INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Subdivision of Eccentric Fuzzy Graphs

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OPEN ACCESS

Received: 12-10-2023

Accepted: 22-11-2023

Published: 25-12-2023

Citation: Jovita JJ, Umamaheswari O, Meenal N (2023) Subdivision of Eccentric Fuzzy Graphs. Indian Journal of Science and Technology 16(SP4): 8-17. https://doi.org/10.17485/IJST/v16iSP4.ICAMS90

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Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objectives: To present a recursion formula on the order of an eccentric fuzzy graph and the same is found by subdividing the set of lines in it. **Methods:** Here, a special fuzzy graph known as an eccentric fuzzy graph denoted by EF(G): (σ_e, μ_e), which is derived by eccentricity and diameter of a crisp graph G^*:(P,L) with membership function of point set σ_e and line set μ_e is considered. In this article, the wheel and ladder graph were mainly discussed. **Findings:** With membership function of Eccentric fuzzy graph, the order of an eccentric fuzzy ladder graph was established and the same is found by subdividing the lines on the left path, right path and on both. Further, order for an eccentric fuzzy wheel graph by subdividing rim, spoke and both. Additionally, recursion formulas for atop were found. **Novelty:** This is the first article which deals with the order of an eccentric fuzzy graph by subdividing edges.
2020 A.M.S Mathematical Subject Classification: 03E72, 05C07, 05C12, 05C69.

Keywords: Fuzzy sets; Fuzzy Graph; Distance in graph; Vertex degree; subdivision

1 Introduction

Let $G^* : (P,L)$ denote the crisp graph with point set P and line set L, whose order and size are n and m, respectively. Any two distinct end points p_i and p_j can be joined by a line p_ip_j. Two points are said to be adjacent if they are joined by a line. The shortest path between two points p_i and p_j is denoted by $d(p_i, p_j)$ called distance. The farthest distance from a point p_j is defined as eccentricity, and it is denoted by $ecc(p_j)$ that is, ecc(p_j)=max$\{d(p_i, p_j)/p_i, p_j \in P(G^*)\}$. The maximum (minimum) eccentricity of a graph is called the diameter(radius), and it is denoted by dia = max$\{ecc(p_i)/p_i \in P(G^*)\}$ (rad = min$\{ecc(p_i)/p_i \in P(G^*)\}$). The path is denoted by P_α as a sequence of distinct consecutive points and distinct consecutive lines. The closed path is called a cycle $C_{\alpha.}$ The length of the shortest cycle of the graph is called the girth of the graph. A new point u_i is introduced and connect it to every point in a cycle, the new graph thus obtained is called wheel graph and is denoted by $W_{\alpha.}$ The points on the cycle are called rim points, and the newly introduced point is the apex. The new lines are referred to as spokes. If G_1^* and G_2^* are any two graphs, then the cartesian product is denoted by $G_1^* \times G_2^*$ and is defined as the cartesian product of $P(G_1^*)$ and $P(G_2^*)$ where the two points (p_1, p_1') and (p_2, p_2') are adjacent in $G_1^* \times G_2^*$ if and only if either $p_1 = p_2$ and p_1' is adjacent to p_2' in G_2^*. Ladder graph L_α is the cartesian product of two paths P_2 and P_α denoted by $P_2 \times P_\alpha$. Thus, the ladder graph has two paths, one on the left p_α and one on the right $p_\alpha^{'}$ with a line that connecting two.

The idea of Eccentric fuzzy graph was first defined by Meenal et al.[1–3]. This article mainly focused on order of an eccentric fuzzy ladder graph and eccentric fuzzy wheel graph by subdividing the lines on it. Through which the recursion formula were derived. The subdivision on ladder graph is made at a time for right path, left path and both. Further, for the wheel graph the subdivision were made on rim, spoke and both singly.

Notation:

(1). $Nr(O(EF(G^*))$ denotes the numerator part of a graph's eccentricity sum.
(2). $O(EF(S_{P_\alpha}(L_\alpha)))$ denotes the order in subdividing on the left path of a ladder graph.
(3). $O(EF(S_{2P_\alpha}(L_\alpha)))$ denotes the order in subdividing on both the left and right paths of a ladder graph.
(4). $O(EF(S_{P_2}(L_\alpha)))$ denotes the order on the subdividing line between the left path and the right path of a ladder graph.
(5). $O(EF(S(L_\alpha)))$ denotes the order of subdividing on every line of the ladder graph.
(6). $O(EF(S_r(W_\alpha)))$denotes the order in subdividing on the rim of a wheel graph.
(7). $O(EF(S_s(W_\alpha)))$ denotes the order of subdividing on a wheel graph's spoke.
(8). $O(EF(S_{rs}(W_\alpha)))$ denotes the order of subdividing a wheel graph's rim and spokes.

2 Subdivision of EF(G)

Definition 2.1[1,2] An Eccentric Fuzzy Graph $EF(G) : (\sigma_e, \mu_e)$ under the crisp graph $G^* : (P, L)$ with the set of eccentric membership functions as follows,

Eccentric Fuzzy Point set function $\sigma_e(p_i) : P(G*) to [0,1]$ on the point set $P(G^*)$ is specified as,

$$\sigma_e(p_i) = \frac{ecc(p_i)}{dia(G^*)}; \quad \forall p_i \in P(G^*).$$

(ii). Eccentric Fuzzy line set function as $\mu_e(p_i p_j) : L(G^*) to [0,1]$ on the line set $L(G^*)$ is specified as, $\mu_e(p_i p_j) = min\{\sigma_e(p_i), \sigma_e(p_j)\}; \quad \forall p_i p_j \in L(G^*)$ that is, every line is a strong line.

Definition 2.2 Let $EF(G) : (\sigma_e, \mu_e)$ be an Eccentric Fuzzy Graph under the crisp graph $G^* : (P, L)$, then Order and Size are defined as, $O_e(EF(G)) = \sum_{p_j \in P(G^*)} \sigma_e(p_j); \quad \forall p_j \in P(G^*)$

$S_e(EF(G)) = \sum_{p_i p_j \in L(G^*)} \mu_e(p_i p_j); \quad \forall\, p_i p_j \in L(G^*)$.

Theorem 2.1 The order of a eccentric fuzzy ladder graph $EF(L_{2\alpha})$, under the crisp graph $L_\alpha : (P, L)$ whose order is 2α, where $\alpha \geq 4$ is,

$$O(EF(L_\alpha)) = \begin{cases} \frac{\frac{3\alpha^2+2\alpha-1}{2}}{\alpha}; if\ \alpha\ is\ odd \\ \frac{\frac{3\alpha^2+2\alpha}{2}}{\alpha}; if\ \alpha\ is\ even \end{cases} \quad and\ O(EF(L_{\alpha+4})) = \frac{Nr(O(EF(L_\alpha)))}{\alpha+2} + \frac{6\alpha+8}{\alpha+2}$$

Proof. Let $EF(L_\alpha)$ be an eccentric fuzzy ladder graph whose crisp graph L_α with point set $\{p_1, p_2, \ldots, p_\alpha, p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{2\alpha}\}$.

Case(i): If α is odd

The eccentricity of the point in a Ladder graph $L_{2\alpha}$ is, $ecc(p_1) = \alpha = ecc(p_{\alpha+1}) = ecc(p_\alpha) = ecc(p_{2\alpha})$;
$ecc(p_2) = \alpha - 1 = ecc(p_{\alpha+2}) = ecc(p_{\alpha-1}) = ecc\left(p_{(2\alpha-1)}\right); \ldots;$
$ecc(p_{\frac{\alpha-1}{2}}) = ecc(p_{\frac{(3\alpha-1)}{2}}) = ecc(p_{\frac{\alpha+3}{2}}) = \frac{\alpha+3}{2}; ecc(p_{\frac{\alpha+1}{2}}) = ecc(p_{\frac{3\alpha+1}{2}}) = \frac{\alpha+1}{2}$.

Thus, the diameter is $dia(G^*) = \alpha$.

The membership function of a point set is defined by, $\sigma_e(p_i) = \frac{ecc(p_i)}{dia(G^*)} = \frac{ecc(p_i)}{\alpha}; \quad \forall p_i \in P(G^*)$

The order of a ladder graph is,

$$O(EF(L_\alpha)) = 4\left[\frac{\alpha}{\alpha} + \frac{\alpha-1}{\alpha} + \frac{\alpha-2}{\alpha} + \cdots + \frac{\alpha-(\frac{\alpha-3}{2})}{\alpha}\right] + 2\frac{\left[\frac{\alpha+1}{2}\right]}{\alpha}$$
$$= 4\frac{\left[\alpha(\frac{\alpha-1}{2}) - (1+2+\cdots+\frac{\alpha-3}{2})\right]}{\alpha} + 2\frac{\left[\frac{\alpha+1}{2}\right]}{\alpha}$$
$$OEF(L_\alpha)) = \frac{\left[\frac{3\alpha^2+2\alpha-1}{2}\right]}{\alpha}$$

The recursion formula for eccentric fuzzy ladder graph is,

$O(EF(L_\alpha)) = \frac{Nr(O(EF(L_\alpha)))}{\alpha+2} + \frac{4(\alpha+1)+4(\alpha+2)-2\left(\frac{\alpha+3}{2}+\frac{\alpha+1}{2}\right)}{\alpha+2}$

$O(EF(L_\alpha)) = \frac{Nr(O(EF(L_\alpha)))}{\alpha+2} + \frac{6\alpha+8}{\alpha+2}$.

Case(ii): If α is even

The eccentricity of the point in a Ladder graph $L_{2\alpha}$ is,

$$ecc(p_1) = \alpha = ecc(p_{\alpha+1}) = ecc(p_\alpha) = ecc(p_{2\alpha});$$

$$ecc(p_2) = \alpha - 1 = ecc(p_{\alpha+2}) = ecc(p_{\alpha-1}) = ecc\left(p_{(2\alpha-1)}\right);\ldots;$$

$$ecc\left(p_{\frac{\alpha-2}{2}}\right) = ecc\left(p_{\frac{(3\alpha-2)}{2}}\right) = \frac{\alpha+4}{2} = ecc\left(p_{\frac{\alpha+4}{2}}\right) = ecc\left(p_{\frac{(3\alpha+4)}{2}}\right);$$

$ecc(p_{\frac{\alpha}{2}}) = ecc(p_{\alpha+\frac{(\alpha)}{2}}) = \frac{\alpha+2}{2} = ecc(p_{\frac{\alpha+2}{2}}) = ecc(p_{\alpha+\frac{\alpha+2}{2}})$.

Thus, the diameter is $dia(G^*) = \alpha$.

The membership function of a point set is defined by, $\sigma_e(p_i) = \frac{ecc(p_i)}{dia(G^*)} = \frac{ecc(p_i)}{\alpha}; \quad \forall p_i \in L_\alpha$

The order of a ladder graph is,

$O(EF(L_\alpha)) = 4\left[\frac{\alpha}{\alpha} + \frac{\alpha-1}{\alpha} + \frac{\alpha-2}{\alpha} + \cdots + \frac{\alpha-(\frac{\alpha-2}{2})}{\alpha}\right]$

$= 4\frac{\left[\alpha(\frac{\alpha}{2}) - (1+2+\cdots+\frac{\alpha-2}{2})\right]}{\alpha}$

$OEF(L_\alpha)) = \frac{\frac{3\alpha^2+2\alpha}{2}}{\alpha}$.

The recursion formula of the ladder graph,

$O(EF(L_{\alpha+1})) = \frac{Nr(O(EF(L_\alpha)))}{\alpha+2} + \frac{4(\alpha+1)+4(\alpha+2)-4\left(\frac{\alpha+1}{2}\right)}{\alpha+2}$

$O(EF(L_{\alpha+1})) = \frac{Nr(O(EF(L_\alpha)))}{\alpha+2} + \frac{6\alpha+8}{\alpha+2}$.

Remark 2.1

Order of an eccentric fuzzy ladder graph of α=3 is 16 i.e., $O(EF(L_3)) = 16$

Theorem 2.2 The order of a eccentric fuzzy ladder graph by subdividing the lines on left path of ladder graph in an $EF(L_\alpha)$, under the crisp graph $L_\alpha : (P,L)$, whose order is $3\alpha-1$, where $\alpha \geq 3$ is,

$O(EF(S_{P_\alpha}(L_{\alpha+2}))) = \begin{cases} \frac{\frac{9\alpha^2+12\alpha-9}{4}}{\alpha+1}; \; if\ \alpha\ is\ odd \\ \frac{\frac{9\alpha^2+12\alpha-8}{4}}{\alpha+1}; \; if\ \alpha\ is\ even \end{cases} \quad and\ O(EF(S_{P_\alpha}(L_{\alpha+2}))) = \frac{Nr(O(EF(S_{P_\alpha}(L_\alpha))))}{\alpha+3} + \frac{9\alpha+15}{\alpha+3}$

Proof . Consider an eccentric fuzzy ladder graph $EF(L_\alpha)$ with a crisp graph L_α consisting of two paths with point set $(p_1, p_2, \ldots, p_\alpha\}$ denotes left path and $\{p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{2\alpha}\}$ denotes right path. Assume the subdivision of the line is constructed as follows:

(i) Join the points p_i to $p_{\alpha+i}$ where $1 \leq i \leq \alpha$.

(ii). The newly added point on the subdivision of the line in the left path is represented by $p_{(i)(i+1)}$ where $1 \leq i \leq \alpha-1$.

The new point established after subdividing the lines on the left path is,

$\{p_1, p_{(1)(2)}, p_2, p_{(2)(3)}, p_3, \ldots, p_{\alpha-1}, p_{(\alpha-1)(\alpha)}, p_\alpha, p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{2\alpha}\}$.

Case(i): If α is odd.

The eccentricity of the points is given by,

$$ecc(p_1) = ecc(p_\alpha) = ecc\left(p_{(1)(2)}\right) = ecc\left(p_{(\alpha-1)(\alpha)}\right) = \alpha+1;$$

$$ecc(p_2) = ecc(p_{\alpha-1}) = ecc\left(p_{(2)(3)}\right) = ecc(p_{\alpha+1}) = ecc(p_{2\alpha}) = ecc\left(p_{(\alpha-2)(\alpha-1)}\right) = \alpha;$$

$$ecc(p_3) = ecc(p_{\alpha-2}) = ecc\left(p_{(3)(4)}\right) = ecc(p_{\alpha+2}) = ecc\left(p_{(2\alpha-1)}\right) = ecc\left(p_{(\alpha-3)(\alpha-2)}\right) = \alpha-1$$

$$ecc\left(p_{\frac{\alpha-1}{2}}\right)=ecc\left(p_{\frac{\alpha+3}{2}}\right)=ecc\left(p_{\left(\frac{\alpha-1}{2}\right)\left(\frac{\alpha+1}{2}\right)}\right)=ecc\left(p_{\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+3}{2}\right)}\right)=ecc\left(p_{\left(\frac{3\alpha-3}{2}\right)}\right)=ecc\left(p_{\left(\frac{3\alpha+5}{2}\right)}\right)=\frac{\alpha+5}{2};$$

$$ecc(p_{\frac{\alpha+1}{2}})=ecc(p_{(\frac{3\alpha-1}{2})})=ecc(p_{(\frac{3\alpha+3}{2})})=\frac{\alpha+3}{2};ecc(p_{(\frac{3\alpha+1}{2})})=\frac{\alpha+1}{2}.$$

This implies that the diameter is dia$(G^{*})=\alpha$+1.

Thus, the membership value of point set is defined by, $\sigma_e\left(p_i\right)=\frac{ecc(p_i)}{dia(G^*)}=\frac{ecc(p_i)}{\alpha+1};\quad\forall p_i\in$ P(G*)

The order of an eccentric fuzzy ladder graph by subdividing the lines on the left path of the ladder graph is,

$$O(EF(S_{P_\alpha}(L_\alpha)))=6\left[\frac{\alpha}{\alpha+1}+\frac{\alpha-1}{\alpha+1}+\cdots+\frac{\alpha-(\frac{\alpha-5}{2})}{\alpha+1}\right]+4\left[\frac{\alpha+1}{\alpha+1}\right]+3\left[\frac{(\frac{\alpha+3}{2})}{\alpha+1}\right]+\left[\frac{(\frac{\alpha+1}{2})}{\alpha+1}\right]$$
$$=\frac{6}{\alpha+1}\left(\alpha\left(\frac{\alpha-3}{2}\right)-\left(1+2+3+\cdots+\frac{\alpha-5}{2}\right)\right]+4\left(\frac{\alpha+1}{\alpha+1}\right]+\frac{3}{2}\left(\frac{\alpha+3}{\alpha+1}\right]+\frac{1}{2}\left(\frac{\alpha+1}{\alpha+1}\right]$$
$$O(EF(S_{P_\alpha}(L_\alpha)))=\frac{\left[\frac{9\alpha^2+12\alpha-9}{4}\right]}{\alpha+1}.$$

The recursion formula on the order of an eccentric fuzzy ladder graph by subdividing the lines on the left path of the ladder graph is,

$$O\left(EF(S_{P_\alpha}\left(L_{\alpha+1}\right)\right)=\frac{Nr(O(EF(S_{P_\alpha}(L_\alpha))))}{\alpha+3}+\frac{6(\alpha+2)+4(\alpha+3)+2(\alpha+1)-3(\frac{\alpha+5}{2})-2(\frac{\alpha+3}{2})-(\frac{\alpha+1}{2})}{\alpha+3}$$
$$O(EF(S_{P_\alpha}(L_\alpha)))=\frac{Nr(O(EF(S_{P_\alpha}(L_{3\alpha-1}))))}{\alpha+3}+\frac{6\alpha+42}{\alpha+3}.$$

Case(ii): If α is even.

The eccentricity of the points is given by,

$$ecc\left(p_1\right)=ecc\left(p_\alpha\right)=ecc\left(p_{(1)(2)}\right)=ecc\left(p_{(\alpha-1)(\alpha)}\right)=\alpha+1;$$

$$ecc\left(p_2\right)=ecc\left(p_{\alpha-1}\right)=ecc\left(p_{(2)(3)}\right)=ecc\left(p_{\alpha+1}\right)=ecc\left(p_{2\alpha}\right)=ecc\left(p_{(\alpha-2)(\alpha-1)}\right)=\alpha;$$

$$ecc\left(p_3\right)=ecc\left(p_{\alpha-2}\right)=ecc\left(p_{(3)(4)}\right)=ecc\left(p_{\alpha+2}\right)=ecc\left(p_{(2\alpha-1)}\right)=ecc\left(p_{(\alpha-3)(\alpha-2)}\right)=\alpha-1;\ldots;$$

$$ecc\left(p_{\frac{\alpha-2}{2}}\right)=ecc\left(p_{\frac{\alpha+4}{2}}\right)=ecc\left(p_{\left(\frac{\alpha-2}{2}\right)\left(\frac{\alpha}{2}\right)}\right)=ecc\left(p_{\left(\frac{\alpha+2}{2}\right)\left(\frac{\alpha+4}{2}\right)}\right)=ecc\left(p_{\left(\frac{3\alpha-4}{2}\right)}\right)=ecc\left(p_{\left(\frac{3\alpha+6}{2}\right)}\right)=\frac{\alpha+6}{2};$$

$$ecc\left(p_{\frac{\alpha}{2}}\right)=ecc\left(p_{\frac{\alpha+2}{2}}\right)=ecc\left(p_{\left(\frac{\alpha}{2}\right)\left(\frac{\alpha+2}{2}\right)}\right)=ecc\left(p_{\left(\frac{3\alpha-2}{2}\right)}\right)=ecc\left(p_{\left(\frac{3\alpha+4}{2}\right)}\right)=\frac{\alpha+4}{2};$$

$ecc(p_{(\frac{3\alpha}{2})})=ecc(p_{(\frac{3\alpha+2}{2})})=\frac{\alpha+2}{2}.$

This implies that the diameter is dia$(G^{*})=\alpha$+1.

Thus, the membership value of a point set is defined by, $\sigma_e\left(p_i\right)=\frac{ecc(p_i)}{dia(G^*)}=\frac{ecc(p_i)}{\alpha+1};\quad\forall p_i\in$ P(G*)

The order of an eccentric fuzzy ladder graph by subdividing the lines on the left path of the ladder graph is,

$$O(EF(S_{P_\alpha}(L_\alpha)))=6\left[\frac{\alpha}{\alpha+1}+\frac{\alpha-1}{\alpha+1}+\cdots+\frac{\alpha-(\frac{\alpha-6}{2})}{\alpha+1}\right]+4\left[\frac{\alpha+1}{\alpha+1}\right]+5\left[\frac{(\frac{\alpha+4}{2})}{\alpha+1}\right]+2\left[\frac{(\frac{\alpha+2}{2})}{\alpha+1}\right]$$
$$=\frac{6}{\alpha+1}\left[\alpha\left(\frac{\alpha-4}{2}\right)-\left(1+2+3+\cdots+\frac{\alpha-6}{2}\right)\right]+4\left[\frac{\alpha+1}{\alpha+1}\right]+\frac{5}{2}\left[\frac{\alpha+4}{\alpha+1}\right]+\frac{2}{2}\left[\frac{\alpha+2}{\alpha+1}\right]$$
$$O(EF(S_{P_\alpha}(L_\alpha)))=\frac{\left[\frac{9\alpha^2+12\alpha-8}{4}\right]}{\alpha+1}.$$

The recursion formula on the order of an eccentric fuzzy ladder graph by subdividing the lines on the left path of the ladder graph is, $O(EF(S_{P_\alpha})(L_\alpha))=\frac{Nr(O(EF(S_{P_\alpha}(L_\alpha))))}{\alpha+3}+\frac{6(\alpha+2)+4(\alpha+3)+2(\alpha+1)-3(\frac{\alpha+4}{2})-2(\frac{\alpha+2}{2})-(\frac{\alpha+6}{2})}{\alpha+3}$

$$O(EF(S_{P_\alpha}(L_\alpha)))=\frac{Nr(O(EF(S_{P_\alpha}(L_{3\alpha-1}))))}{\alpha+3}+\frac{9\alpha+15}{\alpha+3}.$$

Remark 2.2

The order of an eccentric fuzzy ladder graph on subdivision of left path of a laddar graph is $O\left(EF\left(S_{P_\alpha}\right)\left(L_3\right)\right)=24$ *and* $O\left(EF\left(S_{P_\alpha}\right)\left(L_4\right)\right)=48$.

Theorem 2.3 The order of a eccentric fuzzy ladder graph by subdividing the lines on both path of ladder graph in an $EF(L_\alpha)$, under the crisp graph $L_\alpha:(P,L)$, whose order is $4\alpha-2$, where $\alpha\geq 3$ is,

$O\left(EF\left(S_{2P_\alpha}\left(L_\alpha\right)\right)\right)=\frac{2\alpha(3\alpha-2)}{2\alpha-1}$ *and* $O(EF(S_{2P_\alpha}(L_{\alpha+1})))=\frac{Nr(O(EF(S_{2P_\alpha}(L_\alpha))))}{2\alpha+1}+\frac{2(6\alpha+1)}{2\alpha+1}$

Proof. Consider an eccentric fuzzy ladder graph $EF\left(L_{2\alpha}\right)$with a crisp graph L_α consisting of two paths with point set $(p_1,p_2,..,p_\alpha\}$ denotes left path and $(p_{\alpha+1},p_{\alpha+2},\ldots,p_{2\alpha}\}$ denotes right path. Assume the subdivision of the line is constructed as follows:

(i) Join the points p_i to $p_{\alpha+i}$ where $1\leq i\leq\alpha$.

(ii) The newly added point on the subdivision of line in the left path and right path is represented by, $p_{(i)(i+1)}$ and $p_{(\alpha+i)(\alpha+(i+1))}$, where $1\leq i\leq\alpha-1$.

The new point set after subdividing the lines on the left and right paths is,

$$\{p_1,p_{(1)(2)},p_2,p_{(2)(3)},p_3,\cdots,p_{\alpha-1},p_{(\alpha-1)(\alpha)},p_\alpha,p_{\alpha+1},p_{(\alpha+1)(\alpha+2)},p_{\alpha+2},p_{(\alpha+2)(\alpha+3)},\cdots,p_{\alpha+1},p_{(2\alpha-1))(2\alpha)},p_{2\alpha}\}$$

If α is odd, The eccentricity of the points is given by,

$$ecc\left(p_1\right)=ecc\left(p_\alpha\right)=ecc\left(p_{\alpha+1}\right)=ecc\left(p_{2\alpha}\right)=2\alpha-1;$$

$$ecc\left(p_{(1)(2)}\right)=ecc\left(p_{(\alpha-1)(\alpha)}\right)=ecc\left(p_{(\alpha)(\alpha+1)}\right)=ecc\left(p_{(2\alpha-1)(\alpha+\alpha)}\right)=2\alpha-2;\ldots;$$

$$ecc\left(p_{\left(\frac{\alpha-1}{2}\right)\left(\frac{\alpha+1}{2}\right)}\right)=ecc\left(p_{\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+3}{2}\right)}\right)=ecc\left(p_{\left(\frac{3\alpha-1}{2}\right)\left(\frac{3\alpha+1}{2}\right)}\right)=ecc\left(p_{\left(\frac{3\alpha+1}{2}\right)\left(\frac{3\alpha+3}{2}\right)}\right)=\alpha+1;$$

$$ecc(p_{\left(\frac{\alpha+1}{2}\right)}=ecc(p_{\left(\frac{3\alpha+1}{2}\right)})=\alpha.$$

This implies that the diameter is dia$(G^*)=2\alpha-1$.

Thus, the membership value of a point set is defined by, $\sigma_e\left(p_i\right)=\frac{ecc(p_i)}{dia(G^*)}=\frac{ecc(p_i)}{2\alpha-1};\quad\forall p_i\in$ P(G*)

If α is even, The eccentricity of the points is given by,

$$ecc\left(p_1\right)=ecc\left(p_\alpha\right)=ecc\left(p_{\alpha+1}\right)=ecc\left(p_\alpha\right)=2\alpha-1;$$

$$ecc\left(p_{(1)(2)}\right)=ecc\left(p_{(\alpha-1)(\alpha)}\right)=ecc\left(p_{(\alpha)(\alpha+1)}\right)=ecc\left(p_{(2\alpha-1)(2\alpha)}\right)=2\alpha-2;\ldots;$$

$$ecc\left(p_{\frac{\alpha}{2}}\right)=ecc\left(p_{\frac{\alpha+2}{2}}\right)=ecc\left(p_{\left(\frac{3\alpha}{2}\right)}\right)=ecc\left(p_{\left(\frac{3\alpha+2}{2}\right)}\right)=\alpha+1;$$

$ecc(p_{\left(\frac{\alpha}{2}\right)\left(\frac{\alpha+2}{2}\right)})=ecc(p_{\left(\frac{3\alpha}{2}\right)\left(\frac{3\alpha+2}{2}\right)})=\alpha.$

This implies that the diameter is dia$(G^*)=2\alpha$-1.

Thus, the membership value of a point set is defined by, $\sigma_e\left(p_i\right)=\frac{ecc(p_i)}{dia(G^*)}=\frac{ecc(p_i)}{2\alpha-1};\quad\forall p_i\in$ P(G*)

For both the cases of α(either odd or even). The order of an eccentric fuzzy ladder graph by subdividing the lines on both left path and right path of ladder graph is,

$$\begin{aligned}O(EF(S_{2P_\alpha}(L_\alpha)))&=2\left[\frac{\alpha+1}{2\alpha-1}+\frac{\alpha+2}{2\alpha-1}+\cdots+\frac{\alpha+(\alpha-1)}{2\alpha-1}\right]+4\left[\frac{\alpha}{2\alpha-1}\right]\\&=\frac{2}{2\alpha-1}\left(\alpha(\alpha-1)+(1+2+3+\cdots+(\alpha-1)\right]+2\left[\frac{\alpha}{2\alpha-1}\right]\end{aligned}$$

$$O(EF(S_{2P_\alpha}(L_\alpha)))=\left[\frac{2(3\alpha^2-2\alpha)}{2\alpha-1}\right].$$

The recursion formula on the order of an eccentric fuzzy ladder graph by subdividing the lines on both left and right path of the ladder graph for both the odd and even case of α is,

$O(EF(S_{2P_\alpha}(L_\alpha))) = \frac{Nr(O(EF(S_{2P_\alpha}(L_\alpha))))}{2\alpha+1} + \frac{4(2\alpha)+4(2\alpha+1)-2(\alpha+1)-2\alpha}{2\alpha+1}$

$O(EF(S_{2P_\alpha}(L_\alpha))) = \frac{Nr(O(EF(S_{2P_\alpha}(L_\alpha))))}{2\alpha+1} + \frac{2(6\alpha+1)}{2\alpha+1}$.

Theorem 2.4 The order of a ladder graph by subdividing the lines between left and right path in an EF(L_α), under the crisp graph $L_\alpha : (P,L)$, whose order is 3α, where $\alpha \geq 4$ is given by,

$$O(EF(S_{P_2}(L_\alpha))) = \begin{cases} \frac{\frac{9\alpha^2+18\alpha-3}{4}}{\alpha+1}; if\ \alpha\ is\ odd \\ \frac{\frac{9\alpha^2+18\alpha}{4}}{\alpha+1}; if\ \alpha\ is\ even \end{cases} \quad and$$

$$O(EF(S_{P_2}(L_{\alpha+2}))) = \frac{Nr(O(EF(S_{P_2}(L_\alpha))))}{\alpha+3} + \frac{9\alpha+18}{\alpha+3}$$

Proof. Consider an eccentric fuzzy ladder graph $EF(L_\alpha)$ with a crisp graph L_α consisting of two paths with point set $(p_1, p_2, \ldots, p_\alpha\}$ denotes left path and $\{p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{2\alpha}\}$ denotes right path. Assume the subdivision of the line is constructed as follows:

(i) Join the points p_i to $p_{\alpha+i}$ where $1 \leq i \leq \alpha$.

(ii) The newly added point on the subdividing of the line between the left and right paths is represented by $p_{i(\alpha+i)}$ where $1 \leq i \leq \alpha$. Now, the new point established after subdividing the lines between the left and right paths is, $\{p_1, p_{(1)(\alpha+\alpha)}, p_2, p_{(2)(\alpha+(\alpha-1))}, p_3, \ldots, p_{\alpha-1}, p_{(\alpha-1)(\alpha+2)}, p_\alpha, p_{(\alpha)(\alpha+1)}, p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{2\alpha}\}$.

Case(i): If α is odd.

The eccentricity of the points is given by,

$$ecc(p_1) = ecc(p_\alpha) = ecc(p_{\alpha+1}) = ecc(p_{2\alpha}) = ecc\left(p_{(1)(\alpha+\alpha)}\right) = ecc\left(p_{(\alpha)(\alpha+1)}\right) = \alpha+1;$$

$$ecc(p_2) = ecc\left(p_{(2\alpha-1)}\right) = ecc\left(p_{(2)(2\alpha-1)}\right) = ecc(p_{\alpha-1}) = ecc(p_{\alpha+2}) = ecc\left(p_{(\alpha-1)(\alpha+2)}\right) = \alpha; \ldots$$

$$ecc\left(p_{\frac{\alpha-1}{2}}\right) = ecc\left(p_{\left(\frac{3\alpha+3}{2}\right)}\right) = ecc\left(p_{\left(\frac{\alpha-1}{2}\right)\left(\frac{3\alpha+3}{2}\right)}\right) = ecc\left(p_{\left(\frac{\alpha+3}{2}\right)\left(\frac{3\alpha-1}{2}\right)}\right) = ecc\left(p_{\left(\frac{3\alpha-1}{2}\right)}\right) = ecc\left(p_{\frac{\alpha+3}{2}}\right) = \frac{\alpha+5}{2};$$

$ecc(p_{\frac{\alpha+1}{2}}) = ecc(p_{(\frac{3\alpha+1}{2})}) = ecc(p_{(\frac{\alpha+1}{2})(\frac{3\alpha+1}{2})}) = \frac{\alpha+3}{2}$.

This implies that the diameter is dia$(G^*) = \alpha$+1.

Thus, the membership value of a point set is defined by, $\sigma_e(p_i) = \frac{ecc(p_i)}{dia(G^*)} = \frac{ecc(p_i)}{\alpha+1}$;$\forall p_i \in P(S_{P_2}(L_\alpha))$

The order of an eccentric fuzzy ladder graph by subdividing the lines between left and right path of the ladder graph is,

$O(EF(S_{P_2}(L_\alpha))) = 6\left[\frac{\alpha-1}{\alpha+1} + \frac{\alpha-2}{\alpha+1} + \cdots + \frac{\alpha-(\frac{\alpha-5}{2})}{\alpha+1}\right] + 6\left[\frac{\alpha+1}{\alpha+1}\right] + 6\left[\frac{\alpha}{\alpha+1}\right] + 3\left[\frac{(\frac{\alpha+3}{2})}{\alpha+1}\right]$

$= \frac{6}{\alpha+1}\left[\alpha\left(\frac{\alpha-5}{2}\right) - \left(1+2+3+\cdots+\frac{\alpha-5}{2}\right)\right] + 6\left[\frac{\alpha+1}{\alpha+1}\right] + 6\left[\frac{\alpha}{\alpha+1}\right] + 3\left[\frac{(\frac{\alpha+3}{2})}{\alpha+1}\right]$

$O(EF(S_{P_2}(L_\alpha))) = \frac{\left[\frac{9\alpha^2+18\alpha+15}{4}\right]}{\alpha+1}$.

The recursion formula on the order of an eccentric fuzzy ladder graph by subdividing the lines between left and right path of the ladder graph is,

$O(EF(S_{P_2}(L_{\alpha+2}))) = \frac{Nr(O(EF(S_{P_2}(L_\alpha)))}{\alpha+3} + \frac{6(\alpha+2)+6(\alpha+3)-3(\frac{\alpha+5}{2})-3(\frac{\alpha+3}{2})}{\alpha+3}$

$O(EF(S_{P_2}(L_{\alpha+2}))) = \frac{Nr(O(EF(S_{P_2}(L_\alpha)))}{\alpha+3} + \frac{9\alpha+18}{\alpha+3}$.

Case(ii): If α is even.

The eccentricity of the points is given by,

$ecc(p_1) = ecc(p_\alpha) = ecc(p_{\alpha+1}) = ecc(p_{2\alpha}) = ecc\left(p_{(1)(2\alpha)}\right) = ecc\left(p_{(\alpha)(\alpha+1)}\right) = \alpha+1;$

$$ecc(p_2) = ecc\left(p_{(2\alpha-1)}\right) = ecc\left(p_{(2)(2\alpha-1)}\right) = ecc(p_{\alpha-1}) = ecc(p_{\alpha+2}) = ecc\left(p_{(\alpha-1)(\alpha+2)}\right) = \alpha; ..;$$

$ecc(p_{\frac{\alpha}{2}}) = ecc(p_{\frac{\alpha+2}{2}}) = ecc(p_{(\frac{\alpha}{2})(\frac{3\alpha+2}{2})}) = ecc(p_{(\frac{3\alpha+2}{2})}) = ecc(p_{(\frac{\alpha+2}{2})(\frac{3\alpha+4}{2})}) = \frac{\alpha+4}{2}$.

This implies that the diameter is dia$(G^*) = \alpha+1$.

Thus, the membership value of a point set is defined by, $\sigma_e(p_i) = \frac{ecc(p_i)}{dia(G^*)} = \frac{ecc(p_i)}{\alpha+1}; \forall p_i \in P(S_{P_2}(L_\alpha))$.

The order of an eccentric fuzzy ladder graph by subdividing the lines between left and right path of the ladder graph is,

$$O(EF(S_{P_2}(L_\alpha))) = 6\left[\frac{\alpha-1}{\alpha+1} + \frac{\alpha-2}{\alpha+1} + \cdots + \frac{\alpha-\left(\frac{\alpha-4}{2}\right)}{\alpha+1}\right] + 6\left[\frac{\alpha+1}{\alpha+1}\right] + 6\left[\frac{\alpha}{\alpha+1}\right]$$
$$= \frac{6}{\alpha+1}\left[\alpha\left(\frac{\alpha-4}{2}\right) - \left(1+2+3+\cdots+\frac{\alpha-4}{2}\right)\right] + 6\left[\frac{\alpha+1}{\alpha+1}\right] + 6\left[\frac{\alpha}{\alpha+1}\right]$$
$$O(EF(S_{P_2}(L_\alpha))) = \frac{\left(\frac{9\alpha^2+18\alpha}{4}\right]}{\alpha+1}.$$

The recursion formula on order of eccentric fuzzy ladder graph by subdividing the lines between left and right path of the ladder graph is,

$$O(EF(S_{P_2}(L_{\alpha+2}))) = \frac{Nr(O(EF(S_{P_2}(L_\alpha)))}{\alpha+3} + \frac{6(\alpha+3)+6(\alpha+2)-6\left(\frac{\alpha+4}{2}\right)}{\alpha+3}$$
$$O(EF(S_{P_2}(L_{\alpha+2}))) = \frac{Nr(O(EF(S_{P_2}(L_\alpha)))}{\alpha+3} + \frac{9\alpha+18}{\alpha+3}.$$

Theorem 2.5 The order of a ladder graph by subdividing the lines on both paths and between left and right path in an EF(L_α) under the crisp graph $L_\alpha : (P, L)$, whose order is $5\alpha - 2$, where $\alpha \geq 3$ is given by,

$$O(EF(S(L_\alpha))) = \begin{cases} \frac{\frac{15\alpha^2+2\alpha-5}{2}}{2\alpha}; if\ \alpha\ is\ odd \\ \frac{\frac{15\alpha^2+2\alpha-4}{2}}{2\alpha}; if\ \alpha\ is\ even \end{cases} \quad and O(EF(S(L_{\alpha+2}))) = \frac{Nr(O(EF(S(L_\alpha))))}{2\alpha+4} + \frac{30\alpha+32}{2\alpha+4}$$

Proof. Consider an eccentric fuzzy ladder graph $EF(L_\alpha)$ with a crisp graph L_α consisting of two paths with point set denotes $\{p_1, p_2, \ldots, p_\alpha\}$ left path and $\{p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{2\alpha}\}$ denotes right path. Assume the subdivision of the line is constructed as follows:

(i) Join the points p_i to $p_{\alpha+i}$ where $1 \leq i \leq \alpha$.

(ii) The newly added point on subdivision of lines on the left path, the right path, and lines between the left and right paths are represented by $p_{(i)(i+1)}, p_{(\alpha+i)(\alpha+(i+1))}$, where $1 \leq i \leq \alpha-1$ and $p_{(i)(\alpha+i)}$ where $1 \leq i \leq \alpha$. Now, the new point established after subdividing the lines on the left path, the right path, and the lines connecting the left and right paths is,

$$\{p_1, p_{(1)(2)}, p_2, p_{(2)(3)}, p_3, \cdots, p_{\alpha-1}, p_{(\alpha-1)(\alpha)}, p_\alpha, p_{\alpha+1}, p_{(\alpha+1)(\alpha+2)}, p_{\alpha+2}, p_{(\alpha+2)(\alpha+3)}, \cdots, p_{2\alpha-1}, p_{(2\alpha-1)(2\alpha)},$$

$p_{\alpha+\alpha}, p_{(1)(\alpha+1)}, p_{(2)(\alpha+2)}, , p_{(\alpha)(2\alpha)}\}$.

Case(i): If α is odd.

The eccentricity of the points is given by,

$ecc(p_1) = ecc(p_{\alpha+\alpha}) = ecc\left(p_{(1)(\alpha+\alpha)}\right) = ecc(p_\alpha) = ecc(p_{\alpha+1}) = ecc\left(p_{(\alpha)(\alpha+1)}\right) = 2\alpha;$

$ecc\left(p_{(1)(2)}\right) = ecc\left(p_{(\alpha-1)(\alpha)}\right) = ecc\left(p_{(\alpha+1)(\alpha+2)}\right) = ecc\left(p_{(\alpha+(\alpha-1))(\alpha+\alpha)}\right) = 2\alpha - 1;$

$ecc(p_2) = ecc\left(p_{\alpha+(\alpha-1)}\right) = ecc\left(p_{2(\alpha+(\alpha-1))}\right) = ecc(p_{\alpha-1}) = ecc(p_{\alpha+2}) = ecc\left(p_{(\alpha-1)(\alpha+2)}\right) = 2\alpha - 2; \ldots;$

$ecc\left(p_{\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+3}{2}\right)}\right) = ecc\left(p_{\left(\frac{\alpha+3}{2}\right)\left(\frac{\alpha+5}{2}\right)}\right) = ecc\left(p_{\left(\frac{3\alpha+1}{2}\right)\left(\frac{3\alpha+3}{2}\right)}\right) = ecc\left(p_{\left(\frac{3\alpha+3}{2}\right)\left(\frac{3\alpha+5}{2}\right)}\right) = \alpha + 2;$

$ecc(p_{\frac{\alpha+1}{2}}) = ecc(p_{\left(\frac{3\alpha+1}{2}\right)}) = ecc(p_{\left(\frac{\alpha+1}{2}\right)\left(\frac{3\alpha+1}{2}\right)}) = \alpha + 1.$

This implies that the diameter is dia$(G^*) = 2\alpha$.

Thus, the membership value of a point set is defined by, $\sigma_e(p_i) = \frac{ecc(p_i)}{dia(G^*)} = \frac{ecc(p_i)}{2\alpha}; \forall p_i \in P(S(L_\alpha))$.

The order of an eccentric fuzzy ladder graph by subdividing the lines on right path, left path and between left and right path of the ladder graph is,

$$O(EF(S(L_\alpha))) = 6\left[\frac{\alpha+3}{2\alpha} + \frac{\alpha+5}{2\alpha} + \cdots + \frac{(\alpha+\alpha)}{2\alpha}\right] + 4\left[\frac{\alpha+2}{2\alpha} + \frac{\alpha+4}{2\alpha} + \cdots + \frac{(\alpha+(\alpha-1))}{2\alpha}\right] + 3\left[\frac{\alpha+1}{2\alpha}\right]$$
$$= \frac{6}{2\alpha}\left[\alpha\left(\frac{\alpha-1}{2}\right) + (3+5+7+\ldots+\alpha)\right] + \frac{4}{2\alpha}\left[\alpha\left(\frac{\alpha-1}{2}\right) + (2+4+6+\cdots+(\alpha-1))\right] + 3\left[\frac{\alpha+1}{2\alpha}\right]$$
$$= \frac{6}{2\alpha}\left[\alpha\left(\frac{\alpha-1}{2}\right) + \left(\frac{\alpha-1}{4}\right)\left(2(3) + \left(\frac{\alpha-1}{2} - 1\right)2\right)\right] + \frac{4}{2\alpha}\left[\alpha\left(\frac{\alpha-1}{2}\right) + \left(\frac{\alpha-1}{4}\right)\left(2(2) + \left(\frac{\alpha-1}{2} - 1\right)2\right)\right] + 3\left[\frac{\alpha+1}{2\alpha}\right]$$
$$O(EF(S(L_\alpha))) = \frac{\left[\frac{15\alpha^2+2\alpha-5}{2}\right]}{2\alpha}.$$

The recursion formula on the order of an eccentric fuzzy ladder graph by subdividing the lines on left path, right path and lines between left and right path of the ladder graph is,

$$O(EF(S(L_{\alpha+2}))) = \frac{Nr(O(EF(S(L_\alpha))))}{2\alpha+4} + \frac{4(2\alpha+1)+6(2\alpha+2)+4(2\alpha+3)+6(2\alpha+4)-3(\alpha+1)-4(\alpha+2)-3(\alpha+3)}{2\alpha+4}$$
$$O(EF(S(L_{\alpha+2}))) = \frac{Nr(O(EF(S(L_\alpha))))}{2\alpha+4} + \frac{30\alpha+32}{2\alpha+4}.$$

Case(ii): If α is even.

The eccentricity of the points is given by,

$ecc\,(p_1) = ecc\,(p_{\alpha+\alpha}) = ecc\left(p_{(1)(\alpha+\alpha)}\right) = ecc\,(p_\alpha) = ecc\,(p_{\alpha+1}) = ecc\left(p_{\alpha()(\alpha+1)}\right) = 2\alpha;$
$ecc\left(p_{(1)(2)}\right) = ecc\left(p_{(\alpha-1)(\alpha)}\right) = ecc\left(p_{(\alpha+1)(\alpha+2)}\right) = ecc\left(p_{(2\alpha-1)(2\alpha)}\right) = 2\alpha - 1;$

$$ecc\,(p_2) = ecc\left(p_{(2\alpha-1)}\right) = ecc\left(p_{(2)(2\alpha-1)}\right) = ecc\,(p_{\alpha-1}) = ecc\,(p_{\alpha+2}) = ecc\left(p_{(\alpha-1)(\alpha+2)}\right) = 2\alpha - 2;\ldots;$$

$$ecc\left(p_{\frac{\alpha}{2}}\right) = ecc\left(p_{\frac{\alpha+2}{2}}\right) = ecc\left(p_{\left(\frac{\alpha}{2}\right)\left(\frac{3\alpha+2}{2}\right)}\right) = ecc\left(p_{\left(\frac{3\alpha}{2}\right)}\right) = ecc\left(p_{\left(\frac{3\alpha+2}{2}\right)}\right) = ecc\left(p_{\left(\frac{\alpha+2}{2}\right)\left(\frac{3\alpha}{2}\right)}\right) = \alpha+2;$$

$ecc(p_{\left(\frac{\alpha}{2}\right)\left(\frac{\alpha+2}{2}\right)}) = ecc(p_{\left(\frac{3\alpha}{2}\right)\left(\frac{3\alpha+2}{2}\right)}) = \alpha+1.$

This implies that the diameter is $\text{dia}(G^*) = 2\alpha$.

Thus, the membership value of a point set is defined by, $\sigma_e(p_i) = \frac{ecc(p_i)}{dia(G^*)} = \frac{ecc(p_i)}{2\alpha}; \forall p_i \in P(S(L_{\alpha+2}))$.

The order of an eccentric fuzzy ladder graph by subdividing the lines on left path, right path and lines between the left and right path of the ladder graph is,

$O(EF(S(L_\alpha))) = 6\left[\frac{\alpha+2}{2\alpha} + \frac{\alpha+4}{2\alpha} + \cdots + \frac{\alpha+\alpha}{2\alpha}\right] + 4\left[\frac{\alpha+3}{2\alpha} + \frac{\alpha+5}{2\alpha} + \cdots + \frac{(\alpha+(\alpha-1))}{2\alpha}\right] + 2\left[\frac{\alpha+1}{2\alpha}\right]$
$= \frac{6}{2\alpha}\left[\alpha\left(\frac{\alpha}{2}\right) + (2+4+6+\cdots+\alpha)\right] + \frac{4}{2\alpha}\left[\alpha\left(\frac{\alpha-2}{2}\right) + (3+5+7+\cdots+(\alpha-1))\right] + 2\left[\frac{\alpha+1}{2\alpha}\right]$
$= \frac{6}{2\alpha}\left[\alpha\left(\frac{\alpha}{2}\right) + \left(\frac{\alpha}{4}\right)(2(2) + \left(\frac{\alpha-1}{2} - 1\right)2)\right] + \frac{4}{2\alpha}\left[\alpha\left(\frac{\alpha-2}{2}\right) + \left(\frac{\alpha-2}{4}\right)(2(3) + \left(\frac{\alpha-2}{2} - 1\right)2)\right] + 2\left[\frac{\alpha+1}{2\alpha}\right]$
$O(EF(S(L_\alpha))) = \frac{\left[\frac{15\alpha^2+2\alpha-4}{2}\right]}{2\alpha}$.

The recursion formula on the order of an eccentric fuzzy ladder graph by subdividing the lines on left path, right path and between left and right path of the ladder graph is,

$O(EF(S(L_{\alpha+2}))) = \frac{Nr(O(EF(S(L_\alpha))))}{2\alpha+4} + \frac{4(2\alpha+1)+6(2\alpha+2)+4(2\alpha+3)+6(2\alpha+4)-2(\alpha+1)-6(\alpha+2)-2(\alpha+3)}{2\alpha+4}$
$O(EF(S(L_{\alpha+2}))) = \frac{Nr(O(EF(S(L_\alpha))))}{2\alpha+4} + \frac{30\alpha+32}{2\alpha+4}$.

Illustration 2.1 Consider a ladder graph L_{11} with point set $\{p_1, p_2, \ldots, p_{11}, p_{12}, p_{13}, \ldots, p_{22}\}$. Assume that, if the lines are subdivided once, then the new point set for α=11 is,

$\{p_1, p_{(1)(2)}, p_2, p_{(2)(3)}, p_3, \ldots, p_6, p_{(6)(7)}, p_7, p_8, p_{(8)(9)}, p_9, \ldots, p_{13}, p_{(13)(14)}, p_{14}, p_{(1)(22)}, p_{(2)(21)}, \ldots, p_{(11)(11)}\}$.

From Theorem 2.5, the eccentricity and membership value of points on subdivision of ladder graph were defined and thus the diameter is 2α. If α=11 which is odd, then the order of an eccentric fuzzy ladder graph by subdividing the lines on left path, right path and lines between the left and right path of the ladder graph is,

$O(EF(S(L_{11}))) = 6\left[\frac{11+3}{22} + \frac{11+5}{22} + \cdots + \frac{(11+11)}{22}\right] + 4\left[\frac{11+2}{22} + \frac{11+4}{22} + \cdots + \frac{(11+(11-1))}{22}\right] + 3\left[\frac{11+1}{22}\right] =$
$\frac{6}{22}\left[\alpha\left(\frac{11-1}{2}\right) + 3+5+7+\cdots+11\right] + \frac{4}{22}\left[\alpha\left(\frac{11-1}{2}\right) + (2+4+6+\cdots+(11-1))\right] + 3\left[\frac{11+1}{22}\right]$
$O(EF(S(L_{11}))) = \frac{\left[\frac{15(11)^2+2(11)-5}{2}\right]}{22} = \frac{916}{22}$.

The recursion formula on the order of an eccentric fuzzy ladder graph by subdividing the lines on left path, right path and lines between left and right path of the ladder graph is,

$O(EF(S(L_{11+2}))) = \frac{Nr(O(EF(S(L_{11}))))}{2(11)+4} + \frac{4(2(11)+1)+6(2(11)+2)+4(2(11)+3)+6(2(11)+4)-3(11+1)-4(11+2)-3(11+3)}{2(11)+4}$
$O(EF(S(L_{13}))) = \frac{Nr(O(EF(S(L_{11}))))}{26} + \frac{30(11)+32}{26} = \frac{1278}{26}$.

If α=12 which is even, then the order of an eccentric fuzzy ladder graph by subdividing the lines on left path, right path and lines between the left and right path of the ladder graph is,

$O(EF(S(L_{12}))) = 6\left[\frac{12+2}{24} + \frac{12+4}{24} + \cdots + \frac{12+12}{24}\right] + 4\left[\frac{12+3}{24} + \frac{12+5}{24} + \cdots + \frac{(12+(12-1))}{24}\right] + 2\left[\frac{12+1}{24}\right]$
$= \frac{6}{24}\left[12\left(\frac{12}{2}\right) + (2+4+6+\cdots+12)\right] + \frac{4}{24}\left[12\left(\frac{12-2}{2}\right) + (3+5+7+\cdots+(12-1))\right] + 2\left[\frac{12+1}{24}\right]$
$= \frac{\left[\frac{15(12)^2+2(12)-4}{2}\right]}{24} = \frac{684+380+26}{24}$
$O(EF(S(L_{12}))) = \frac{1090}{24}$.

The recursion formula on the order of an eccentric fuzzy ladder graph by subdividing the lines on left path, right path and lines between left and right path of the ladder graph is,

$O(EF(S(L_{12+2}))) = \frac{Nr(O(EF(S(L_{12}))))}{2(12)+4} + \frac{4(2(12)+1)+6(2(12)+2)+4(2(12)+3)+6(2(12)+4)-3(12+1)-4(12+2)-3(12+3)}{2(12)+4}$
$O(EF(S(L_{14}))) = \frac{Nr(O(EF(S(L_{12}))))}{2(12)+4} + \frac{30(12)+32}{2(12)+3} = \frac{1482}{28}$.

Theorem 2.6 The order of an eccentric fuzzy wheel graph EF(W_α), under the crisp graph W_α whose order is α+1, where $\alpha \geq 4$ is given by, $O(EF(W_\alpha)) = \frac{2\alpha+1}{2}$ *and* $O(EF(W_{\alpha+2})) = O(EF(W_\alpha)) + 1$

Proof . Let $EF(W_\alpha)$ be an eccentric wheel graph with points $\{p_1, p_2, \ldots, p_\alpha\}$ and central point $p_{\alpha+1}$. The eccentricity of the points are given by,

$ecc(p_1) = ecc(p_2) = \cdots = ecc(p_\alpha) = 2; ecc(p_{\alpha+1}) = 1$.

The diameter of wheel graph is 2.

Thus, the membership value of point are, $\sigma_e(p_1) = \sigma_e(p_2) = \cdots = \sigma_e(p_\alpha) = \frac{2}{2}; \sigma_e(p_{\alpha+1}) = \frac{1}{2}$.

The order of a wheel graph is,

$O(EF(W_\alpha)) = \left(\frac{1}{2}\right) + (\alpha)\left(\frac{2}{2}\right)$

$O(EF(W_\alpha)) = \frac{2\alpha+1}{2}$.

The recursion formula on order of the eccentric fuzzy wheel graph is,

$O(EF(W_{\alpha+1})) = O(EF(W_\alpha)) + \left(\frac{2}{2}\right)$

$O(EF(W_{\alpha+1})) = O(EF(W_\alpha)) + 1$.

Remark 2.2 Order of eccentric fuzzy wheel graph of α=3 is 4 i.e., $O(EF(W_3)) = 4$.

Theorem 2.7 The order of a wheel graph by subdividing the lines on rim in EF(W_α) under the crisp graph W_α whose order is $2\alpha+1$, where $\alpha \geq 4$ is given by,

$O(EF(S_r(W_\alpha))) = \frac{7\alpha+2}{4}$ *and* $O(EF(S_r(W_{\alpha+1}))) = O(EF(S_r(W_\alpha))) + \left(\frac{7}{4}\right)$

Proof . Let $EF(W_\alpha)$ be an eccentric wheel graph with points $\{p_1, p_2, \ldots, p_\alpha\}$ and central point $p_{\alpha+1}$. Assume the subdivision of line is constructed as follows,

(i) Connect the cycle p_1 to p_α and $p_{\alpha+1}$ to p_i where $1 \leq i \leq \alpha$.

(ii) The newly added point on subdivision on rim is represent by $p_{(i)(i+1)}$ where $1 \leq i \leq \alpha - 1$.

The eccentricity of the points is given by,

$$ecc(p_1) = ecc(p_2) = \cdots = ecc(p_\alpha) = 3;$$

$ecc(p_{(1)(2)} = ecc(p_{(2)(3)}) = \cdots = ecc(p_{(\alpha-1)(\alpha)}) = ecc(p_{(\alpha)(1)}) = 4; ecc(p_{\alpha+1}) = 2$.

The diameter of wheel graph is 4. Now the new points on subdivision of rim on wheel graph is $\{p_1, p_{(1)(2)}, p_2, p_{(2)(3)}, p_3, \ldots, p_{(\alpha-2)(\alpha-1)}, p_{\alpha-1}, p_{(\alpha-1)(1)}, p_\alpha\}$.

Thus, the membership value of point are, $\sigma_e(p_i) = \frac{ecc(p_i)}{dia(G^*)} = \frac{ecc(p_i)}{4}; \forall p_i \in P(S_r(W_\alpha))$.

$$\sigma_e(p_1) = \sigma_e(p_2) = \cdots = \sigma_e(p_\alpha) = \frac{3}{4};$$

$\sigma_e(p_{(1)(2)} = \sigma_e(p_{(2)(3)}) = \cdots = \sigma_e(p_{(\alpha-1)(\alpha)}) = \sigma_e(p_{(\alpha)(1)}) = \frac{4}{4}; \sigma_e(p_{\alpha+1}) = \frac{2}{4}$.

The order of wheel graph on subdivision of rim is,

$O(EF_r(W_\alpha)) = \left(\frac{2}{4}\right) + \alpha\left(\frac{3}{4}\right) + \alpha\left(\frac{4}{4}\right)$

$O(EF_r(W_\alpha)) = \frac{7\alpha+2}{4}$.

The recursion formula on subdividing rim on an eccentric fuzzy wheel graph is,

$O(EF_r(W_{\alpha+1})) = O(EF_r(W_\alpha)) + \left(\frac{3}{4}\right) + \left(\frac{4}{4}\right)$

$O(EF_r(W_{\alpha+1})) = O(EF_r(W_\alpha)) + \left(\frac{7}{4}\right)$.

Remark 2.2 Order of eccentric fuzzy wheel graph on subdivision of rim of α=3 is $\frac{20}{3}$ i.e., $O(EF_r(W_3)) = \frac{20}{3}$.

Theorem 2.8 The order of a wheel graph by subdividing the lines on rim and spoke in an EF(W_α) under the crisp graph W_α whose order is $3\alpha+1$, where $\alpha \geq 5$ is given by,

$O(EF(S_{rs}(W_\alpha))) = \frac{15\alpha+3}{6}$ *and* $O(EF(S_{rs}(W_{\alpha+2}))) = O(EF(S_{rs}(L_\alpha))) + \left(\frac{15}{6}\right)$

Proof. Let $EF(W_\alpha)$ be an eccentric wheel graph with points $\{p_1, p_2, \ldots, p_\alpha\}$ and central point $p_{\alpha+1}$. Assume the subdivision of line is constructed as follows,

(i). Connect the cycle p_1 to p_α and $p_{\alpha+1}$ to p_i where $1 \leq i \leq \alpha$.

(ii). The newly added point on subdivision on rim and spoke is represent by $p_{(i)(i+1)}$ where $1 \leq i \leq \alpha - 1$ and $p_{(i)(\alpha+1)}$ where $1 \leq i \leq \alpha$. Now the new points on subdivision of rim and spoke on wheel graph is $\{p_1, p_{(1)(2)}, p_2, p_{(2)(3)}, p_3, \ldots, p_{(\alpha-2)(\alpha-1)}, p_{\alpha-1}, p_{(\alpha-1)(1)}, p_\alpha, p_{(1)(\alpha)}, p_{(2)(\alpha)}, \ldots, p_{(\alpha-1)(\alpha)}\}$.

The eccentricity of the points is given by,

$$ecc(p_1) = ecc(p_2) = \cdots = ecc(p_\alpha) = 5;$$

$$ecc(p_{(1)(2)}) = ecc(p_{(2)(3)}) = \cdots = ecc(p_{(\alpha-1)(\alpha)}) = ecc(p_{(\alpha)(1)}) = 6;$$

$ecc\left(p_{(1)(\alpha+1)}\right) = ecc\left(p_{(2)(\alpha+1)}\right) = \cdots = ecc(p_{(\alpha)(\alpha+1)}) = 4; ecc(p_{\alpha+1}) = 3.$

The diameter of wheel graph is 6. Thus, the membership value of point are,

$\sigma_e(p_i) = \frac{ecc(p_i)}{dia(G^*)} = \frac{ecc(p_i)}{6}; \forall p_i \in P(S_{rs}(W_\alpha))$

$$\sigma_e(p_1) = \sigma_e(p_2) = \cdots = \sigma_e(p_\alpha) = \frac{5}{6};\ \sigma_e(p_{(1)(2)} = \sigma_e(p_{(2)(3)}) = \cdots = \sigma_e(p_{(\alpha-1)(\alpha)}) = \sigma_e(p_{(\alpha)(1)}) = \frac{6}{6};$$

$\sigma_e\left(p_{(1)(\alpha)}\right) = \sigma_e\left(p_{(2)(\alpha)}\right) = \cdots = \sigma_e(p_{(\alpha)(\alpha+1)}) = \frac{4}{6}; \sigma_e(p_{\alpha+1}) = \frac{3}{6}.$

The order of wheel graph on subdivision of rim and spoke is,

$O\left(EF\left(S_{rs}\left(W_\alpha\right)\right)\right) = \left(\frac{3}{6}\right) + \alpha\left(\frac{4}{6}\right) + \alpha\left(\frac{5}{6}\right) + \alpha\left(\frac{6}{6}\right)$

$O(EF(S_{rs}(W_\alpha))) = \frac{15\alpha+3}{6}.$

The recursion formula on subdividing rim and spoke on eccentric fuzzy ladder graph,

$O(EF(S_{rs}(W_{\alpha+1}))) = O(EF(S_{rs}(W_\alpha))) + \left(\frac{4}{6}\right) + \left(\frac{5}{6}\right) + \left(\frac{6}{6}\right)$

$O(EF(S_{rs}(W_{\alpha+1}))) = O(EF(S_{rs}(W_\alpha))) + \left(\frac{15}{6}\right).$

Remark 2.3

1. Order of eccentric fuzzy wheel graph on subdivision of rim and spoke of α=3 is $\frac{33}{4}$ i.e., $O\left(EF_{rs}\left(W_3\right)\right) = \frac{33}{4}$.
2. Order of eccentric fuzzy wheel graph on subdivision of rim and spoke of α=4 is $\frac{55}{5}$ i.e., $O\left(EF_{rs}\left(W_4\right)\right) = \frac{55}{5}$.

Illustration 2.2

Let us consider a wheel graph W_9 by subdividing every line, then new points are

$\{p_1, p_{(1)(2)}, p_2, p_{(2)(3)}, p_3, p_{(3)(4)},\ p_4, p_{(8)(9)}, p_9, p_{(9)(1)}, \cdots, p_{(1)(10)}, p_{(2)(10)}, \cdots,\ p_{(9)(10),}\}$

From Theorem 2.8, the eccentricity and membership value of points on subdivision of wheel graph were defined and thus the diameter is 6. The order of wheel graph on subdivision of rim and spoke is,

$O\left(EF\left(S_{rs}\left(W_9\right)\right)\right) = \left(\frac{3}{6}\right) + 9\left(\frac{4}{6}\right) + 9\left(\frac{5}{6}\right) + 9\left(\frac{6}{6}\right)$

$O(EF(S_{rs}(W_9))) = \frac{15(9)+3}{6} = \frac{138}{6}.$

The recursion formula on subdividing rim and spoke of an eccentric fuzzy wheel graph,

$O(EF(S_{rs}(W_{9+1}))) = O(EF(S_{rs}(W_9))) + \left(\frac{4}{6}\right) + \left(\frac{5}{6}\right) + \left(\frac{6}{6}\right)$

$O(EF(S_{rs}(W_{10}))) = O(EF(S_{rs}(W_9))) + \left(\frac{15}{6}\right) = \frac{153}{6}.$

3 Conclusion

A special fuzzy graph known as an eccentric fuzzy graph is considered in this with a defined membership function of point set and line set and with the eccentricity and diameter of a crisp graph. Further, the recursion formula on order by subdividing lines of left path, right path and on both on the ladder graph and rim, spoke and both on wheel graph were found with membership function of eccentric fuzzy graph.

4 Declaration

Presented in 'International Conference on Applied Mathematical Sciences' (ICAMS 2022) during 21 & 22 Dec. 2022, organized virtually by the Department of Mathematics, JJ College of Arts and Science, Pudukkottai, Tamil Nadu, India. The Organizers claim the peer review responsibility.

References

1) Meenal N. A First Look at Eccentric Fuzzy Graphs. *International Journal of Research and Analytical Reviews (IJRAR)*. 2019;6(1):188–194. Available from: https://ijrar.org/papers/IJRAR19J1029.pdf.
2) Meenal N, Jovita JJ. On Regular Inverse Eccentric Fuzzy Graph. *Applications and Applied Mathematics*. 2022;17(3):119–128. Available from: https://digitalcommons.pvamu.edu/aam/vol17/iss3/10/.
3) Meenal N, Jovita JJ. A first look at inverse eccentric fuzzy graph. In: INTERNATIONAL CONFERENCE ON RECENT ADVANCES IN MATHEMATICS AND COMPUTATIONAL ENGINEERING: ICRAMCE 2022, 6–7 January 2022, Chennai, India;vol. 2829, Issue 1 of AIP Conference Proceedings. AIP Publishing. 2023. Available from: https://doi.org/10.1063/5.0156803.