exactly the same as for case 1.

Here, the modular coloring for all edges $S\left(v_{ij}\right)\neq S\left(u_{ij}\right)\forall\ v_{ij}u_{ij}\in E(J^{\circ}_{n,m}\,J_{t,s})$ is different. Hence, $J_{n,m}\circ\ J_{t,s}$ admits modular 3-coloring. Thus, the modular chromatic number of a graph $J_{n,m}\circ\ J_{t,s}$ is 3 when n and t are even and $\mathscr{S}\equiv 0\,(mod\ 3)$.

Example:

$m_c(J^{\circ}_{6,5}\,J_{4,3})=3$

$(J^{\circ}_{6,5}\,J_{4,3})$

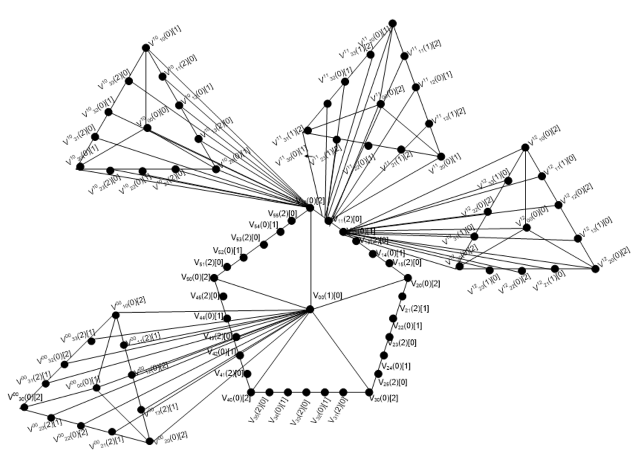

Fig 1. Modular coloring of $\left(j_{6,5}\circ j_{4,3}\right)$

3.2 Lemma

For any integer $m,n,s,t \geq 3$, where n and t are even and $S \equiv 1\,(mod\ 3)$, then $J_{n,m} \circ\ J_{t,s}$ is modular 3-coloring.

Proof

Let G be the corona product of generalized Jahangir graph $(J_{n,m} \circ\ J_{t,s})$. Let m be any positive integer, n and t are even, and $S \equiv 1\,(mod\ 3)$. To prove the lemma, we have the following three cases.

Case 1. $t \equiv 0(mod\ 3)$

The proof of this case is exactly the same as case 1 of lemma 3.1.

Case 2. $t \equiv 1(mod\ 3)$

Subcase 2.1. $t \equiv 4(mod\ 12)$

Consider the coloring for $J_{n,m}$ is defined as follows:

$$C(v_{ij}) = \begin{cases} 0 & if\ 0 \leq i \leq m,\ j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq n-1,\ where\ j\ is\ even \\ 1 & 1 \leq i \leq m, 1 \leq j \leq n-1,\ where\ j\ is\ odd \end{cases}.$$

Let the coloring of $j_{t,s}^{h}$ be defined by the following two subcases.

Subcase 2.1.1. When $h = \{(i,j) : i,j = 0\}$,

$$C\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ i,j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq n-1,\ where\ j \equiv 2,3(mod4) \\ 1 & 1 \leq i \leq m, 1 \leq j \leq n-1,\ where\ j \equiv 1(mod4) \\ 2 & if\ 1 \leq i \leq m,\ j=0\ and\ 1 \leq i \leq m,\ 1 \leq j \leq n-1,\ where\ j \equiv 0(mod4) \end{cases}.$$

The above coloring yields the modular coloring of $J_{t,s}^{h}$. The coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$ is 0; then the modular coloring of $J_{t,s}^{h}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right) = \begin{cases} 1 & if\ 1 \leq i \leq m,\ j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq n-1, where\ j\ is\ even \\ 2 & if\ i,j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq\ n-1, where\ j\ is\ odd \end{cases}.$$

Subcase 2.1.2. When $h = \{(i,j) : 1 \leq i \leq m, 0 \leq j \leq n-1\}$,

$$C\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ i,\ j=0\ and\ 1 \leq i \leq m, j=0\ \&\ 1 \leq j \leq n-1,\ where\ j \equiv 0,2,3\ (mod4) \\ 2 & 1 \leq i \leq m, 1 \leq j \leq n-3,\ where\ j \equiv 1(mod4) \end{cases}.$$

Here the coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$ is 0 and 1. Let the modular coloring of $J_{t,s}^{h}$ be, respectively, as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ i,j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq n-1,\ where\ j\ is\ odd \\ 2 & if\ 1 \leq i \leq m,\ j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq n-1,\ where\ j\ is\ even \end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ 1 \leq i \leq m,\ j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq n-1, where\ j\ is\ even \\ 1 & if\ i,j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq n-1, where\ j\ is\ odd \end{cases}.$$

From the above, it results that the modular coloring of $J_{n,m}$ is as follows:

$$\mathscr{S}(v_{ij}) = \begin{cases} 0 & if\ i,j=0 \\ 1 & if\ 1 \leq i \leq m,\ j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq n-1, where\ j\ is\ even \\ 2 & if\ 1 \leq i \leq m, 1 \leq j \leq n-1, where\ j\ is\ odd \end{cases}.$$

Subcase 2.2. $t \equiv 10(mod\ 12)$

Consider the coloring for $J_{n,m}$ is $C(v_{ij}) = 0,\ for\ i,j=0\ and\ 1 \leq i \leq m, 0 \leq j \leq n-1$, and the coloring of $j_{t,s}^{h}$ defined by the following two subcases.

Subcase 2.2.1. When $h = \{(i,j) : i,j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq\ n-1, where\ j\ is\ odd\}$,

$$C\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ 1 \leq i \leq m,\ \ j=0\ and\ 1 \leq j \leq n-1,\ where\ j \equiv 0,2,3\ (mod4) \\ 1 & 1 \leq i \leq m, 1 \leq j \leq n-1,\ where\ j \equiv 1(mod4) \\ 2 & if\ i,j=0 \end{cases}.$$

The above coloring and the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$ gives the modular coloring of $J_{t,s}^{h}$. The coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$ is 0; then the modular coloring of $J_{t,s}^{h}$ is as follows:

$$\mathscr{S}(v_{ij}) = \begin{cases} 0 & if\ i,j=0 \\ 1 & if\ 1 \leq i \leq m,\ j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq n-1, where\ j\ is\ even \\ 2 & if\ 1 \leq i \leq m, 1 \leq j \leq n-1, where\ j\ is\ odd \end{cases}.$$

Subcase 2.2.2.

When $h = \{(i,j) : 1 \leq i \leq m,\ j=0\ and\ 1 \leq i \leq m, 1 \leq j \leq\ n-1, where\ j\ is\ even\}$,

$$C\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ 1 \leq i \leq m,\ j=0\ and\ 1 \leq j \leq n-1,\ where\ j \equiv 0,2,3\ (mod4) \\ 1 & if\ i,j=0 \\ 2 & 1 \leq i \leq m, 1 \leq j \leq n-1,\ where\ j \equiv 1(mod4) \end{cases}.$$

The above coloring and the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$ gives the modular coloring of $J^h_{t,s}$. The coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$ is 0; then the modular coloring of $J^h_{t,s}$ is as follows:

$$\mathscr{S}\left(v^h_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

By applying the above coloring, the modular coloring of $J_{n,m}$ is assigned as follows:

$$\mathscr{S}(v_{ij})=\begin{cases}1 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\\ 2 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq\ n-1, where\ j\ is\ odd\end{cases}.$$

Case 3. $t\equiv 2(mod\ 3)$

Consider the coloring for $J_{n,m}$ is $C(v_{ij})=0,\ for\ i,j=0\ and\ 1\leq i\leq m, 0\leq j\leq n-1$, and the coloring of $j^h_{t,s}$ is defined by the following two types.

Subcase 3.1. When $h=\{(i,j): i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq\ n-1, where\ j\ is\ odd\}$,

$$C\left(v^h_{ij}\right)=\begin{cases}0 & if\ 0\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ even\\ 2 & 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ odd\end{cases}.$$

Subcase 3.2.

When $h=\{(i,j): 1\leq i\leq m, j=0\ and\ 1\leq i\leq m, 1\leq j\leq\ n-1, where\ j\ is\ even\}$,

$$C\left(v^h_{ij}\right)=\begin{cases}0 & if\ 0\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ even\\ 1 & 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ odd\end{cases}.$$

The modular coloring of $J_{t,s}$ for the above two subcases follows from subcase 2.2.1 and 2.2.2, respectively. From the above coloring, the modular coloring of $J_{n,m}$ is as follows:

$$\mathscr{S}(v_{ij})=\begin{cases}1 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\\ 2 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq\ n-1, where\ j\ is\ odd\end{cases}.$$

Here, the modular coloring for all edges $S(v_{ij})\neq S(u_{ij})\forall\ \ v_{ij}u_{ij}\in E(J_{n,m}\circ\ J_{t,s})$ is different. Hence, $(J_{n,m}\circ\ J_{t,s})$ admits modular 3-coloring. Thus, the modular chromatic number of a graph $(J_{n,m}\circ\ J_{t,s}))$ is 3 when n and t are even and $S\equiv 1\,(mod\ 3)$.

3.3 Lemma

For any integer $m,n,s,t\geq 3$, where n and t are even and $s\equiv 2\,(mod\ 3)$, then $J_{n,m}\circ\ J_{t,s}$ is modular 3-coloring.

Proof

Let G be the corona product of the generalized Jahangir graph $(J_{n,m}\circ\ J_{t,s})$. Let n and t be even, $S\equiv 2\,(mod\ 3)$, and m is any positive integer. The following cases must be considered to prove the lemma.

Case 1. $t\equiv 0(mod\ 3)$

The proof of this case follows from case 1 of lemma 3.1.

Case 2. $t\equiv 1(mod\ 3)$

Consider the coloring for $J_{n,m}$ is defined by $C(v_{ij})=0$, for all $i,j=0\ and\ 1\leq i\leq m$, $1\leq j\leq\ n-1$, and the coloring of $j^h_{t,s}$ is defined by the following two types.

Subcase 2.1.

When $h=\{(i,j): 1\leq i\leq m, j=0\ and\ 1\leq i\leq m, 1\leq j\leq\ n-1,\ where\ j\ is\ even\}$,

$$C\left(v^h_{ij}\right)=\begin{cases}0 & if\ 0\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ even\\ 2 & 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ odd\end{cases}.$$

The above coloring gives the modular coloring of $J^h_{t,s}$. The coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$ is 0; then the modular coloring of $J^h_{t,s}$ is as follows:

$$\delta\left(v^h_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 1 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

Subcase 2.2. When $h=\{(i,j): i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq\ n-1, where\ j\ is\ odd\}$,

$$C\left(v^h_{ij}\right)=\begin{cases}0 & if\ i,j=0\\ 1 & if\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ even\end{cases}.$$

The above coloring and the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$ gives the modular coloring of $J^h_{t,s}$. The coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$ is 0; then the modular coloring of $J^h_{t,s}$ is as follows:

$$\mathscr{S}\left(v^h_{ij}\right)=\begin{cases}1 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

It gives the modular coloring of $J_{n,m}$ as follows:

$$\mathscr{S}(v_{ij}) = \begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le\ n-1, where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \end{cases}.$$

Case 3. $t \equiv 2(mod\ 3)$

Consider the coloring for $J_{n,m}$ is defined as follows:

$$C(v_{ij}) = \begin{cases} 0 & if\ 0\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \\ 1 & 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \end{cases},$$

and the coloring of $j^h_{t,s}$ is defined by the following two types.

Subcase 3.1. When $h = \{(i,j) : 1\le i\le m, 0\le j\le n-1\}$,

$$C\left(v^h_{ij}\right) = \begin{cases} 0 & if\ 0\le i\le m,\ j=0\ \ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \\ 1 & 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \end{cases}.$$

The modular coloring of $J^h_{t,s}$ depends on the above coloring and the vertex coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$. That vertex v_{ij} may receive any of the two colors 0 and 1. According to the given coloring, the modular coloring of $J^h_{t,s}$ is, respectively, as follows:

$$\mathscr{S}(v_{ij}) = \begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le\ n-1, where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \end{cases}.$$

$$\mathscr{S}(v_{ij}) = \begin{cases} 0 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \\ 1 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le\ n-1, where\ j\ is\ odd \end{cases}.$$

Subcase 3.2. When $h = \{(i,j) : i,j=0\}$,

$$C\left(v^h_{ij}\right) = \begin{cases} 0 & if\ i,j=0 \\ 1 & if\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \end{cases}.$$

The above coloring and the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$ gives the modular coloring of $J^h_{t,s}$. The coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$ is 0; then the modular coloring of $J^h_{t,s}$ is as follows:

$$\mathscr{S}(v_{ij}) = \begin{cases} 1 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le\ n-1, where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \end{cases}$$

It gives the modular coloring of $J_{n,m}$ as follows:

$$\mathscr{S}(v_{ij}) = \begin{cases} 0 & if\ i,j=0 \\ 1 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \\ 2 & if\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \end{cases}.$$

Here, the modular coloring for all edges $S(v_{ij}) \neq S(u_{ij}) \forall\ v_{ij}u_{ij} \in E(J_{n,m} \circ\ J_{t,s})$ is different. Hence $(J^{\circ}_{n,m}\ J_{t,s})$ admits modular 3-coloring. Thus, the modular chromatic number of a graph $\forall\ v_{ij}u_{ij} \in E(J_{n,m} \circ\ J_{t,s}))$ is 3 when n and t are even and $S \equiv 2\,(mod\ 3)$.

3.4 Lemma

For any integer $m,n,s,t \ge 3$, where n is odd and t is even and $s \equiv 0(mod3)$; then $J_{n,m} \circ\ J_{t,s}$ is modular 3-coloring.

Proof

When n is odd and t is even

Since t is even, from theorem 3.1, $m_c(J_{t,s}) = 2$ [5], each vertex of $J^h_{t,s}$ is adjacent to the h_{th} vertex of $J_{n,m}$, so we need a minimum of 3 colors to color $J_{n,m} \circ\ J_{t,s}$. That is $m_c(J_{n,m} \circ\ J_{t,s}) \ge 3$.

Case 1. $t \equiv 0(mod3)$

Consider the coloring for $J_{n,m}$ is defined as follows:

$$C(v_{ij}) = \begin{cases} 0 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \\ 1 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \end{cases},$$

and the coloring of $j^h_{t,s}$ is defined by the following two types.

Subcase 1.1. When $h = \{(i,j) : 0\le i\le m,\ j=0\ and\ 1\le i\le m, 2<j\le n-1\}$,

This case follows from subcase 1.1 of lemma 3.1.

Subcase 1.2. When $h = \{(i,j) : 1\le i\le m, j=1$,

$$C\left(v^h_{ij}\right) = \begin{cases} 0 & if\ 0\le i\le m,\ j=0and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \\ 1 & 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \end{cases}$$

The modular coloring of $J^h_{t,s}$ depends on the above coloring and the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$, while the vertex v_{ij} receives the color 0. The modular coloring of $J^h_{t,s}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

Let the modular coloring of $J_{n,m}$ be as follows:

$$\mathscr{S}\left(v_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ even\\ 1 & if\ 1\leq i\leq m, j=1\\ 2 & if\ 1\leq i\leq m, j=0\ and\ 1<j\leq n-1,\ where\ j\ is\ odd\end{cases}.$$

This indicates that every adjacent vertex receives a distinct color sum.

Case 2. $t\equiv 1(mod3)$

Case 3. $t\equiv 2(mod3)$

The proof of the above two cases exactly follows from case 1.

From the above, it is clear that the end point of every edge receives a different color sum, which satisfies modular 3-coloring. Therefore, $m_c\left(J_{n,m}\circ J_{t,s}\right)=3$.

3.5 Lemma

For any integer $m,n,s,t\geq 3$, where n is odd and t is even and $s\equiv 1(mod3)$, then $m_c\left(J_{n,m}\circ J_{t,s}\right)=3$ is modular 3-coloring.

Proof:

Case 1. $t\equiv 0(mod3)$

The proof of this case follows from case 1 of lemma 3.4.

Case 2. $t\equiv 1(mod3)$

Subcase 2.1. $t\equiv 4(mod12)$

When n is odd $(n>5)$ and t is even

Consider the coloring for $J_{n,m}$ is $C:v\left(J_{n,m}\right)\rightarrow\{0,1,2\}$, which is defined as follows:

$$C\left(v_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,\ j=0,1\ \&\ 2\\ 1 & if\ 1\leq i\leq m,3\leq j\leq n-1,\ where\ j\ is\ even\\ 2 & if\ 1\leq i\leq m,3\leq j\leq n-2,\ where\ j\ is\ odd\end{cases}$$

and the coloring of $j_{t,s}^{h}$ is defined by the following three types.

Subcase 2.1.1. When $h=\{(i,j):1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, j=n-1\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 0\leq i\leq m,\ j=0\ \&\ 1\leq i\leq m,1\leq j\leq n-1,\ where\ j\ \equiv 0,2,3(mod4)\\ 1 & if\ 1\leq i\leq m,1\leq j\leq n-1,\ where\ j\ \equiv 1(mod4)\end{cases}.$$

Here the coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$ is 0 and 1. Let the modular coloring of $J_{t,s}^{h}$ be as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ odd\\ 1 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}1 & if\ i,j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

Subcase 2.1.2. When $h=\{(i,j):1\leq i\leq m, j=2\&3\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 0\leq i\leq m,\ j=0\ \&\ 1\leq i\leq m,1\leq j\leq n-1,\ where\ j\ \equiv 0,2,3(mod4)\\ 2 & if\ 1\leq i\leq m,1\leq j\leq n-1,\ where\ j\ \equiv 1(mod4)\end{cases}.$$

The coloring of the h_{th} adjacent vertex v_{ij} in $J_{n,m}$ is 0 and 2. Let the modular coloring of $J_{t,s}^{h}$ be as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

.

Subcase 2.1.3. When $h=\{(i,j):i,j=0\ and\ 1\leq i\leq m, j=1\ \&\ 4\leq j\leq n-2\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,\ 1\leq j\leq n-1,\ where\ j\ \equiv 2,3(mod4)\\ 1 & if\ 1\leq i\leq m,1\leq j\leq n-1,\ where\ j\ \equiv 1(mod4)\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1,\ where\ j\ \equiv 0(mod4)\end{cases}.$$

Furthermore, the h_{th} vertex v_{ij} of $J_{n,m}$ colored by 0, 1, and 2, respectively, then the modular coloring of $J_{t,s}^{h}$ adjacent to the h_{th} vertex v_{ij} of $J_{n,m}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}1 & if\ 1\leq i\leq m,\ j=0and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ even\\ 2 & if\ i,j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ odd\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m,1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\\ 1 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\end{cases}.$$

Let the modular coloring of $J_{n,m}$ be as follows:

$$\mathscr{S}\left(v_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,\ j=1,3\ \&\ n-1\\ 1 & if\ 1\leq i\leq m, 2\leq j\leq n-3,\ where\ j\ is\ even\\ 2 & if\ 1\leq i\leq m, j=0\ and\ 4<j\leq n-2\ where\ j\ is\ odd\end{cases}.$$

Subcase 2.2. $t\equiv 10(mod12)$

When n is odd ($n>5$) and t is even

Consider the coloring for $J_{n,m}$ is $C: v\left(J_{n,m}\right)\rightarrow\{0,1,2\}$, which is defined as follows:

$$C\left(v_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,\ j=0\ \&\ 3\leq j\leq n-1,\ where\ j\ is\ odd\\ 1 & if\ 1\leq i\leq m, 3\leq j\leq n-1,\ where\ j\ is\ even\\ 2 & if\ 1\leq i\leq m,\ j=1\ \&\ 2\end{cases},$$

and the coloring of $j_{t,s}^{h}$ is defined by the following three types.

Subcase 2.2.1. When $h=\{(i,j): i,j=0,\ 1\leq i\leq m,\ j=2\ \&\ 3\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 0\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ \equiv 0,2,3(mod4)\\ 1 & if\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ \equiv 1(mod4)\end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ follows from subcase 2.1.1.

Subcase 2.2.2. When $h=\{(i,j): 1\leq i\leq m, j=0\ \&\ 1\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,\ j=0\ and\ 1\leq i\leq m, j=0\ \&\ 1\leq j\leq n-1,\ where\ j\ \equiv 0,2,3(mod4)\\ 2 & if\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ \equiv 1(mod4)\end{cases}$$

, and the modular coloring of $J_{t,s}^{h}$ follows from subcase 2.1.2.

Subcase 2.2.3. When $h=\{(i,j):\ 1\leq i\leq m, 4\leq j\leq n-1\ \}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq j\leq n-1,\ where\ j\ \equiv 0,1,2(mod4)\\ 1 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\ \equiv 3(mod4)\end{cases}.$$

Let the modular coloring of $J_{t,s}^{h}$ depend on the color of its h_{th} adjacent vertex v_{ij} in $J_{n,m}$ and let 0 and 1 be the, respective, colors.

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 1 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}1 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

According to the above coloring, the modular coloring of $J_{n,m}$ is as follows:

$$\mathscr{S}\left(v_{ij}\right)=\begin{cases}0 & if\ \ 1\leq i\leq m, j=1\ and\ 4\leq j\leq n-1,\ where\ j\ is\ even\\ 1 & if\ 1\leq i\leq m, j=0\ and\ j=2\\ 2 & if\ i,j=0\ and\ 1\leq i\leq m,\ 3\leq j\leq n-1,\ where\ j\ is\ odd\end{cases}.$$

Case 3. $t\equiv 2(mod3)$

Subcase 3.1. $t\equiv 2(mod12)$

Consider the coloring for $J_{n,m}$ is defined as follows:

$$C\left(v_{ij}\right)=\begin{cases}0 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\\ 2 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\end{cases},$$

and the coloring of $j_{t,s}^{h}$ is defined by the following two types.

Subcase 3.1.1. When $h=\{(i,j):\ 1\leq i\leq m,\ j=n-1\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq j\leq n-1\ where\ j\ \equiv 0,2,3(mod4)\\ 1 & if\ 1\leq i\leq m, 1\leq j\leq n-1\ where\ j\ \equiv 1(mod4)\\ 2 & if\ i,j=0\end{cases}.$$

From the above coloring, the modular coloring of $J_{t,s}^{h}$ depends on the coloring of its h_{th} adjacent vertex v_{ij} in $J_{n,m}$, and 0, 1, and 2 are the respective colors.

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 1 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}1 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 1\leq i\leq m,\ j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ even\\ 2 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1, where\ j\ is\ odd\end{cases}.$$

Subcase 3.1.2. When $h=\{(i,j): i,j=0\ and\ 1\leq i\leq m, 0\leq j\leq n-2\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ 1\le i\le m,\ j=0\ and\ 1\le j\le n-1,\ where\ j\ \equiv 0,2,3(mod4) \\ 1 & if\ i,j=0 \\ 2 & if\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ \equiv 1(mod4) \end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ depends on the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$, while the vertex v_{ij} receives the colors 0, 1, and 2. The modular coloring of $J_{t,s}^{h}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \\ 1 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ odd \end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 1 & if\ 1\le i\le m,\ j=0 and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \\ 2 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ odd \end{cases}.$$

Let the modular coloring of $J_{n,m}$ be defined as follows:

$$\mathscr{S}\left(v_{ij}\right)=\begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \\ 1 & if\ 1\le i\le m, j=0\ and\ 3<j\le n-1\ where\ j\ is\ odd \\ 2 & if\ 1\le i\le m, j=\ 1 \end{cases}.$$

Subcase 3.2. $t\equiv 8(mod12)$

Consider the coloring for $J_{n,m}$ is defined by $(v_{ij})=0,\ for\ \ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1$, and the coloring of $j_{t,s}^{h}$ is defined by the following three types.

Subcase 3.2.1.

When $h=\{(i,j):\ 1\le i\le m,\ j=0\}$,

$$C\left(v_{ij}^{h}\right)=\begin{pmatrix} 0 & if\ 0\le i\le m,\ j=0\ \&\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ \equiv 0,2,3(mod4) \\ 1 & if\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ \equiv 1(mod4) \end{pmatrix}.$$

The modular coloring of $J_{t,s}^{h}$ follows from subcase 2.1.1.

Subcase 3.2.2.

When $h=\{(i,j): 1\le i\le m, 1\le j\le n-1, where\ j\ is\ odd\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ i,\ j=0\ and\ 1\le i\le m,\ j=0\ \&\ 1\le j\le n-1,\ where\ j\ \equiv 0,2,3(mod4) \\ 2 & if\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ \equiv 1(mod4) \end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ follows from subcase 2.1.2.

Subcase 3.2.3.

When $h=\{(i,j):\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even\ \}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ \equiv 2,3(mod4) \\ 1 & if\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ \equiv 1(mod4) \\ 2 & if\ 1\le i\le m, j=0\ and\ 1\le j\le n-1,\ where\ j\ \equiv 0(mod4) \end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ follows from subcase 2.1.3.

Let the modular coloring of $J_{n,m}$ be as follows:

$$\mathscr{S}\left(v_{ij}\right)=\begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \\ 1 & if\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ odd \\ 2 & if\ 1\le i\le m, j=0 \end{cases}.$$

From the above cases, it is very clear that the end point of every edge of $J_{n,m}\ \circ\ J_{t,s}$ receives a different color sum, which satisfies modular 3-coloring. Therefore, $m_c\left(J_{n,m}\ \circ\ J_{t,s}\right)=3$.

3.6 Lemma

For any integer $m,n,s,t\ge 3$, where n is odd and t is even and $s\equiv 2(mod3)$, then $J_{n,m}\circ\ J_{t,s}$ is modular 3-coloring.

Proof

Case 1. $t\equiv 0(mod3)$

The proof of this case follows from case 1 of lemma 3.4.

Case 2. $t\equiv 1(mod3)$

Consider the coloring for $J_{n,m}$ is defined by $C\left(v_{ij}\right)=0$ and the coloring of $j_{t,s}^{h}$ is defined by the following three types.

Subcase 2.1.

When $h=\{(i,j): 1\le i\le m, j=0\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ 0\le i\le m,\ j=0\ \ and\ 1\le i\le m, 1\le j\le n-1,\ wherej\ is\ even \\ 2 & 1\le i\le m, 1\le j\le n-1,\ wherej\ is\ odd \end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ depends on the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$, while the vertex v_{ij} receives the color 0. The modular coloring of $J_{t,s}^{h}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ odd \\ 1 & if\ 1\le i\le m,\ j=0 and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ even \end{cases}.$$

Subcase 2.2.

When $h=\{(i,j):1\le i\le m, 0\le j\le n-1,\ where\ j\ is\ odd\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ 0\le i\le m,\ j=0\ \ and\ 1\le i\le m, 1\le j\le n-1,\ where j\ is\ even \\ 1 & 1\le i\le m, 1\le j\le n-1,\ where j\ is\ odd \end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ depends on the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$, while the vertex v_{ij} receives the color 0. The modular coloring of $J_{t,s}^{h}$ is as follows:

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ 0\le i\le m,\ j=0\ \ and\ 1\le i\le m, 1\le j\le n-1,\ where j\ is\ even \\ 1 & 1\le i\le m, 1\le j\le n-1,\ where j\ is\ odd \end{cases}.$$

Subcase 2.3. When $h=\{(i,j):i,j=0\ and\ 1\le i\le m, 1\le j\le\ n-1, where\ j\ is\ even\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ i,j=0 \\ 1 & if\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ even \end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ depends on the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$, while the vertex v_{ij} receives the color 0. The modular coloring of $J_{t,s}^{h}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 1 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ \ where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ \ where\ j\ is\ even \end{cases}.$$

Let the modular coloring of $J_{n,m}$ be as follows:

$$\mathscr{S}\left(v_{ij}\right)=\begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \\ 1 & if\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0 \end{cases}.$$

Case 3. $t\equiv 2(mod 3)$

When n is odd $(n>5)$ and t is even

Consider the coloring for $J_{n,m}$ is $C:v\left(J_{n,m}\right)\to\{0,1,2\}$, which is defined as follows:

$$C\left(v_{ij}\right)=\begin{pmatrix} 0 & if\ i,j=0\ and\ \ 1\le i\le m,\ j=0,1\ \&2 \\ 1 & if\ \ 1\le i\le m, 3\le j\le n-1,\ where\ j\ is\ even \\ 2 & if\ \ 1\le i\le m, 3\le j\le n-1,\ where\ j\ is\ odd \end{pmatrix},$$

and the coloring of $j_{t,s}^{h}$ is defined by the following three types.

Subcase 3.1.

When $h=\{(i,j):1\le i\le m, j=2\ \&\ 3\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ 0\le i\le m,\ j=0\ \ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \\ 1 & 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ depends on the coloring of the h_{th} adjacent vertex v_{ij} of $J_{n,m}$, while the vertex v_{ij} receives the color 0 and 2. The modular coloring of $J_{t,s}^{h}$ is, respectively, as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ \ where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 1 & if\ 1\le i\le m,\ j=0 and\ 1\le i\le m, 1\le j\le n-1,\ \ where\ j\ is\ even \\ 2 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1, where\ j\ is\ odd \end{cases}.$$

Subcase 3.2.

When $h=\{(i,j):1\le i\le m, j=0\ \&\ n-1\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ 0\le i\le m,\ j=0\ \ and\ 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ even \\ 2 & 1\le i\le m, 1\le j\le n-1,\ where\ j\ is\ odd \end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ depends on the coloring of its h_{th} adjacent vertex v_{ij} in $J_{n,m}$. For the respective colors 0 and 1 of the vertex v_{ij}, the modular coloring is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ \ where\ j\ is\ odd \\ 1 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ \ where\ j\ is\ even \end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 1 & if\ i,j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ \ where\ j\ is\ odd \\ 2 & if\ 1\le i\le m,\ j=0\ and\ 1\le i\le m, 1\le j\le n-1,\ \ where\ j\ is\ even \end{cases}.$$

Subcase 3.3.

When $h=\{(i,j): i,j=0 \; and \; 1\le i\le m,\, j=1 \,\&\, 4\le j\le \, n-2\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\; i,j=0 \\ 1 & if\; 1\le i\le m, 1\le j\le n-1,\; where\; j\; is\; odd \\ 2 & if\; 1\le i\le m,\; j=0\; and\; 1\le i\le m, 1\le j\le n-1,\; where\; j\; even \end{cases}.$$

The modular coloring of $J_{t,s}^{h}$ depends on the coloring of its h_{th} adjacent vertex v_{ij} in $J_{n,m}$. For the respective colors 0, 1, and 2 of the vertex v_{ij}, the modular coloring is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 1 & if\; i,j=0\; and\; 1\le i\le m, 1\le j\le n-1,\;\; where\; j\; is\; odd \\ 2 & if\; 1\le i\le m,\; j=0\; and\; 1\le i\le m, 1\le j\le n-1,\;\; where\; j\; is\; even \end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\; 1\le i\le m,\; j=0\; and\; 1\le i\le m, 1\le j\le n-1, where\; j\; is\; even \\ 2 & if\; i,j=0\; and\; 1\le i\le m, 1\le j\le n-1, where\; j\; is\; odd \end{cases}$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases} 0 & if\; i,j=0\; and\; 1\le i\le m, 1\le j\le n-1, where\; j\; is\; odd \\ 1 & if\; 1\le i\le m,\; j=0\; and\; 1\le i\le m, 1\le j\le n-1, where\; j\; is\; even \end{cases}$$

Let the modular coloring of $J_{n,m}$ be as follows:

$$\mathscr{S}\left(v_{ij}\right)=\begin{cases} 0 & if\; i,j=0\; and\; 1\le i\le m,\; j=1,3\,\&\, n-1 \\ 1 & if\; 1\le i\le m, 1\le j\le n-3,\; where\; j\; is\; even \\ 2 & if\; 1\le i\le m,\; j=0\; and\; 1\le i\le m, 4\le j\le n-2,\; where\; j\; is\; odd \end{cases}.$$

From the above cases, it is very clear that the end point of every edge of $(J_{n,m} \circ J_{t,s})$ receives a different color sum, which satisfies modular 3-coloring. Therefore, $m_c\left(J_{n,m}^{\circ} \;\; J_{t,s}\right)=3$.

3.7 Theorem

The modular chromatic number of the corona product of a generalized Jahangir graph $(J_{n,m} \circ J_{t,s})$ is 3 when t is even (excluding when t is odd).

Proof

Let $(J_{n,m} \circ J_{t,s})$ be the corona product of a generalized Jahangir graph, in that all the outer graphs $J_{t,s}^{h}$ are modular 2-colorable; as t is even, we require a minimum of 3 colors to color the center graph $J_{n,m}$.The mapping of the coloring follows from lemmas 3.1, 3.2, and 3.3, respectively; the resulting modular 3-coloring is given by the following tables.

Let t be even; the proof of the theorem is discussed according to the value of n in the following two cases.

Case 1. When n and t are even

Table 1. Coloring of $J_{n,m}$

Values of s	**Values of** t	**C(v_{ij}) of** $J_{n,m}$		
		v_{00}	v_{i0}	v_{ij}
$s\equiv 0(mod3)$	For all t	1	0	$\{2,0,2,0,\ldots,2\}$
$s\equiv 1\,(mod3)$ & $s\equiv 2\,(mod3)$	$t\equiv 0(mod3)$	1	0	$\{2,0,2,0,\ldots,2\}$
$s\equiv 1\,(mod3)$	$t\equiv 1(mod3)$ & $t\equiv 4(mod12)$	0	0	$\{1,0,1,0,\ldots,1\}$
	$t\equiv 1(mod3)$ & $t\equiv 10(mod12)$	0	0	$\{0,0,0,0,\ldots,0\}$
	$t\equiv 2(mod3)$	0	0	$\{0,0,0,0,\ldots,0\}$
$s\equiv 2\,(mod3)$	$t\equiv 1(mod3)$	0	0	$\{0,0,0,0,\ldots,0\}$
	$t\equiv 2(mod3)$	0	0	$\{1,0,1,0,\ldots,1\}$

Table 2. Coloring of $j_{t,s}^{h}$

Cases	Subcases	Types	C(v_{ij}) of $J_{t,s}^{h}$		
			v_{00}^{h}	v_{i0}^{h}	v_{ij}^{h}
$s\equiv 0(mod3)$	For all t	Type-1	0	0	{2,0,2,0,..., 2}
		Type-2	0	0	{1,0,1,0,..., 1}
$s\equiv 1\,(mod3)$ & $s\equiv 2\,(mod3)$	$t\equiv 0(mod3)$	Type-1	0	0	{2,0,2,0,..., 2}
		Type-2	0	0	{1,0,1,0,..., 1}
	$t\equiv 1(mod3)$& $t\equiv 4(mod12)$	Type-1	0	2	{1,0,0,2,..., 1,0,0}
		Type-2	0	0	{1,0,0,0,..., 1,0,0}
$s\equiv 1\,(mod3)$					

Continued on next page

Table 2 continued

	$t \equiv 1(mod3)\&$ $t \equiv 10(mod12)$	Type-1	2	0	{1,0,0,0,…, 1}
		Type-2	1	0	{2,0,0,0,…, 2}
	$t \equiv 2(mod3)$	Type-1	0	0	{2,0,2,0,…, 2}
		Type-2	0	0	{1,0,1,0,…, 1}
$s \equiv 2\,(mod3)$	$t \equiv 1(mod3)$	Type-1	0	0	{2,0,2,0,…, 2}
		Type-2	0	2	{1,2,1,2,…, 1}
	$t \equiv 2(mod3)$	Type-1	0	0	{1,0,1,0,…, 1}
		Type-2	0	2	{1,2,1,2,…, 1}

Table 3. Modular coloring of $J_{n,m}$

Values of s	**Values of** t	$\mathscr{S}(v_{ij})$ of $J_{n,m}$		
		v_{00}	v_{i0}	v_{ij}
$s \equiv 0(mod3)$	For all t	0	2	$\{0,1,0,1,\ldots,0\}$
$s \equiv 1\,(mod3)$ & $s \equiv 2\,(mod3)$	$t \equiv 0(mod3)$	0	2	$\{0,1,0,1,\ldots,0\}$
$s \equiv 1\,(mod3)$	$t \equiv 1(mod3)\&$ $t \equiv 4\,(mod12)$	2	1	$\{2,1,2,1,\ldots,2\}$
	$t \equiv 1(mod3)\&$ $t \equiv 10(mod12)$	2	1	$\{2,1,2,1,\ldots,2\}$
	$t \equiv 2(mod3)$	2	1	$\{2,1,2,1,\ldots,2\}$
$s \equiv 2\,(mod3)$	$t \equiv 1(mod3)$	0	2	$\{0,2,0,2,\ldots,0\}$
	$t \equiv 2(mod3)$	0	1	$\{2,1,2,1,\ldots,2\}$

Table 4. Modular coloring of $j^h_{t,s}$

Cases	**Cases**	**Types**		$\mathscr{S}(v_{ij})$ of $J^h_{t,s}$		
				v^h_{00}	v^h_{i0}	v^h_{ij}
$s \equiv 0(mod3)$	For all t	Type-1	0	0	1	{0,1,0,1,…,0}
			1	1	2	{1,2,1,2,…,1}
		Type-2	0	0	2	{0,2,0,2,…,0}
			2	2	1	{2,1,2,1,…,2}
$s \equiv 1\,(mod3)\&s \equiv 2\,(mod3)$	$t \equiv 0(mod3)$	Type-1	0	0	1	{0,1,0,1,…,0}
			1	1	2	{1,2,1,2,…,1}
		Type-2	0	0	2	{0,2,0,2,…,0}
			2	2	1	{2,1,2,1,…,2}
$s \equiv 1\,(mod3)$	$t \equiv 1(mod3)$ & $t \equiv 4(mod12)$	Type-1	0	2	1	{2,1,2,1,…,2}
		Type-2	0	0	2	{0,2,0,2,…,0}
			1	1	0	{1,0,1,0,…,1}
	$t \equiv 1(mod3)$ & $t \equiv 10(mod12)$	Type-1	0	0	1	{0,1,0,1,…,0}
		Type-2	0	0	2	{0,2,0,2,…,0}
	$t \equiv 2(mod3)$	Type-1	0	0	1	{0,1,0,1,…,0}
		Type-2	0	0	2	{0,2,0,2,…,0}
$s \equiv 2\,(mod3)$	$t \equiv 1(mod3)$	Type-1	0	0	1	{0,1,0,1,…,0}
		Type-2	0	1	2	{1,2,1,2,…,1}
	$t \equiv 2(mod3)$	Type-1	0	0	2	{0,2,0,2,…,0}
			1	1	0	{1,0,1,0,…,1}
		Type-2	0	1	2	{1,2,1,2,…,1}

Case 2. When n is odd t is even

Table 5. Coloring of $J_{n,m}$

Values of s	**Values of** t	$C(v_{ij})$ **of** $J_{n,m}$		
		v_{00}	v_{i0}	v_{ij}
$s \equiv 0(mod3)$	For all t	0	2	$\{1,0,2,0,2,\ \ldots,0\}$

Continued on next page

Table 5 continued

$s\equiv 1\,(mod3)\,\&\,s\equiv 2\,(mod3)$	$t\equiv 0\,(mod3)$	0	2	$\{1,0,2,0,2,\,\ldots,0\}$
$s\equiv 1\,(mod3)$	$t\equiv 1(mod3)\&\ t\equiv 4(mod12)$	0	2	$\{0,2,1,2,1,\,\ldots,0,1,0\}$
	$t\equiv 1(mod3)\&\ t\equiv 10(mod12)$	0	2	$\{0,1,2,0,2,0,\,\ldots,2,0\}$
	$t\equiv 2(mod3)\ \&\ t\equiv 2(mod12)$	0	1	$\{0,1,0,1,0,1,0,\,\ldots,2\}$
	$t\equiv 2(mod3)\ \&\ t\equiv 8(mod12)$	0	2	$\{1,0,1,0,\,\ldots,10\}$
$s\equiv 2\,(mod3)$	$t\equiv 1(mod3)$	0	2	$\{1,0,1,0,\,\ldots,1,0\}$
	$t\equiv 2(mod3)$	0	2	$\{1,0,1,1,2,1,2,\,\ldots,0\}$

Table 6. Coloring of $j^h_{t,s}$

Cases	**Cases**	**Types**	C(v_{ij}) of $J^h_{t,s}$ v^h_{00}	v^h_{i0}	v^h_{ij}
$s\equiv 0(mod3)$	for all t	Type-1	0	0	{2,0,2,0,…, 2}
		Type-2	0	0	{1,0,1,0,…, 1}
$s\equiv 1\,(mod3)$ & $s\equiv 2\,(mod3)$	$t\equiv 0(mod3)$	Type-1	0	0	{2,0,2,0,…, 2}
		Type-2	0	0	{1,0,1,0,…, 1}
$s\equiv 1\,(mod3)$	$t\equiv 1\,(mod3)$& $t\equiv 4(mod12)$	Type-1	0	0	{1,0,0,0,…,1,0,0}
		Type-2	0	0	{2,0,0,0,…,2,0,0}
		Type-3	0	2	{1,0,0,2,…,1,0,0}
	$t\equiv 1(mod3)$& $t\equiv 10(mod12)$	Type-1	0	0	{1,0,0,0,…, 1}
		Type-2	1	0	{2,0,0,0,…, 2}
		Type-3	1	0	{0,0,1,0,…,0}
	$t\equiv 2(mod3)$& $t\equiv 2(mod12)$	Type-1	2	0	{1,0,0,0,…,1}
		Type-2	1	0	{2,0,0,0,…, 2}
	$t\equiv 2(mod3)$& $t\equiv 8(mod12)$	Type-1	0	0	{1,0,0,0,…,1,0,0}
		Type-2	0	0	{2,0,0,0,…,2,0,0}
		Type-3	0	2	{1,0,0,2,…,1,0,0}
$s\equiv 2\,(mod3)$	$t\equiv 1(mod3)$	Type-1	0	0	{1,0,1,0,…, 1}
		Type-2	0	0	{2,0,2,0,…, 2}
		Type-3	0	2	{1,2,1,2,…, 1}
	$t\equiv 2(mod3)$	Type-1	0	0	{1,0,1,0,…, 1}
		Type-2	0	0	{2,0,2,0,…, 2}
		Type-3	0	2	{1,2,1,2,…, 1}

Table 7. Modular coloring of $J_{n,m}$

Values of s	**Values of t**	$S(v_{ij})$ $_{of}J_{n,m}$ v_{00}	v_{i0}	v_{ij}
$s\equiv 0(mod3)$	For all t	1	0	$\{0,1,0,1,\ldots,0\}$
$s\equiv 1\,(mod3)\,\&s\equiv 2\,(mod3)$	$t\equiv 0(mod3)$	1	0	$\{0,1,0,1,\ldots,0\}$
$s\equiv 1\,(mod3)$	$t\equiv 1(mod3)\&$ $t\equiv 4(mod12)$	0	0	$\{1,2,1,2,\ldots,2,0,0\}$
	$t\equiv 1(mod3)\&$ $t\equiv 10(mod12)$	0	0	$\{2,2,0,1,\ldots,0,1\}$
	$t\equiv 2(mod3)\ \&$ $t\equiv 2(mod12)$	2	0	$\{2,0,2,0,\ldots,2,0\}$
	$t\equiv 2(mod3)\ \&$ $t\equiv 8(mod12)$	0	0	$\{0,0,0,0,\ldots,0,0\}$

Continued on next page

Table 7 continued

$s \equiv 2\,(mod3)$	$t \equiv 1(mod3)$	0	0	$\{0,0,0,0,\ldots,0,0\}$
	$t \equiv 2(mod3)$	0	0	$\{0.0.2,1,\ldots,2,1\}$

Table 8. Modular coloring of $j^h_{J_{t,s}}$

Cases	Cases	Types	C$-(h_{th}v_{ij})$	$\mathscr{S}\left(v_{ij}\right)$ of $J^h_{t,s}$ v^h_{00}	v^h_{i0}	v^h_{ij}
$s \equiv 0(mod3)$	For all t	Type-1	0	0	1	{0,1,0,1,…,0}
			1	1	2	{1,2,1,2,…,1}
		Type-2	0	0	2	{0,2,0,2,…,0}
$s \equiv 1\,(mod3)$ & $s \equiv 2\,(mod3)$	$t \equiv 0(mod3)$	Type-1	0	0	1	{0,1,0,1,…,0}
			1	1	2	{1,2,1,2,…,1}
		Type-2	0	0	2	{0,2,0,2,…,0}
	$t \equiv 1(mod3)$& $t \equiv 4(mod12)$	Type-1	0	0	1	{0,1,0,1,...,0}
			1	1	2	{1,2,1,2,…,1}
		Type-2	0	0	2	{0,2,0,2,…,0}
			2	2	1	{2,1,2,1,…,2}
		Type-3	0	2	1	{2,1,2,1,…,2}
			1	0	2	{0,2,0,2,…,0}
			2	1	0	(1,0,1,0,…,1)
	$t \equiv 1(mod3)$& $t \equiv 10(mod12)$	Type-1	0	0	1	{0,1,0,1,...,0}
			1	1	2	{1,2,1,2,…,1}
		Type-2	0	0	2	{0,2,0,2,…,0}
			2	2	1	{2,1,2,1,…,2}
		Type-3	0	0	1	{0,1,0,1,…,0}
			1	1	2	{1,2,1,2,…,1}
$s \equiv 1\,(mod3)$	$t \equiv 2(mod3)$& $t \equiv 2(mod12)$	Type-1	0	0	1	{0,1,0,1,…,0}
			1	1	2	{1,2,1,2,…,1}
			2	2	0	{2,0,2,0,…,2}
		Type-2	0	0	2	{0,2,0,2,…,0}
			1	1	0	{1,0,1,0,…,1}
			2	2	1	{2,1,2,1,…,2}
	$t \equiv 2(mod3)$& $t \equiv 8(mod12)$	Type-1	0	0	1	{0,1,0,1,…,0}
			1	1	2	{1,2,1,2,…,1}
		Type-2	0	0	2	{0,2,0,2,…,0}
			2	2	1	{2,1,2,1,…,2}
		Type-3	0	2	1	{2,1,2,1,…,2}
			1	0	2	{0,2,0,2,…,2}
			2	1	0	{1,0,1,0,…,1}
$s \equiv 2\,(mod3)$	$t \equiv 1(mod3)$	Type-1	0	0	2	{0,2,0,2,…,0}
		Type-2	0	0	1	{0,1,0,1,…,0}
		Type-3	0	1	2	{1,2,1,2,…,1}
	$t \equiv 2(mod3)$	Type-1	0	0	2	{0,2,0,2,…,0}
			1	1	0	{1,0,1,0,…,1}
			2	2	1	{2,1,2,1,…,2}
		Type-2	0	0	1	{0,1,0,1,…,0}
			1	1	2	{1,2,1,2,…,1}
		Type-3	0	1	2	{1,2,1,2,…,1}
			1	2	0	{2,0,2,0,…,2}
			2	0	1	{0,1,0,1,…,0}

A centre vertex, rim vertices are colored using Z_3 colors, and the petal vertices are colored using Z_3 colors in cyclic bases, from the [Tables 3, 4, 5, 6, 7 and 8] it is evident that each $S(v_{ij}) \neq S(u_{ij})$, where v_{ij} & u_{ij} are adjacent vertices in $(J_{n,m} \circ J_{t,s})$; that is, the modular coloring of every vertex is different, which satisfies modular 3-coloring. Therefore, $M_c\left(J^{\circ}_{n,m}\, J_{t,s}\right) = 3$(excluding t is odd).

3.8 Lemma

For any integer $m,n,s,t \geq 3$ and n and t are odd when $t \equiv 1,3\ (mod 4)$, then $J^{\circ}_{n,m}\, J_{t,s}$ is modular 4-coloring.

Proof

Let G be the corona product of a generalized Jahangir graph, which is denoted by $(J^{\circ}_{n,m}\, J_{t,s})$; it has a center graph $J_{n,m}$ and h copies of outer graphs $J_{t,s}$, while each $J_{t,s}$ is adjacent to the h_{th} vertex of $J_{n,m}$, where the h_{th} copy of $J_{t,s}$ is denoted by $J^{h}_{t,s}$ .Since t is odd, by theorem 3.1, $m_c(J_{t,s}) = 3$ [5]. Each vertex of $J^{h}_{t,s}$ is adjacent to the h_{th} vertex of $J_{n,m}$; to achieve modular coloring for $(J^{\circ}_{n,m}\, J_{t,s})$, a minimum of 4 colors is required to color the graph $J_{n,m}$. That is, $m_c\left(J^{\circ}_{n,m}\, J_{t,s}\right) \geq 4$.

The coloring of the graph, defined by $C : V(J_{n,m}) \to z_4$, is as follows:

$$C(v_{ij}) = \begin{cases} 0 & if\ 1 \leq i \leq m,\ j = 0, and\ 1 \leq\ i \leq m, 3 \leq j \leq n-1,\ where\ j\ is\ odd \\ 1 & if\ 1 \leq\ i \leq m, j = 1 \\ 2 & if\ i, j = 0 \\ 3 & if\ \ 1 \leq\ i \leq m, 1 \leq j \leq n-1,\ where\ js\ evevn. \end{cases}.$$

The modular coloring of $J_{n,m}$ depends on the modular coloring of the outer graphs $J^{h}_{t,s}$ as discussed in the following cases.

Case 1.$t \equiv 1\ (mod 4)$

The coloring of the graph $J^{h}_{t,s}$ is defined by the following two subcases.

Subcase 1.1.

When $h = \{ij : 1 \leq i \leq m, 3 < j \leq n-1,\ where\ j\ is\ even\ \}$,

$$C\left(v^{h}_{ij}\right) = \begin{pmatrix} 0 & if\ 1 \leq i \leq m,\ j = 0\ \&\ 1 \leq j \leq n-1,\ where\ j \equiv 2,3\ (mod\ 4) \\ 1 & if\ \ i, j = 0\ and\ 1 \leq\ i \leq m,\ 1 \leq j \leq n-1,\ \ where\ j \equiv 1(mod\ 4) \\ 3 & if\ 1 \leq i \leq m, 1 \leq j \leq n-1,\ where\ j \equiv 0(mod\ 4) \end{pmatrix}.$$

We can obtain that the modular coloring of $J^{h}_{t,s}$ depends on the above coloring and the coloring of the h_{th} vertex v_{ij} of $J_{n,m}$. Here, all the h_{th} vertices v_{ij} of $J_{n,m}$ receive the color 3; then the modular coloring of $J^{h}_{t,s}$ is as follows:

$$\mathscr{S}\left(v^{h}_{ij}\right) = \begin{cases} 0 & if\ \ 1 \leq i \leq m,\ j = 0\ and\ 1 \leq\ i \leq m,\ 2 \leq j \leq n-2,\ where\ j\ is\ even \\ 2 & if\ 1 \leq i \leq m,\ 2 \leq j \leq n-1,\ where\ j\ is\ odd \\ 3 & if\ i, j = 0\ and\ 1 \leq i \leq m, j = 1\ and\ j = n-1 \end{cases}.$$

Subcase 1.2.

When $h = \{ij : 0 \leq i \leq m,\ j = 0\ ;\ 1 \leq i \leq m, j = 2\ and\ 1 \leq j \leq n-1,\ where\ j\ is\ odd\ \}$,

$$C\left(v^{h}_{ij}\right) = \begin{cases} 0 & if\ i,\ j = 0; 1 \leq i \leq m, j = 0\ \&\ \ 3\ , 3 \leq j \leq n-1,\ where\ j \equiv 0,1\ (mod\ 4) \\ 1 & if\ 1 \leq\ i \leq m, j = 2\ and\ \ 3 < j \leq n-1,\ \ where\ j \equiv 3\,(mod\ 4) \\ 3 & if\ 1 \leq i \leq m,\ j = 1\ and\ \ 3 \leq j \leq n-1,\ where\ j \equiv 2\,(mod\ 4) \end{cases}.$$

Using the above coloring and the coloring of the h_{th} vertex v_{ij}, we assign the modular coloring of $J^{h}_{t,s}$. According to the respective colors 0, 1, 2, and 3 of the h_{th} vertex v_{ij}, the modular coloring of $J^{h}_{t,s}$ is given as follows:

$$\mathscr{S}\left(v^{h}_{ij}\right) = \begin{cases} 0 & if\ \ i, j = 0\ \ and\ 1 \leq\ i \leq m,\ j = 4 \\ 1 & if\ 1 \leq i \leq m, j = 1,\ 3\ \&\ n-1\ ;\ 1 \leq\ i \leq m,\ 4 < j \leq n-2,\ where\ j\ is\ even \\ 3 & if\ \ 1 \leq i \leq m,\ j = 0\ , 2\ and\ 4 < j \leq n-2,\ where\ j\ is\ odd \end{cases}.$$

$$\mathscr{S}\left(v^{h}_{ij}\right) = \begin{cases} 0 & if\ \ 1 \leq i \leq m,\ j = 0\ , 2\ and\ 4 < j \leq n-2,\ where\ j\ is\ odd \\ 1 & if\ i, j = 0\ and\ 1 \leq i \leq m, j = 4 \\ 2 & if\ 1 \leq\ i \leq m,\ j = 1, 3\&\ n-1\ and\ \ 4 < j \leq n-2,\ where\ j\ is\ even \end{cases}.$$

$$\mathscr{S}\left(v^{h}_{ij}\right) = \begin{cases} 1 & if\ \ 1 \leq i \leq m,\ j = 0\ , 2\ and\ 4 < j \leq n-2,\ where\ j\ is\ odd \\ 2 & if\ i, j = 0\ and\ 1 \leq i \leq m, j = 4 \\ 3 & if\ 1 \leq i \leq m,\ j = 1,\ 3\&n-1\ and\ 4 < j \leq n-2,\ where\ j\ is\ even \end{cases}.$$

$$\mathscr{S}\left(v^{h}_{ij}\right) = \begin{cases} 0 & if\ 1 \leq\ i \leq m,\ j = 1,\ 3\&n-1\ and\ 4 < j \leq n-2,\ where\ j\ is\ even \\ 2 & if\ \ 1 \leq i \leq m,\ j = 0\ , 2\ and\ 4 < j \leq n-2,\ where\ j\ is\ odd \\ 3 & if\ \ i, j = 0\ and\ 1 \leq i \leq m, j = 4 \end{cases}.$$

Applying the above colorings, we obtain the modular coloring of $J_{n,m}$ as follows:

$$\mathscr{S}(v_{ij}) = \begin{cases} 0 & if\ i, j = 0\ and\ 1 \leq\ i \leq m,\ 3 \leq j \leq n-1,\ where\ j\ is\ even \\ 1 & if\ 1 \leq\ i \leq m,\ 1 \leq j \leq n-1,\ where\ j\ is\ even \\ 2 & if\ \ 1 < i \leq m, j = 0,\ \&\ 3 \leq j \leq n-1,\ where\ j\ is\ odd \\ 3 & if\ 1 \leq\ i \leq m, j = 1 \end{cases}.$$

Case 2.$t \equiv 3\ (mod 4)$

The coloring of the graph $J_{t,s}$ is defined by the following two subcases.

Subcase 1.1.

When $h = \{ij : 1 \le i \le m, 3 < j \le n-1,\ where\ j\ is\ even\}$,

$$C\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ 1 \le i \le m,\ \ j = 0\ \&\ 1 \le j \le n-1,\ \ where\ j \equiv 1,2(mod4) \\ 1 & if\ i, j = 0\ and\ 1 \le\ i \le m, 1 \le j \le n-1,\ where\ j \equiv 3(mod4) \\ 3 & if\ 1 \le\ i \le m, 1 \le j \le n-1,\ where\ j \equiv 0(mod4) \end{cases}.$$

We can obtain that the modular coloring of $J_{t,s}^{h}$ depends on the above coloring and the coloring of the h_{th} vertex v_{ij} of $J_{n,m}$. Here, all the h_{th} vertices v_{ij} of $J_{n,m}$ receive the color 3; then the modular coloring of $J_{t,s}^{h}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ 0 \le i \le m,\ j = 0\ \&\ 2 \le j \le n-2,\ where\ j\ is\ even \\ 2 & if\ 1 \le\ i \le m, 2 \le j \le n-2,\ where\ j\ is\ odd \\ 3 & if\ i, j = 0\ and\ 1 \le i \le m,\ \ j = 1\ \&\ n-1 \end{cases}.$$

Subcase 1.2.

When $h = \{ij : 0 \le i \le m,\ j = 0\,;\ 1 \le i \le m, j = 2\ and\ 1 \le j \le n-1,\ where\ j\ is\ odd\ \}$,

$$C\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ i,\ j = 0\ \&\ \ 1 \le i \le m,\ \ j = 0\ \&\ 3 \le j \le n-1,\ where\ j \equiv 2,3\,(mod4) \\ 1 & if\ 1 \le\ i \le m,\ \ j = 2\ and\ 3 \le j \le n-1,\ where\ j \equiv 1\,(mod4) \\ 3 & if\ 1 \le\ i \le m,\ \ j = 1\ \&\ \ 3 \le j \le n-1,\ where\ j \equiv 0\,(mod4) \end{cases}.$$

Using the above coloring and the coloring of the h_{th} vertex v_{ij}, we assign the modular coloring of $J_{t,s}^{h}$. According to the respective colors 0, 1, 2, and 3 of the h_{th} vertex v_{ij}, the modular coloring of $J_{t,s}^{h}$ is given as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ i, j = 0\ and\ 1 \le i \le m,\ \ j = 3 \\ 1 & if\ 1 \le\ i \le m, j = 1\ and\ \ 3 \le j \le n-1,\ where\ j\ is\ even \\ 3 & if\ 1 \le i \le m,\ \ j = 0\ ,2\ and\ 5 \le j \le n-1,\ where\ j\ is\ odd \end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ 1 \le i \le m,\ \ j = 0\ ,2\ and\ 5 \le j \le n-1,\ where\ j\ is\ odd \\ 1 & if\ i, j = 0\ and\ 1 \le i \le m,\ \ j = 3 \\ 2 & if\ 1 \le\ i \le m, j = 1\ and\ \ 3 \le j \le n-1,\ where\ j\ is\ even \end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right) = \begin{cases} 1 & if\ 1 \le i \le m,\ \ j = 0\ ,2\ and\ 5 \le j \le n-1,\ where\ j\ is\ odd \\ 2 & if\ i, j = 0\ and\ 1 \le i \le m,\ \ j = 3 \\ 3 & if\ 1 \le\ i \le m, j = 1\ and\ \ 3 \le j \le n-1,\ where\ j\ is\ even \end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right) = \begin{cases} 0 & if\ \ 1 \le\ i \le m, j = 1\ and\ 3 \le j \le n-1,\ where\ j\ is\ even \\ 2 & if\ 1 \le i \le m,\ \ j = 0\ ,2\ and\ 5 \le j \le n-1,\ where\ j\ is\ odd \\ 3 & if\ i, j = 0\ and\ 1 \le i \le m,\ \ j = 3 \end{cases}.$$

Applying the above coloring, we get the modular coloring of $J_{n,m}$ to be the same as in case 1. Therefore, the result follows from case 1.

3.9 Lemma

For any integer $m, n, s, t \ge 3$ and n is even and t is odd, where $t \equiv 1,3\,(mod4)$, $J_{n,m}^{\circ}\, J_{t,s}$ is modular 4-coloring.

Proof

Let G be the corona product of a generalized Jahangir graph denoted by $\left(J_{n,m}^{\circ}\, J_{t,s}\right)$; it has a center graph $J_{n,m}$ and h copies of outer graphs $J_{t,s}$, while each $J_{t,s}$ is adjacent to the h_{th} vertex of $J_{n,m}$, where the h_{th} copy of $J_{t,s}$ is denoted by $J_{t,s}^{h}$. As t is odd, by theorem 3.1, $m_c\left(J_{t,s}\right) = 3$ [5]. Each vertex of $J_{t,s}^{h}$ is adjacent to the h_{th} vertex of $J_{n,m}$; to achieve modular coloring for $\left(J_{n,m}^{\circ}\, J_{t,s}\right)$, a minimum of 4 colors is required to color the graph $J_{n,m}$. That is, $m_c\left(J_{n,m}^{\circ}\, J_{t,s}\right) \ge 4$.

The coloring of the graph is defined by $C : V\left(J_{n,m}\right) \to z_4$ as follows:

$$C\left(v_{ij}\right) = \begin{cases} 0 & if\ 1 \le i \le m,\ \ j = 0\ and\ 1 \le\ i \le m, 1 \le j \le n-1\ where\ j\ is\ even \\ 2 & if\ i, j = 0\ and\ 1 \le\ i \le m, j = 1\ and\, j = n-1 \\ 3 & if\ 1 \le\ i \le m, 2 \le j \le n-2\ where\ j\ is\ odd \end{cases}$$

The modular coloring of $J_{n,m}$ depends on the modular coloring of outer graphs $J_{t,s}^{h}$ as discussed in the following cases.

Case 1. $t \equiv 1\,(mod4)$

The coloring of the graph $J_{t,s}$ is defined by the following two subcases.

Subcase 1.1.

When $h = \{i, j : 1 \le\ i \le m,\ \ j = 2,\ \ j = n-2\ and\ 3 \le j < (n-2)\ where\ j\ is\ odd\ \}$,

$$C\left(v_{ij}^{h}\right) = \begin{pmatrix} 0 & if\ 1 \le i \le m,\ \ j = 0\ and\ 1 \le\ i \le m, 1 \le j \le n-1,\ \ where\ j \equiv 2,3(mod4) \\ 1 & if\ i, j = 0\ and\ 1 \le\ i \le m, 1 \le j \le n-1,\ \ where\ j \equiv 1(mod4) \\ 3 & if\ 1 \le\ i \le m, 1 \le j \le n-1,\ \ where\ j \equiv 0(mod4) \end{pmatrix}.$$

We can obtain that the modular coloring of $J_{t,s}^{h}$ depends on the above coloring and the coloring of the h_{th} vertex v_{ij} of $J_{n,m}$. Here, all the h_{th} vertices v_{ij} of $J_{n,m}$ receive the colors 0 and 3; then the modular coloring of $J_{t,s}^{h}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 1\leq i\leq m, j=1\ and\ 3\leq j\leq n-1,\ where\ j\ is\ even\\ 2 & if\ 1\leq i\leq m,\ j=0\ ,2\ and\ 5\leq j\leq n-1,\ where\ j\ is\ odd\\ 3 & if\ i,j=0\ and\ 1\leq i\leq m,\ \ j=3\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 1\leq i\leq m,\ j=0\ \&\ 2\leq j\leq n-2,\ where\ j\ is\ even\\ 2 & if\ 1\leq i\leq m, 2\leq j\leq n-2,\ where\ j\ is\ odd\\ 3 & if\ i,j=0\ and\ 1\leq i\leq m,\ j=1\ \&\ n-1\end{cases}.$$

Subcase 1.2.

When $h=\{ij:\ 0\leq i\leq m,\ j=0; 1\leq i\leq m, j=1, n-1\ \&\ 3<j<(n-2)\ where\ j\ is\ even\}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,\ j=0\ \&\ 1\leq i\leq m, j=0,\ 3; \quad 3\leq j\leq n-1,\ where\ j\equiv 0,1\,(mod4)\\ 1 & if\ 1\leq i\leq m, j=2; \quad 4\leq j\leq n-1,\ where\ j\equiv 3\,(mod4)\\ 3 & if\ 1\leq i\leq m, j=1; \quad 3\leq j\leq n-1,\ where\ j\equiv 2\,(mod4)\end{cases}.$$

Using the above coloring and the coloring of the h_{th} vertex v_{ij}, we assign the modular coloring of $J_{t,s}^{h}$. According to the respective colors 0 and 2 of the h_{th} vertex v_{ij}, the modular coloring of $J_{t,s}^{h}$ is given as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if i,j=0\ and\ 1\leq i\leq m,\ j=4\\ 1 & if 1\leq i\leq m, j=1,3\ and\ 4<j\leq n-1,\ where\ j\ is\ even\\ 3 & if\ 1\leq i\leq m,\ j=0,2\ \ and\ \ 4<j\leq n-1,\ where\ j\ is\ odd\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}1 & i\,if\ 1\leq i\leq m,\ j=0,2\ \ and\ \ 4<j\leq n-1,\ \ where\ j\ is\ odd\\ 2 & if i,j=0\ and\ 1\leq i\leq m,\ j=4\\ 3 & if 1\leq i\leq m, j=1,3\ and\ 4<j\leq n-1,\ where\ j\ is\ even\end{cases}.$$

Applying the above colorings, we obtain the modular coloring of $J_{n,m}$ as follows:

$$\mathscr{S}\left(v_{ij}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m, j=1\ and\ j=n-1\\ 1 & if\ 1\leq i\leq m\ ,2\leq j\leq n-2,\ where\ j\ is\ odd\\ 2 & if\ 1\leq i\leq m,\ j=0\ and\ \ 2\leq j\leq n-2,\ where\ j\ is\ even\end{cases}.$$

It is very clear that the end point of the edges receive a distinct color sum. Therefore, $M_c\left(J_{n,m}^{\circ}\, J_{t,s}\right)=4$.

Case 2.$t\geq 5$ and $t\equiv 3\,(mod4)$

The coloring of the graph $J_{t,s}$ is defined by the following two subcases.

Subcase 2.1.

When $h=\{i,j: 1\leq i\leq m,\ j=2,\ n-2\ and\ 3\leq j<(n-2),\ where\ j\ is\ odd\ \}$,

$$C\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 1\leq j\leq m, j=0\ and\ 1\leq j\leq n-1,\ where\ j\equiv 1,2(mod4)\\ 1 & if\ i,j=0\ and\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\equiv 3(mod4)\\ 3 & if\ 1\leq i\leq m, 1\leq j\leq n-1,\ where\ j\equiv 0(mod4)\end{cases}.$$

We can obtain that the modular coloring of $J_{t,s}^{h}$ depends on the above coloring and the coloring of the h_{th} vertex v_{ij} of $J_{n,m}$. Here, all the h_{th} vertices v_{ij} of $J_{n,m}$ receive the colors 0 and 3; then the modular coloring of $J_{t,s}^{h}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if i,j=0\ and\ 1\leq i\leq m,\ j=1\ \&\ n-1\\ 1 & if 1\leq j\leq m, j=0\ and\ \ 2\leq j\leq n-2,\ where\ j\ is\ even\\ 3 & if\ 1\leq i\leq m, 2\leq j\leq n-2,\ where\ j\ is\ odd\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ 1\leq j\leq m, j=0\ and\ \ 2\leq j\leq n-2,\ where\ j\ is\ even\\ 2 & if\ 1\leq i\leq m, 2\leq j\leq n-2,\ where\ j\ is\ odd\\ 3 & if i,j=0\ and\ 1\leq i\leq m,\ j=1\ \&\ n-1\end{cases}.$$

Subcase 2.2.

When $h=\{ij: 1\leq i\leq m, j=0,\ 1,\ \&\ n-1\ and\ 3<j<(n-2),\ where\ j\ is\ even\}$,

$$C\left(v_{ij}^{h}\right)=\begin{pmatrix}0 & if\ i,j=0; 1\leq i\leq m, j=0\ \&\ 3\leq j\leq n-1,\ where\ j\equiv 2,3\,(mod4)\\ 1 & if\ 1\leq i\leq m, j=2\ and\ 3\leq j\leq n-1,\ where\ j\equiv 1\ (mod4)\\ 3 & if\ 1\leq i\leq m, j=1\ and\ 3\leq j\leq n-1,\ where\ j\equiv 0\,(mod4)\end{pmatrix}.$$

Using the above coloring and the coloring of the h_{th} vertex v_{ij}, we assign the modular coloring of $J_{t,s}^{h}$. According to the respective colors 0 and 2 of the h_{th} vertex v_{ij}, the modular coloring of $J_{t,s}^{h}$ is as follows:

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}0 & if\ i,j=0\ and\ 1\leq i\leq m,\ j=3\\ 1 & if\ \ 1\leq i\leq m, j=1\ and\ \ 3\leq j\leq n-1,\ where\ j\ is\ even\\ 3 & if\ 1\leq i\leq m, j=0\ ,2\ and\ 3\leq j\leq n-1,\ where\ j\ is\ odd\end{cases}.$$

$$\mathscr{S}\left(v_{ij}^{h}\right)=\begin{cases}1 & if\ 1\leq i\leq m, j=0\ ,2\ and\ 3\leq j\leq n-1,\ where\ j\ is\ odd\\ 2 & if\ i,j=0\ and\ 1\leq i\leq m,\ j=3\\ 3 & if\ \ 1\leq\ i\leq m, j=1\ and\ \ 3\leq j\leq n-1,\ where\ j\ is\ even\end{cases}.$$

It is very clear that the end point of the edges receive a distinct color. Therefore, $m_c\left(J_{n,m}^{\circ}\,J_{t,s}\right)=4$. Applying the above coloring, the modular coloring of $J_{n,m}$ is obtained to be the same as the above case 1. Therefore, the result follows from case 1.

3.10 Theorem

The modular chromatic number of the corona product of a generalized Jahangir graph $(J_{n,m}^{\circ}\,J_{t,s})$ is 4 (excluding when t is even).

Proof

Let $J_{n,m}$ be a generalized Jahangir graph, for $m\geq 3$. Let $\left(J_{n,m}^{\circ}\,J_{t,s}\right)$ be the corona product of the generalized Jahangir graph in that all the outer graphs $J_{t,s}^{h}$ are modular 3-colorable; as t is odd, we require a minimum of 4 colors to color the center graph $J_{n,m}$.

The mapping of the coloring follows from lemmas 3.8 and 3.9, respectively; the resulting modular 4-coloring of $\left(J_{n,m}^{\circ}\,J_{t,s}\right)$ is given in the following tables.

Let t be odd; the proof of the theorem is discussed according to the value of n in the following two cases.

Case 1. When n and t are odd

Table 9. Coloring of $\left(J_{n,m}^{\circ}\,J_{t,s}\right)$

V_{ij}	C(v_{ij}) of $J_{n,m}$	V_{ij}^{h}	C(v_{ij}^{h}) of $J_{t,s}^{h}$	
			Type-1	**Type-2**
v_{00}	2	v_{00}^{h}	1	0
v_{i0}	0	v_{i0}^{h}	0	0
v_{ij}	{1,3,0,3,0,3,...,0,3}	v_{ij}^{h}	$\{1,0,0,3,\ 1,0,0,3,\ldots\}$	$\{3,1,0,0,\ 3,1,0,0,\ldots\}$

Table 10. Modular coloring of($J_{n,m}^{\circ}\,J_{t,s}$) when $t\equiv 1\ (mod 4)$ and $t\equiv 3\ (mod 4)$

$\mathscr{S}(J_{n,m})$		Cases	Types	C($h_{th}v_{ij}$)	v_{00}^{h}	v_{i0}^{h}	v_{ij}^{h}
V_{ij}	$S(v_{ij})$ **for** $J_{n,m}$						
			Type-1	3	3	0	$\{3,0,2,3,\ldots,3\}$
				0	0	3	$(1,3,1,0,...,0\}$
v_{00}	1	$t\equiv 1(mod\ 4)$	Type-2	1	1	0	$\{2,0,2,1,\ldots,1\}$
				2	2	1	$\{2,0,2,1,\ldots,1\}$
				3	3	2	$\{0,2,0,3\ldots,3\}$
v_{i0}	2		Type-1	3	2	3	$\{2,3,1,3,1,2,\ldots,2\}$
				0	0	3	$(1,3,0,1,3,1,...,1\}$
v_{ij}	{3,1,2,1,2,…,1,2,1}	$t\equiv 3(mod\ 4)$	Type-2	1	1	0	$\{2,0,1,2,0,2,\ldots,2\}$
				2	2	1	$(\ 3,1,2,0,1,3,\ldots 3\}$
				3	3	2	$\{0,2,3,1,2,0,\ldots,0\}$

Case 2. When n is even and t is odd

Table 11. Coloring of $\left(J_{n,m}^{\circ}\,J_{t,s}\right)$

V_{ij}	C(v_{ij}) of $J_{n,m}$	V_{ij}^{h}	C(v_{ij}) of $J_{t,s}^{h}$	
			Type-1	**Type-2**
v_{00}	2	v_{00}^{h}	1	0
v_{i0}	0	v_{i0}^{h}	0	0
v_{ij}	{2,0, 3,0,…,3,0,2}	v_{ij}^{h}	$\{(1,0,0,3),(1,0,0,3)\ldots\}$	$\{(3,1,0,0),(3,1,0,0),\ldots\}$

A centre vertex, rim vertices are colored using Z_4 colors, and the petal vertices are colored using Z_4 colors in cyclic bases; from the [Tables 10 and 12] it is evident that each $S(v_{ij})\neq\ S(u_{ij})$, where v & u are adjacent vertices in $J_{n,m}^{\circ}\,J_{t,s}$. That is, the modular coloring of every vertex is different, satisfying modular 4-coloring. Therefore, $M_c\left(J_{n,m}^{\circ}\,J_{t,s}\right)=4$ (excluding t is even).

Table 12. Modular coloring of($J^{\circ}_{n,m} J_{t,s}$) when $t \equiv 1 \pmod 4$ and $t \equiv 3 \pmod 4$

$S(J_{n,m})$ V_{ij}	$S(v_{ij})$ for $J_{n,m}$	Cases	Types	$\mathrm{C}(h_{th} v_{ij})$	v^h_{00}	v^h_{i0}	v^h_{ij}
v_{00}	0	$t \equiv 1(mod\ 4)$	Type-1	0	0	1	$\{0,1,3,0\,0,1,3,0,...\}$
				3	3	1	$\{3,0,2,3\,3,0,2,3,\ldots\}$
			Type-2	0	0	3	$\{0,1,3,1\,0,1,3,1,\ldots\}$
				2	2	1	{2,3,1,3 2,3,1,3,…}
v_{i0}	2	$t \equiv 3(mod\ 4)$	Type-1	0	0	1	$\{0,13,13,0,\ldots,0\}$
				3	3	0	$\{3,0,2,0,2,3,\ldots,3\}$
v_{ij}	$\{0,2,1,2,\ldots,1,2,0\}$		Type-2	0	0	3	$(1,3,0,1,3,1,...,1\}$
				2	2	1	$(3,1,2,0,1,3,\ldots 3\}$

4 Conclusion

In this study, we analyzed the corona product of a generalized Jahangir graph using various cases and determined the modular chromatic number is 3 and 4 for any positive integer $m,n,s,t \geq 3$ and $k \in z_k$. Previously, we obtained the result that $\chi(J_{n,m}) = m_c(J_{n,m})$; here, we arrived at the conclusion that $m_c(J_{n,m}) < m_c\left(J^{\circ}_{n,m} J_{t,s}\right)$. Readers can focus on various product graphs and find different bounds. Still some open problems are available[1].

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