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RESEARCH ARTICLE

Dynamics of Bianchi Type-III Universe with Time-Varying Deceleration Parameter in $f(G)$ Gravity

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Abstract

Objectives: To explore Bianchi type-III cosmological models in $f(G)$ gravity theory with time-varying deceleration parameters. **Methods:** In this paper, to get the exact solution of the field equation in the presence of perfect fluid, we consider: (i) the relation that shear scalar $(\sigma)$is proportional to the expansion scalar(θ), resulting in $C = B^n$. (ii) time-varying deceleration parameter, $q = -1 + \frac{1}{\sqrt{t}}$ suggested by Dewri[(1)], indicating an accelerated expansion of the universe. (iii) the power-law $f(G)$ model, which is compatible with the recent observational data. Also, along with the graphical representation, the geometrical and physical aspects are examined. **Findings:** In this study, the model transit from the early decelerating phase$(q > 0)$ to the present accelerating phase $(q < 0)$, and in support of that, violation of the strong energy condition with negative pressure is observed, ultimately indicating the presence of the dark energy. It is worthwhile to mention that the contribution of dark energy initially lies in the phantom domain$(\omega < -1)$. After a period of time, it extends to the quintessence domain$(\omega > -1)$ and later approaches to Λ cold dark matter (CDM) model $(\omega = -1)$ at a late time. **Novelty:** The universe's evolution has been explored in the framework of $f(G)$ gravity, identifying the cosmological parameters of the model obtained in this work. The negative pressure and the falling of strong energy conditions suggest its consistency with the recent cosmological observation of the dark energy universe, and the findings of this study may contribute to a deeper comprehension of the universe's accelerated expansion.

Keywords: f (G) gravity; Bianchi typeIII; Time-varying deceleration; Dark energy; Power-law

1 Introduction

The current understanding of the universe in modern cosmology is expanding and accelerating, making it a prominent topic nowadays. Numerous observational data point to this unknown energy, sometimes referred to as dark energy, being the source of unexpected changes in the cosmos with tremendous negative pressure. Thus, to comprehend the process underlying the existence of dark energy and late- time acceleration in the universe, scientists have developed several modified theories of gravity, including $f(R)$ gravity, $f(R,T)$ gravity, $f(G)$ gravity, $f(R,\phi)$ gravity, $f(Q)$ gravity, and so on. These modified theories of gravity are well-developed through alternating scalar invariants and replacing the appropriate generic function of this scalar in the action function. The modified Gauss-Bonnet theory of gravity, also known as the $f(G)$ gravity theory [2], is one of Einstein's modified theories of gravity generated by incorporating the Gauss-Bonnet invariant in the Einstein Hilbert action in which $f(G)$ is a generic function of the Gauss-Bonnet (GB) term G. Compared to other modified theories of gravity, the $f(G)$ gravity theory for the specific choice of function f, produces the most realistic result of the late-time cosmic acceleration bypassing the solar system test. Also, it may allow us to examine the transition phases of the universe from early deceleration to late acceleration or from the non-phantom phase to the phantom phase. In $f(G)$ gravity, anisotropic cosmological models have been discussed by Shamir [3] [4], reported that the model starts as a point-type singularity, and the Equation of State (EoS) parameter has observed that the model has a phantom dark energy during the evolution process. This indicates that the model eventually ends up with a finite-time future singularity. Taser and Dogru [5] constructed a model of $f(G)$ gravity for locally rotationally symmetric (LRS) Bianchi type-I universe shows an expanding or collapsing universe with deceleration corresponding to multiple choice of arbitrary constant in the case of massless and massive scalar field. The role of holographic dark energy in $f(G)$ gravity is stable throughout the evolution of time with hybrid expansion-law for anisotropic LRS Bianchi type-I model but, the model can be seen as unstable in the future evolution with power-law expansion [6]. Furthermore, holographic dark energy with a variable deceleration parameter in $f(G)$ gravity indicates that the Gauss-Bonnet term acts similarly to the Λ cold dark matter (CDM) model for linear and quadratic models in the context of the isotropic and homogeneous Friedmann Lemaitre Walker (FRW) cosmological model [7]. Bhatti et al. [8] also discussed the dynamical analysis of a self-gravitating star in $f(G)$ gravity for spherically symmetric fluid exhibiting locally anisotropic pressure along with the energy density. The singularity-free solution is another approach in the context of $f(G)$ gravity studied by Bhatti et al. [9]. To explore singularity-free solution theory for a self-gravitating object called a gravastar, they assumed the irrotational cylindrical metric and isotropic pressure with perfect fluid, and the repulsive force produced by dark energy prevents the formation of singularities in gravastar. Through the use of the zero complexity factor and the quasi-homologous evolution, Yousaf et al. [10] discussed certain features of a dynamically charged relativistic system using $f\left(G,T^2\right)$ theory, which is another proposed modified gravity theory. A term $T^2 \equiv T_{\mu\nu}\, T^{\mu\nu}$ was added to the standard action of $f(G)$ gravity to give the $f(G,T^2)$ theory. In [11],they also tried to explore the $f(G,T^2)$ theory upon the complexity of time-dependent dissipative as well as non-dissipative spherically symmetric celestial structures. The concept of complexity for dynamical spherically symmetric dissipative self-gravitating configuration is also generalized in the scenario of $f(G)$ gravity by Yousaf et al. [12]. Whereas in the dissipative scenario, the fluid is still geodesic but shearing and has a large range of solutions, an isotropic, geodesic, homogeneous, and shear-free fluid distribution that satisfies the vanishing complexity constraint and proceeds homologously corresponds to such a fluid. Munyeshyaka et al. [13] investigated cosmological perturbations in $f(G)$ gravity and examined energy density perturbations for a Gauss-Bonnet field-dust system as well as for a Gauss-Bonnet field-radiation system in three distinct (trigonometric, exponential, logarithmic) $f(G)$ models.

In recent years, cosmological models with deceleration parameters (q) are of great interest to researchers. For depending on its sign, the deceleration parameter, which is a geometric parameter, represents the dynamics of acceleration or deceleration of the cosmos. The deceleration parameter was widely extended by researchers to investigate the accelerating universe's dynamic behaviour. The time-dependent deceleration parameter considered by Vinutha and Kavya [14] in R^n gravity plays an important role in accelerating expansion of the universe depending on the choice of value of $'n'$. Pawar and Shahare [15] investigated the dynamics of tilted the Bianchi type -III cosmological model, and the solution of the field equation of $f(R,T)$ gravity provides the expanding nature of the universe with a constant deceleration parameter. Tiwari et al. [16] used time-varying deceleration parameter to explore a Bianchi type-I cosmological model in $f(R,T)$ theory in the context of a different law of variation. Additionally, the variable deceleration parameter shown in the literature [17], provides the transitional behavior of anisotropic Bianchi type-III cosmological models from the early deceleration to the late time acceleration in the presence of a magnetic field for Sen-Dunn scalar-tensor theory. They also showed the model bounds from the decelerating phase to the de-sitter expansion or the exponential expansion phase of the universe.

So, motivated by the above discussion, here we have studied the Bianchi type-III cosmological model in $f(G)$ gravity. As the current scenario of the universe is accelerating from early-time decelerating, so, we consider a special form of time-varying deceleration parameter. Furthermore, considering the power-law $f(G)$ model the current study has been investigated. The paper's outline follows: In Sect. 2, Field Equations in $f(G)$ gravity and solutions of the field equations are discussed. The Results and discussion part is mentioned in Sect. 3. In Sect. 4, concluding remarks of the papers are given.

2 Field equations in $f(G)$ gravity

The action of Modified Gauss-Bonnet gravity is

$$S=\frac{1}{2k}\int d^4x\sqrt{-g}\,[R+f(G)]+S_M\left(g^{\mu\nu},\psi\right) \tag{1}$$

Where k is the coupling constant, g is the determinant of the metric tensor $g_{\mu\nu}$ and $S_M\left(g^{\mu\nu},\psi\right)$ is the matter action, in which matter is minimally coupled to the metric tensor, and ψ denotes the matter fields. This coupling of matter to the metric tensor suggests that $f(G)$ gravity is a purely metric theory of gravity. In pure $f(G)$ gravity there is a ghost mode occurs when the field equation contains more than two derivatives; it also appears when cosmological perturbation is taken into account. Because of the Ostrogradsky instability, higher-derivative theories typically have ghost degrees of freedom (d.o.f.). With these concerns in mind, scientists looked into possible ways to get rid of the ghosts in $f(G)$ gravity theory. They specially employed the Lagrange multiplier technique, and by means of constraints they are able to eliminate the ghost modes from Gauss-Bonnet theories of the forms $f(G)$.

The $f(G)$ is an arbitrary function of the Gauss-Bonnet (GB) invariant G:

$$G\equiv R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} \tag{2}$$

Where R is the Ricci scalar and $R_{\mu\nu}$ and $R_{\mu\nu\sigma\rho}$ denote the Ricci and Riemannian tensor.

Gravitational field equations are obtained by varying the action in Equation (1) concerning the metric tensor

$$\begin{aligned}&R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+8\left[R_{\mu\rho\nu\sigma}+R_{\rho\nu}g_{\sigma\mu}-R_{\rho\sigma}g_{\nu\mu}-R_{\mu\nu}g_{\sigma\rho}+R_{\mu\sigma}g_{\nu\rho}+\frac{1}{2}R\left(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\sigma}g_{\nu\rho}\right)\right]\\&\times\nabla^{\rho}\nabla^{\sigma}F+\left(Gf_G-f\right)g_{\mu\nu}=kT_{\mu\nu}\end{aligned} \tag{3}$$

where the operator ∇_μ denotes the covariant derivative and f_G represents the derivative of f concerning to G.

Both theoretical and experimental data support a homogeneous and isotropic universe and the typical FRW model's nature is homogenous, isotropic, and broadly consistent with the state of the universe today. Nevertheless, it fails to provide an explicit account of the universe's early stages of evolution. Models of the Bianchi type describe the early universe in a physically plausible way. It is well-known that, near the Big Bang, the universe is neither isotropic nor spherically symmetric. Thus, anisotropic models play an important role in cosmology. The Bianchi type-III model is the most basic anisotropic universe model among those proposed models to describe the anisotropy of the universe. Therefore, it is extensively studied due to its distinct geometric properties, which distinguishes it from the other Bianchi-type models in explaining the fundamental aspects of the universe.

The spatially homogeneous, anisotropic, and Bianchi type-III space-time is given by

$$ds^2=dt^2-A^2dx^2-B^2e^{-2\alpha x}dy^2-C^2dz^2 \tag{4}$$

Where A, B, and C are cosmic scale factors.

The Bianchi type-III cosmological models with electromagnetic field tensor and relativistic charged has been discussed by Mete et al. [18] in Brans-Dicke theory of gravity indicates that the model with vector field components is isotropic, non-singular, and convergent. In general theory of relativity, the Bianchi type-III model in the presence of anisotropic dark fluid with massive scalar meson field was studied by Raju et al. [19] addressed the EoS parameter of dark energy model varies in phantom region initially and approaches to a constant value in the quintessence region. Considering three different types of Bianchi models (Bianchi type-I, Bianchi type-III and Kantowski Sachs space-time) the issue of an accelerated universe can be explained in the vacuum field equation of $f(R,\phi)$ gravity [20]. The solution of this model can indicate that the universe's expansion will be completely stopped in the future.

The Ricci scalar and Gauss-Bonnet invariant are

$$R=-2\left[\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}-\frac{\alpha^2}{A^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}\right] \tag{5}$$

$$G=8\left[\frac{\dot{A}\dot{B}\ddot{C}}{ABC}+\frac{\dot{A}\ddot{B}\dot{C}}{ABC}+\frac{\ddot{A}\dot{B}\dot{C}}{ABC}-\frac{\alpha^2}{A^2}\bullet\frac{\ddot{C}}{C}\right] \tag{6}$$

Where dot represents the derivative with respect to $'t'$.

Here, the energy-momentum tensor is taken as

$$T_{\mu\nu} = (\rho + p)\,u_\mu u_\nu - p g_{\mu\nu} \tag{7}$$

where u_μ is the four-velocity vector, and ρ, p are the energy density and pressure of the universe, respectively.

The average scale factor $a(t)$ and the volume scale factor V are defined by

$$V = a^3 = ABC \tag{8}$$

The average Hubble parameter is defined by

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{9}$$

The expansion scalar θ is defined by

$$\theta = 3H = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \tag{10}$$

The mean anisotropy parameter A_m and shear scalar σ^2 are defined by

$$A_m = \frac{1}{3}\sum\left(\frac{\triangle H_i}{H}\right)^2;\ i = 1,2,3 \tag{11}$$

where $\triangle H_i = H_i - H;\ i = 1,2,3$ and, $H_1 = \frac{\dot{A}}{A}, H_2 = \frac{\dot{B}}{B},\ H_3 = \frac{\dot{C}}{C}$ are directional Hubble parameters along x, y, and z axes, respectively.

And,

$$\sigma^2 = \frac{3}{2}A_m H^2 \tag{12}$$

Now, using Equations (4) and (7), the field Equation (3) takes the form

$$-\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC} + 8\left(\frac{\dot{B}\ddot{C}}{BC} + \frac{\ddot{B}\dot{C}}{BC}\right)\dot{f}_G + 8\frac{\dot{B}\dot{C}}{BC}\ddot{f}_G - Gf_G + f = \kappa p \tag{13}$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} - \frac{\dot{A}\dot{C}}{AC} + 8\left(\frac{\dot{A}\ddot{C}}{AC} + \frac{\ddot{A}\dot{C}}{AC}\right)\dot{f}_G + 8\frac{\dot{A}\dot{C}}{AC}\ddot{f}_G - Gf_G + f = \kappa p \tag{14}$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + \frac{\alpha^2}{A^2} + 8\left(\frac{\dot{A}\ddot{B}}{AB} + \frac{\ddot{A}\dot{B}}{AB}\right)\dot{f}_G + 8\left(\frac{\dot{A}\dot{B}}{AB} - \frac{\alpha^2}{A^2}\right)\ddot{f}_G - Gf_G + f = \kappa p \tag{15}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{\alpha^2}{A^2} - 24\frac{\dot{A}\dot{B}\dot{C}}{ABC}\dot{f}_G + 8\frac{\alpha^2}{A^2}\bullet\frac{\dot{C}}{C}\dot{f}_G + Gf_G - f = \kappa\rho \tag{16}$$

$$\alpha\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = 0 \tag{17}$$

These are complicated and highly non-linear differential equations. In the next subsection 2.1, we will try to find the solution to the field equation.

2.1 Solutions of the field equation

From Equation (17), we get

$$A = c_1 B \tag{18}$$

Here c_1 is the integration constant. Without loss of generality, we can consider $c_1 = 1$.

Now, using this condition, Equation (18) can be written as

$$A = B \tag{19}$$

With the Equation (19), the field Equations (13), (14), (15) and (16) now take the form

$$-\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC} + 8\left(\frac{\dot{B}\ddot{C}}{BC} + \frac{\ddot{B}\dot{C}}{BC}\right)\dot{f}_G + 8\frac{\dot{B}\dot{C}}{BC}\ddot{f}_G - Gf_G + f = \kappa p \tag{20}$$

$$-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{\alpha^2}{B^2} + 16\frac{\dot{B}\ddot{B}}{B^2}\dot{f}_G + 8\left(\frac{\dot{B}^2}{B^2} - \frac{\alpha^2}{B^2}\right)\ddot{f}_G - Gf_G + f = \kappa p \tag{21}$$

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{B}\dot{C}}{BC} - \frac{\alpha^2}{B^2} + 8\left(\frac{\alpha^2}{B^2} \bullet \frac{\dot{C}}{C} - 3\frac{\dot{B}^2\dot{C}}{B^2C}\right)\dot{f}_G + Gf_G - f = \kappa\rho \tag{22}$$

Here, we use a physical condition that the shear scalar σ and the expansion scalar θ are proportional to each other ($\sigma \propto \theta$), which leads to

$$C = B^n \tag{23}$$

where n is an arbitrary real number and $n \neq 0, 1$ for nontrivial solutions.

Thus, with Equation (23), the field Equations (20), (21) and (22) reduces to

$$-(n+1)\frac{\ddot{B}}{B} - n^2\frac{\dot{B}^2}{B^2} + 8\left(2n\frac{\dot{B}\ddot{B}}{B^2} + n(n-1)\frac{\dot{B}^3}{B^3}\right)\dot{f}_G + 8n\frac{\dot{B}^2}{B^2}\ddot{f}_G - Gf_G + f = \kappa p \tag{24}$$

$$-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{\alpha^2}{B^2} + 16\frac{\dot{B}\ddot{B}}{B^2}\dot{f}_G + 8\left(\frac{\dot{B}^2}{B^2} - \frac{\alpha^2}{B^2}\right)\ddot{f}_G - Gf_G + f = \kappa p \tag{25}$$

$$(1+2n)\frac{\dot{B}^2}{B^2} - \frac{\alpha^2}{B^2} + 8n\left(\frac{\alpha^2}{B^2} \bullet \frac{\dot{B}}{B} - 3\frac{\dot{B}^3}{B^3}\right)\dot{f}_G + Gf_G - f = \kappa\rho \tag{26}$$

Since the field Equations (24), (25) and (26) are highly non-linear, thus to solve the field equation, we assume the time-varying deceleration parameter [1] as

$$q = -1 + \frac{1}{\sqrt{t}} \tag{27}$$

The Hubble's parameter is defined as $H = \frac{\dot{a}}{a}$ and from the above equation, we obtain

$$H = \frac{\dot{a}}{a} = \frac{1}{2\sqrt{t} - 1} \tag{28}$$

Where the integrating constant is assumed as unity.

Now, integrating Equation (28), we have found

$$a = \left(2\sqrt{t} - 1\right)^{\frac{1}{2}} e^{\sqrt{t}} \tag{29}$$

Using Equation (29) in Equation (8), we get the metric potential as

$$A = B = \left[\left(2\sqrt{t}-1\right)^{\frac{1}{2}} e^{\sqrt{t}} \right]^{\frac{3}{(n+2)}} \tag{30}$$

$$C = \left[\left(2\sqrt{t}-1\right)^{\frac{1}{2}} e^{\sqrt{t}} \right]^{\frac{3n}{(n+2)}} \tag{31}$$

So, model Equation (4) with Equations (30) and (31) takes the form

$$ds^2 = -dt^2 - \left[(2\sqrt{t}-1)^{1/2} e^{\sqrt{t}}\right]^{6/(n+2)} dx^2 - \left[(2\sqrt{t}-1)^{1/2} e^{\sqrt{t}}\right]^{6/(n+2)} e^{-2\alpha x} dy^2 - [(2\sqrt{t}- 1)^{1/2} e^{\sqrt{t}}\Big]^{6n/(n+2)} dz^2 \tag{32}$$

For this solution, the Ricci scalar and the Gauss-Bonnet invariant become

$$R = -\frac{2\left(9\sqrt{t}-3n-6\right)}{(n+2)\sqrt{t}\left(2\sqrt{t}-1\right)^2} - \frac{18\left(n^2+n+1\right)}{(n+2)^2\left(2\sqrt{t}-1\right)^2} + \frac{2\alpha^2}{\left(\left(2\sqrt{t}-1\right)^{\frac{1}{2}} e^{\sqrt{t}}\right)^{\frac{6}{(n+2)}}} \tag{33}$$

$$G = \frac{648n(\sqrt{t}-1)}{(n+2)^3\sqrt{t}(2\sqrt{t}-1)^4} + \frac{24n\alpha^2(n+2-3n\sqrt{t})\left(\left(2\sqrt{t}-1\right)^{\frac{1}{2}} e^{\sqrt{t}}\right)^{-\frac{6}{(n+2)}}}{(n+2)^2\sqrt{t}(2\sqrt{t}-1)^2} \tag{34}$$

2.2 Physical and geometrical properties of the cosmological model

In this subsection 2.2, we will discuss the physical and geometrical properties of the model with some important parameters considering the $f(G)$ model with

$$f(G) = \beta G^{m+1} \tag{35}$$

Where β and m are arbitrary constants.

This power-law $f(G)$ models are compatible with the observational evidence and predict that early-time inflation and late-time acceleration are expected to occur.

From Equation (35), it follows that

$$f_G(G) = \beta(m+1)G^m \tag{36}$$

Here, we choose $\beta = \frac{1}{(m+1)}$ for further analysis. From Equation (36), we may obtain

$$\dot{f}_G = mG^{m-1}\dot{G}, \qquad \ddot{f}_G = m\left(G^{m-1}\ddot{G} + (m-1)G^{m-2}\dot{G}^2\right) \tag{37}$$

where $G = \frac{648n(\sqrt{t}-1)}{(n+2)^3\sqrt{t}(2\sqrt{t}-1)^4} + \frac{24n\alpha^2(n+2-3n\sqrt{t})\left(\left(2\sqrt{t}-1\right)^{\frac{1}{2}} e^{\sqrt{t}}\right)^{-\frac{6}{(n+2)}}}{(n+2)^2\sqrt{t}(2\sqrt{t}-1)^2}$

Now, with the above Equations (30), (34), (35), (36) and (37), we get the expression of energy density and pressure as follows

$$\rho=\frac{9(1+2n)}{(n+2)^2\left(2\sqrt{t}-1\right)^2}-\frac{\alpha^2}{\left[\left(2\sqrt{t}-1\right)^{\frac{1}{2}e^{\sqrt{t}}}\right]^{\frac{6}{(n+2)}}+8n\left[\frac{3\alpha^2\left[\left(2\sqrt{t}-1\right)^{\frac{1}{2}e^{\sqrt{t}}}\right]^{-\frac{6}{(n+2)}}}{(n+2)\left(2\sqrt{t}-1\right)}-\frac{81}{(n+2)^3\left(2\sqrt{t}-1\right)^3}\right]\dot{f}_G}$$
$$+\left[\frac{648n(\sqrt{t}-1)}{(n+2)^3\sqrt{t}(2\sqrt{t}-1)^4}+\frac{24n\alpha^2(n+2-3n\sqrt{t})\left(\left(2\sqrt{t}-1\right)^{\frac{1}{2}e^{\sqrt{t}}}\right)^{-\frac{6}{(n+2)}}}{(n+2)^2\sqrt{t}(2\sqrt{t}-1)^2}\right]f_G-f \tag{38}$$

$$p=-\frac{(n+3)\left(9\sqrt{t}-3n-6\right)}{2(n+2)^2\sqrt{t}\left(2\sqrt{t}-1\right)^2}-\frac{9\left(n^2+1\right)}{2(n+2)\left(2\sqrt{t}-1\right)}$$
$$+\frac{\alpha^2}{2\left[\left(2\sqrt{t}-1\right)^{\frac{1}{2}e^{\sqrt{t}}}\right]^{\frac{6}{(n+2)}}}+4\left[\frac{6(n+1)\left(9\sqrt{t}-3n-6\right)}{(n+2)^3\left(2\sqrt{t}-1\right)^3\sqrt{t}}+\frac{27n(n-1)}{(n+2)^3\left(2\sqrt{t}-1\right)^3}\right]\dot{f}_G$$
$$+4\left[\frac{9(n+1)}{(n+2)^2\left(2\sqrt{t}-1\right)^2}-\frac{\alpha^2}{\left(\left(2\sqrt{t}-1\right)^{\frac{1}{2}e^{\sqrt{t}}}\right]^{\frac{6}{(n+2)}}}\right]\ddot{f}_G-Gf_G+f \tag{39}$$

As we know that the expression of the equation of state (EoS) parameter is defined as $\omega=\frac{p}{\rho}$, so from Equations (38) and (39), it becomes

$$\omega=[-\frac{(n+3)\left(9\sqrt{t}-3n-6\right)}{2(n+2)^2\sqrt{t}\left(2\sqrt{t}-1\right)^2}-\frac{9\left(n^2+1\right)}{2(n+2)\left(2\sqrt{t}-1\right)}+\frac{\alpha^2}{2\left[\left(2\sqrt{t}-1\right)^{\frac{1}{2}e^{\sqrt{t}}}\right]^{\frac{6}{(n+2)}}}+4\left[\frac{6(n+1)\left(9\sqrt{t}-3n-6\right)}{(n+2)^3\left(2\sqrt{t}-1\right)^3\sqrt{t}}+\frac{27n(n-1)}{(n+2)^3\left(2\sqrt{t}-1\right)^3}\right]\dot{f}_G$$
$$+4\left[\frac{9(n+1)}{(n+2)^2\left(2\sqrt{t}-1\right)^2}-\frac{\alpha^2}{\left[\left(2\sqrt{t}-1\right)^{\frac{1}{2}e^{\sqrt{t}}}\right]^{\frac{6}{(n+2)}}}\right]\ddot{f}_G-Gf_G+f]/$$

$$\frac{9(1+2n)}{(n+2)^2\left(2\sqrt{t}-1\right)^2}-\frac{\alpha^2}{\left[\left(2\sqrt{t}-1\right)^{\frac{1}{2}}e^{\sqrt{t}}\right]^{\frac{6}{(n+2)}}+8n\left[\frac{3\alpha^2\left(\left(2\sqrt{t}-1\right)^{\frac{1}{2}}e^{\sqrt{t}}\right]^{-\frac{6}{(n+2)}}}{(n+2)\left(2\sqrt{t}-1\right)}-\frac{81}{(n+2)^3\left(2\sqrt{t}-1\right)^3}\right]\dot{f}_G} \tag{40}$$

$$+\left[\frac{648n(\sqrt{t}-1)}{(n+2)^3\sqrt{t}(2\sqrt{t}-1)^4}+\frac{24n\alpha^2(n+2-3n\sqrt{t})\left(\left(2\sqrt{t}-1\right)^{\frac{1}{2}}e^{\sqrt{t}}\right)^{-\frac{6}{(n+2)}}}{(n+2)^2\sqrt{t}(2\sqrt{t}-1)^2}\right]f_G-f$$

The volume, expansion scalar, mean anisotropy and shear scalar parameters are found to be as follows.

$$V=\left[\left(2\sqrt{t}-1\right)^{\frac{1}{2}}\ e^{\sqrt{t}}\right]^3 \tag{41}$$

$$\theta=\frac{3}{2\sqrt{t}-1} \tag{42}$$

$$A_m=\frac{2(n-1)^2}{(n+2)^2}=constant \tag{43}$$

$$\sigma^2=\frac{3(n-1)^2}{(n+2)^2\left(2\sqrt{t}-1\right)^2} \tag{44}$$

Again, we examine the time-varying behavior of energy conditions for the stability of the solution. The corresponding energy conditions are the null energy condition (NEC), the weak energy condition (WEC), the strong energy condition (SEC), and the dominant energy condition (DEC), which can be expressed as:

(i) NEC: $\rho+p\geq 0$

(ii) WEC: $\rho\geq 0,\ \rho+p\geq 0$

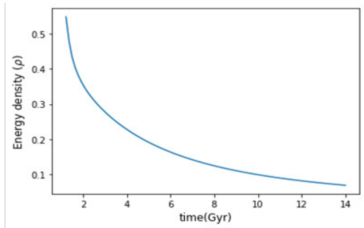
(iii) SEC: $\rho+3p\geq 0, \rho+p\geq 0$

(iv) DEC: $\rho\geq 0,\ \rho\pm p\geq 0$

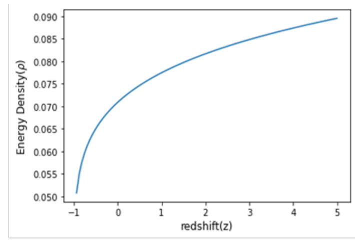
From the above energy conditions, it is well-known that only the violation of SEC $\rho+3p$ indicates the present expansion model of the universe for the existence of dark energy.

3 Results and Discussion

The Bianchi type-III cosmological model with a time-varying deceleration parameter in $f(G)$ gravity is represented by Equation (32). The parametric values $m=5.1018$, $n=1.0432$, and $\alpha=4$ are chosen to analyze the model's behavior throughout the graphical representation. The choice of parametric values provides a favorable result consistent with recent cosmological findings.

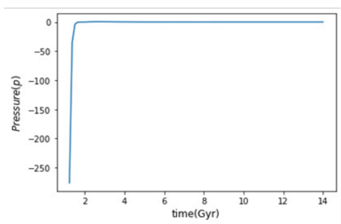


1(a)

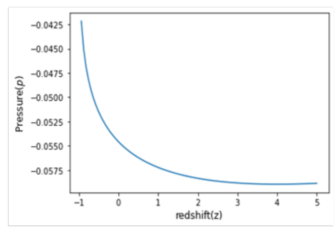


1(b)

Fig 1. The variation of energy density (ρ) **in terms of time (Gyr) (left panel) and redshift** (z) **(right panel) with** $m = 5.1018$, $n = 1.0432$, **and** $\alpha = 4$



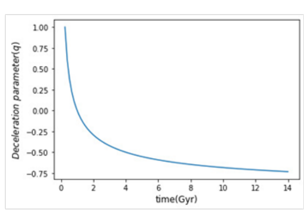
2(a)



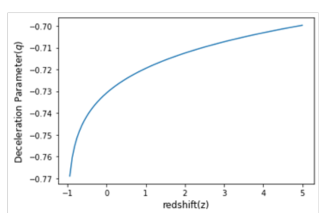
2(b)

Fig 2. The variation of pressure (p) **in terms of time (Gyr) (left panel) and redshift** (z) **(right panel) with** $m = 5.1018$, $n = 1.0432$, **and** $\alpha = 4$

The energy density of the model is positively decreasing function of time 't', and the pressure is always negative, as shown in left panels of Figure 1 (a) and Figure 2 (a). Initially, the energy density is very large while the pressure starts from a very small value, and as time passes, energy density and pressure converge to zero value. But for the time-varying deceleration parameter [(21)], it has also been mentioned about the periodic nature of the energy density and pressure, which is preserved throughout cosmic time. The right panels of Figure 1 (b) and Figure 2 (b) exhibit the energy density of the model is increasing and the pressure is decreasing functions of the redshift (z) and it is clearly seen that the energy density starts with a small positive value, in terms of redshift(z), while the pressure starts with large negative value. Additionally, as time goes on, both energy density and pressure approaches to zero value.



3(a)

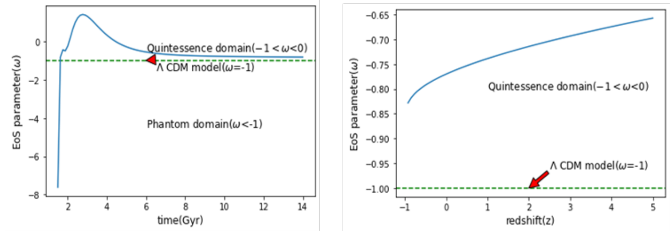


3(b)

Fig 3. The variation of deceleration parameter (q) **in terms of time (Gyr) (left panel) and redshift**(z) **(right panel)**

The deceleration parameter expands from the early deceleration phase $(q > 0)$ for $t < 1$ to the current acceleration phase $(q < 0)$ for $t > 1$. Therefore, the deceleration parameter in this study has a sign-flipping property depending on time $'t'$, as represented in Figure 3 (a). In Shaikh and Wankhade's [(22)] study of $f(R,T)$ gravity, the same sign-flipping deceleration parameter independent of time 't' was also noted for LRS Bianchi type-I. Also, from Equation (27), it is observed that the present value of the deceleration parameter at $t_0 = 13.8$ Gigayear (Gyr), is $q_0(t) = -0.730$ and at $t \to \infty$, it tends to de-sitter expansion

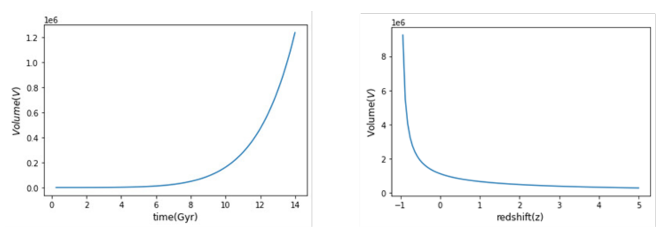
$(q = -1)$. Furthermore, with respect to redshift (z), the deceleration parameter stays in the negative region indicating the model is in accelerating phase $(q < 0)$ and lies in the range $-1 < q < 0$ as shown in Figure 3 (b). Therefore, the model supports the accelerating phase with respect to time 't' and redshift(z).



4(a) 4(b)

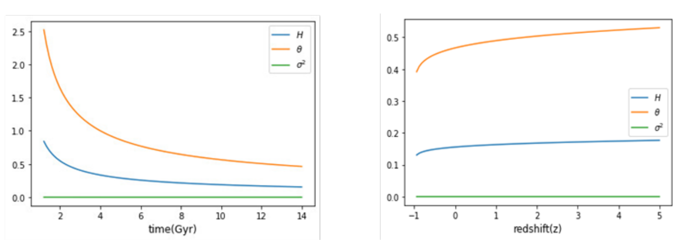
Fig 4. The variation of EoS (ω)parameter of dark energy in terms of time (Gyr) (left panel) and redshift (Z) (right panel) with $m = 5.1018, n = 1.0432,$ *and* $\alpha = 4$

The EoS parameter transits from the phantom $(\omega < -1)$ domain to the quintessence $(\omega > -1)$ domain after crossing the phantom divide line $(\omega = -1)$, and for a period of time, it approaches the radiation dominated era $\left(\omega \approx \frac{1}{3}\right)$. However, after a while, it moves to the quintessence domain $(\omega > -1)$ and at $t \to \infty$, the model approaches to ΛCDM model$(\omega = -1)$. Also, it has been observed that the EoS parameter of Barrow holographic dark energy in $f(Q)$ gravity depicts an early matter-dominated era to the present quintessence era $(\omega > -1)$, which later approaches to the ΛCDM cosmological model $(\omega = -1)$ at the late time [23]. The similar behavior of EoS parameter of dark energy with respect to redshift(z) is observed and the model lies in the quintessence domain $(\omega > -1)$ as depicted in Figure 4 (b). Furthermore, at present $t_0 = 13.8$ Gyr, the value of the EoS parameter corresponding to parametric value $m = 5.1018,\ n = 1.0432$, and $\alpha = 4$ is $\omega_0(t) = -0.819$, the result obtained in this model has also been shown in the literature studied by Koussour and Bennai [24].



5(a) 5(b)

Fig 5. The variation of volume (V) in terms of time (Gyr) (left panel) and redshift (z) (right panel)

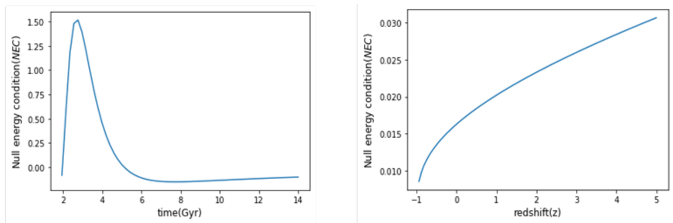


6(a) 6(b)

Fig 6. The variation of Hubble parameter (H) Expansion scalar (θ) Shear scalar (σ^2) in terms of time (Gyr) (left panel) and redshift (z) (right panel)

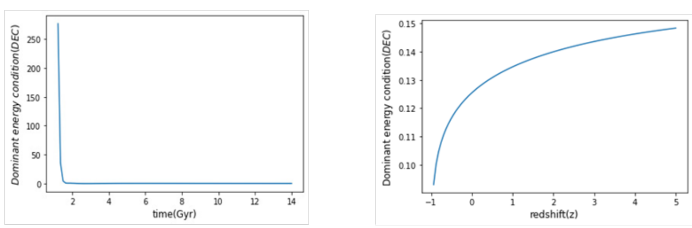
From Equation (41), it is noticed that the spatial volume is zero for $t = 0.25$, while the expansion scalar, Hubble parameter, and the shear scalar given by Equations (28), (42) and (44) are infinite at $t = 0.25$. In our model, $H > 0$, and the positive value of H suggests that the universe is expanding with time. On the other hand, when $t \to \infty$, both the expansion scalar and the Hubble parameter tend to be zero, and the derivative of the Hubble parameter $\frac{dH}{dt} = -\frac{1}{\left(2\sqrt{t}-1\right)^2\sqrt{t}} = \frac{dH}{dt} = 0$ if $t \to \infty$ which implies the fastest rate of expansion of the universe and the greatest value of Hubble's parameter.

The mean anisotropy parameter A_m is decreasing function of time for the model Equation (**??**) and only vanishes in the case of $n = 1$, i.e., $A_m = 0$ if $n = 1$. Also, the shear scalar $\sigma^2 \to 0$ as the time tends to infinity, and therefore, the model found in this study is shear-free and anisotropic for large values of $'t'$. For Bianchi type-III model, along with components of the vector potential, shows the isotropic universe in Brans-Dicke's theory of gravity [18].



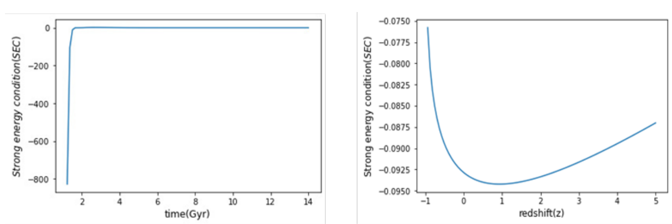
7(a) 7(b)

Fig 7. The variation of null energy condition (NEC) in terms of time (Gyr) (left panel) and redshift (z) (right panel) with $m = 5.1018, n = 1.0432,$ *and* $\alpha = 4$



8(a) 8(b)

Fig 8. The variation of dominant energy condition (DEC) in terms of time (Gyr) (left panel) and redshift (z) (right panel) with $m = 5.1018, n = 1.0432,$ *and* $\alpha = 4$



9(a) 9(b)

Fig 9. The variation of strong energy condition (SEC) in terms of time (Gyr) (left panel) and redshift (z) (right panel) with $m = 5.1018, n = 1.0432,$ *and* $\alpha = 4$

In Figures 7, 8 and 9, the energy conditions are tested against time $'t'$ and redshift(z). All three energy conditions WEC, NEC, DEC are well satisfied, while the violation of SEC is observed for the model. Also, from Figure 7 (b), Figure 8 (b), Figure 9

(b) it is clearly seen that in terms of redshift(z), NEC, DEC are satisfied whereas SEC is violated. Since, NEC gets satisfied, which results in the EoS parameter lying into the quintessence region and the result obtained in the R^n gravity for LRS Bianchi type-I provides the same result for a specific value of $'n'$ [14], and the model leads to the accelerated expansion of the universe.

4 Conclusion

In this paper, a specific power-law model $f(G) = \beta G^{m+1}$ is considered to investigate the solutions of Bianchi type-III space-time in the presence of perfect fluid. Here, the physical and kinematical parameters are determined from the field equation of $f(G)$ gravity. Here, this work employs the time-varying deceleration parameter suggested by Dewri [1] which shows a sign-flipping property from early-time deceleration to late-time acceleration. The model starts to expand from an initial Big Bang singularity at $t = 0.25$ with zero volume to de-sitter expansion at $t \to \infty$ in which $q = -1$ and $\frac{dH}{dt} = 0$, which matches the result of Shaikh and Wankhade's [22]. The evolution of energy density shows a positively decreasing function of time, and the pressure is always negative, which approaches zero at late times. In this study, the characteristic of dark energy indicates the phantom region in the early stage, and with time it approaches quintessence, later approaching the ΛCDM model as time $'t'$ tends to infinity. The study of the energy conditions for the model depicts the violation of the SEC at present, which is exhibited by the presence of dark energy lying in the quintessence epoch of the accelerated expansion of the universe, which is consistent with the present-day observation. The existence of the power-law solutions has been observed in the current study. It is also fascinating to note that in this study, the universe is free from a finite-time future singularity (Big Rip) at the late time. From the findings, it can be deduced that the model obtained in $f(G)$ framework is stable and practical and it describes the current accelerating universe, as it starts from the phantom to quintessence model at present epoch. Further investigation of Bianchi type models in $f(G)$ gravity can deliver the knowledge of the dark energy features and the accelerating universe.

5 Acknowledgement

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