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RESEARCH ARTICLE

Exponential Diophantine Equation $\left(n^2-1\right)^u+n^{2v}=w^2, n=2,3,4,5$

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Abstract

Objectives: Diophantine research focuses on various ways to tackle multivariable and multidegree Diophantine problems. A Diophantine equation is a polynomial equation with only integer solutions. The objective of this manuscript is to find the solutions to a few exponential Diophantine equations $\eta_1:\left(2^2-1\right)^u+\left(2^2\right)^v=w^2, \eta_2:\left(3^2-1\right)^u+\left(3^2\right)^v=w^2, \eta_3:\left(4^2-1\right)^u+\left(4^2\right)^v=w^2$ and $\eta_4:\left(5^2-1\right)^u+\left(5^2\right)^v=w^2$. Also generalize the Exponential equation η_1, η_2, η_3, and η_4 of the form $\left(n^2-1\right)^u+n^{2v}=w^2, n=2,3,4,5$ and explore that it has at least one solution as $(1,0,n)$. **Methods:** Diophantine equations may have finite, infinite or no solutions in integers. There is no universal method for finding solutions to Diophantine equations. The particular type of Exponential Diophantine equation is analysed and generalised by the method of Catalan's conjecture. **Findings:** Exponential Diophantine equations$\eta_1:\left(2^2-1\right)^u+\left(2^2\right)^v=w^2, \eta_2:\left(3^2-1\right)^u+\left(3^2\right)^v=w^2, \eta_3:\left(4^2-1\right)^u+\left(4^2\right)^v=w^2$ and $\eta_4:\left(5^2-1\right)^u+\left(5^2\right)^v=w^2$ has only a finite number of solutions in N_0 (Whole numbers). The solution sets (u,v,w) of η_1, η_2, η_3, and η_4 are, $\{(1,0,2),(2,2,5)\},(1,0,3),(1,0,4),\{(1,0,5),(1,1,7)\}$ respectively. **Novelty:** In this analysis, the particular type of Exponential Diophantine equation is analysed using elementary mathematics concepts instead of higher mathematics also generalize the Exponential equation η_1, η_2, η_3, and η_4 of the form $\left(n^2-1\right)^u+n^{2v}=w^2, n=2,3,4,5$ and explore that it has at least one solution as $(1,0,n)$.

2020 Mathematical Subject Classification: 11D61.

Keywords: Catalan's conjecture; Diophantine equation; Exponential Diophantine equation; Integral solutions; Non-negative integer solution

1 Introduction

Number theory is a pure mathematical area devoted to the development of numbers. Several mathematicians[1,2] have studied various variants of Diophantine equations throughout the last few decades. If variables appear as exponents in a Diophantine equation, it is an exponential Diophantine equation.

For example, the Ramanujan - Nagell equation $2^x-7=x^2$ and the Fermat - Catalan Conjecture equation $a^m+b^n=c^k$.

Janaki G and Gowri Shankari A [3] demonstrated in 2023 that Exponential Diophantine Equation $2^a + n^{2b} = c^2, n = 1,2,3,\ldots$ where a,b, and c are all positive integers has $(a,b,c) = (3,0,3)$ is a unique nonnegative integer solution. G. Janaki and C. Saranya [4] established that positive integer solutions exist to the exponential problem using Jarasandha numbers employing the Catalan conjecture. Several exponential Diophantine equations are solved by many authors in [5–12].

In this paper, the particular type of Exponential Diophantine equation is analysed using elementary mathematics concepts instead of higher mathematics also generalize the Exponential equation η_1, η_2, η_3, and η_4 of the form $\left(n^2-1\right)^u + n^{2v} = w^2, n = 2,3,4,5$ and explore that it has at least one solution is $(1,0,n)$.

2 Methodology

In this part, we establish the four theorems using the factorizable approach and Catalan's conjecture.

Preposition 2.1 (The Catalan's conjecture) $(a,b,x,y) = (3,2,2,3)$ is the only solution for the Diophantine equation $a^x - b^y = 1$, where a,b,x and y are integers with $min(a,b,x,y) \geq 2$.

3 Results and Discussion

Theorem: 3.1

The non-negative integer solution to the Diophantine equation $\eta_1 : \left(2^2-1\right)^u + \left(2^2\right)^v = w^2$ is $\{(1,0,2),(2,2,5)\}$

Proof: Let u, v and w be non-negative integers such that $\eta_1 : 3^u + 4^v = w^2$

Case: 1

If $u = 0$ then η_1 becomes $1 + 2^{2v} = w^2$. Then

$$w^2 - 1 = 2^{2v} = 2^{2v-a}2^a \Rightarrow (w+1) = 2^{2v-a}, (w-1) = 2^a \tag{1}$$

From (Equation (1)),

$$2 = 2^a\left(2^{2v-2a} - 1\right) \tag{2}$$

Here $a = 1$ is the only possible value.

Then from (Equation (2)), $v = \frac{3}{2}$ (not possible).

Therefore, no solution occurs in this case.

Case: 2

If $v = 0$ then η_1 becomes

$$3^u + 1 = w^2 \tag{3}$$

Then

$$w^2 - 1 = 3^u = 3^{u-a}3^a \Rightarrow (w+1) = 3^{u-a}, (w-1) = 3^a \tag{4}$$

From (Equation (4))

$$2 = 3^a\left(3^{u-2a} - 1\right) \tag{5}$$

Here $a = 0$ is the only possible value.

Then from (Equation (5)), $u = 1$.

Apply u in (Equation (3)), one may get $w = 2$.

Therefore, (1,2) is the solution of the equation $3^u + 1 = w^2$.

Case: 3

Assume u, v, and w are all non-negative integers such that

$$\eta_1 : 3^u + 4^v = w^2 \tag{6}$$

In case $2, u \geq 1$.

Then $3^u + 2^{2v} = w^2$ as

$$3^u + (2^v)^2 = w^2 \Rightarrow w^2 - (2^v)^2 = 3^u = 3^{u-a}3^a \tag{7}$$

From (Equation (7)),

$$\left(w+2^{v}\right)=3^{u-a},\left(w-2^{v}\right)=3^{a} \tag{8}$$

Using (Equation (8)) one may get,
$2\left(2^{v}\right)=3^{a}\left(3^{u-2a}-1\right) \Rightarrow 1.2^{v+1}=3^{a}\left(3^{u-2a}-1\right) \Rightarrow a=0$ then

$$2^{v+1}=3^{u}-1 \tag{9}$$

Then, the possible solutions of (Equation (9)) is $\{(1,0,2),(2,2,5)\}$

Hence, the Diophantine equation $\eta_1 : 3^u+4^v=w^2$ has two solutions as $\{(1,0,2),(2,2,5)\}$.

Theorem: 3.2

The non-negative integer solution to the Diophantine equation $\eta_2 : \left(3^2-1\right)^u+\left(3^2\right)^v=w^2$ is $(1,0,3)$.

Proof: Let u,v and w be non-negative integers such that $\eta_2 : 8^u+9^v=w^2$

Case: 1

If $u=0$ then η_2 becomes $1+3^{2v}=w^2$.

Then

$$w^2-1=3^{2v}=3^{2v-a}3^a \Rightarrow (w+1)=3^{2v-a},(w-1)=3^a \tag{10}$$

From (Equation (10)),

$$2=3^a\left(3^{2v-2a}-1\right) \tag{11}$$

Here $a=0$ is the only possible value.

Then from (Equation (11)), $v=\frac{1}{2}$ (not possible).

Therefore, no solution occurs in this case.

Case: 2

If $v=0$ then η_2 becomes

$$8^u+1=w^2 \tag{12}$$

Then

$$w^2-1=8^u=8^{u-a}8^a \Rightarrow (w+1)=8^{u-a},(w-1)=8^a \tag{13}$$

From (Equation (13)),

$$2=8^a\left(8^{u-2a}-1\right) \tag{14}$$

From (Equation (14)), one may get either $a=0$ or $a=\frac{1}{3}$ (not possible)

The only possible is $a=0$ then one may get $8^u=2$ which is not possible.

Therefore, no solution occurs in this case, too.

Case: 3

Assume u,v, and w are all non-negative integers such that

$$\eta_2 : 8^u+9^v=w^2 \tag{15}$$

In case $2, u \geq 1$.

Then (Equation (15)) becomes

$$8^u+\left(3^v\right)^2=w^2 \Rightarrow w^2-\left(3^v\right)^2=8^u=2^{3u}=2^{3u-a}2^a \tag{16}$$

From (Equation (16)),

$$\left(w+3^{v}\right)=2^{3u-a},\left(w-3^{v}\right)=2^{a} \tag{17}$$

Using (Equation (17)) one may get,

$$2\left(3^{v}\right)=2^{a}\left(2^{3u-2a}-1\right) \tag{18}$$

From (Equation (18)), the possible value of $a=1$.

Then (Equation (18)) becomes

$$3^{v}=\left(2^{3u-2a}-1\right) \tag{19}$$

Put $u=1$ then one may get $v=0$.

Now by (Equation (15)), $w=3$

Hence, the Diophantine equation $\eta_2: 8^u+9^v=w^2$ has a single solution as $(1,0,3)$.

Theorem: 3.3

The non-negative integral solution to the Diophantine equation $\eta_3: \left(4^2-1\right)^u+\left(4^2\right)^v=w^2$ is $(1,0,4)$.

Proof:

As discussed in Theorem 3.1 and 3.2, one may get a single solution $(1,0,4)$ for the Diophantine equation $\eta_3: 15^u+16^v=w^2$.

Theorem: 3.4

The non-negative integral solution to the Diophantine equation $\eta_4: \left(5^2-1\right)^u+\left(5^2\right)^v=w^2$ is $\{(1,0,5),(1,1,7)\}$.

Proof:

As discussed in Theorem 3.1 and 3.2, one may get a two solution $\{(1,0,5),(1,1,7)\}$ for the Diophantine equation $\eta_4: 24^u+25^v=w^2$.

4 Remarkable Observation

From Theorem 3.1, 3.2, 3.3, and 3.4, one may explore that the Exponential equation of the form $\left(n^2-1\right)^u+n^{2v}=w^2, n=2,3,4,5$ has at least one solution as $(1,0,n)$.

5 Conclusion

In this work, a few exponential Diophantine equations

$\eta_1: \left(2^2-1\right)^u+\left(2^2\right)^v=w^2, \eta_2: \left(3^2-1\right)^u+\left(3^2\right)^v=w^2, \eta_3: \left(4^2-1\right)^u+\left(4^2\right)^v=w^2$, and

$\eta_4: \left(5^2-1\right)^u+\left(5^2\right)^v=w^2$ has only a finite number of solutions in N_0 is explored. The solution sets (u,v,w) of η_1,η_2,η_3, and η_4 are, $\{(1,0,2),(2,2,5)\},(1,0,3),(1,0,4),\{(1,0,5),(1,1,7)\}$ respectively.

Also, we generalize the Exponential equation of the form $\left(n^2-1\right)^u+n^{2v}=w^2, n=2,3,4,5$ with at least one solution is $(1,0,n)$. One may also find integral solutions to other different exponential Diophantine equations.

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