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Fixed Point Theorems for Non-self Mappings Using Disconnected Graphs

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Abstract

Objectives: To prove the fixed point theorems for non-self mappings using disconnected graphs. **Method:** Graph theoretical approach is adopted to prove the fixed point theorems for non-self mappings. In all the previous works, connected graphs were used for establishing the results, but it is demonstrated in this work that disconnected graphs are best suited, and this new approach simplifies the proofs to a greater extent. **Findings:** The fixed point theorems by Banach, Kannan, Chatterjea, and Bianchini are proved using the new methodology. **Novelty:** An important part of the results concerning fixed point theorems is proving the iterated sequence to be a Cauchy sequence, and this is amalgamated with the edge sequence of the disconnected graph.

Subject Classification: 54H25, 47H10

Keywords: Non-self mapping; Iterated sequence; Disconnected graph; Edge sequence; Fixed point

1 Introduction

Let (X, d) be a metric space and T: X→X be a self mapping. The contraction principles due to Banach, Kannan, Chatterjea, and Bianchini are stated below:

(i) $d(Tx,Ty) \leq \alpha d(x,y)$;
(ii) $d(Tx,Ty) \leq \beta[d(x,Tx)+d(y,Ty)]$;
(iii) $d(Tx,Ty) \leq \beta[d(x,Ty)+d(y,Tx)]$;
(iv) $d(Tx,Ty) \leq \alpha max\{d(x,Tx),d(y,Ty)\}$;
for all $x,\ y\ \in X$, where $\alpha \in (0,1)$ and $\beta \in \left(0,\frac{1}{2}\right)$.

In all the above-mentioned fixed-point results, the mappings under consideration are self-mappings. But in many applications, the mappings may not always be self-mappings, which led to the study of fixed-point results for non-self mappings. One can refer to[1–6] for recent developments. Combining fixed point theory and graph theory has produced interesting results, and using graphs has either simplified or given a different version of the proofs for the existing results. In this paper, we have demonstrated the fixed point results using disconnected graphs, which is distinct from the previous findings. In the ensuing paragraphs, we compare the procedures adopted in the present paper with the previous ones.

A graph is an ordered pair consisting of the vertex set and the edge set. So far, for non-self mappings defined from a non-empty closed subset K of X to X, the vertex set is taken as X, and a sub-graph induced by the graph of X is defined corresponding to the set K. Instead, we can define a graph for K, i.e., consider only the points of K as the vertex set of the graph, thereby eliminating the points of X-K, which are not considered for the discussion of fixed point results.

In all the previous fixed point results on metric spaces endowed with graphs, the edge set of the graph is not explicitly defined; rather, it is defined as one containing the diagonal of X x X. In the present work, the graph for K is defined distinctly as the ordered pair having the vertex set as K and the edge set as the lines joining the points on the boundary with their images. (We use Rothe's boundary condition, which guarantees the images of the boundary points to be in K.)

The length of the edges of the graph associated with K plays a significant role in proving the fixed-point results. An important result that links the length of the edges with the convergence of the iterated sequence is exhibited. This approach simplifies the proof to a greater extent.

The fixed point results in the literature use connected graphs for both self and non-self mappings. When graphs are defined for sets on which self-mappings are defined, connectedness can be guaranteed. But when non-self mappings on a subset of X are considered, the concept of connectedness of graphs is quite ambiguous, as there are points for which the images fall outside K. In this paper, we define graphs whose edges are lines connecting points on the boundary with their image. i.e., if T: K→X is a non-self mapping, then (x, Tx) where $x \in \partial K$, is an edge of the graph and there cannot be two edges sharing a common vertex, i.e., no two edges are adjacent, which leads to a disconnected graph.

The methodology for bringing out the results is designated in the second section, and the third section elaborates on the fixed point results for non-self mappings using the methods discussed in the second section.

2 Methodology

Preliminary concepts, important definitions, and a lemma that plays a crucial part in the proof of the results are illustrated below.

Let X be a Banach space and K be a non-empty closed subset of X. Let T: K→X be a non-self mapping satisfying Rothe's boundary condition, i.e., T(∂K)⊂K. Since T is a non-self mapping, we have, for some x∈K, Tx $\notin$ K. Then we can choose a y∈ ∂K such that y=$(1-\lambda)x+\lambda Tx$, 0<λ<1.

This implies,

$$d(x,Tx) = d(x,y) + d(y,Tx),\ y \in \partial K. \tag{1}$$

Here d(x,y)=||x-y||.

We now give an important property which can be found in the literature concerning fixed point results for non-self mappings that uses Rothe's boundary condition:

$$\left\{ \begin{array}{c} For\ any\ x \in K\ such\ that\ Tx \notin K,\ the\ inequality \\ d(y,Ty) \leq d(x,Tx) \\ holds\ for\ atleast\ one\ corresponding\ y \in \partial K\ given\ by\ equation\ (1) \end{array} \right. \tag{L}$$

Definition 2 .1. Let X be a Banach space and K be a non-empty closed subset of X. Let T: K→X be a non-self mapping satisfying Rothe's boundary condition. The disconnected graph G associated with K is defined as below:

G=(V,E) where V=K and E={(x,Tx)/x∈ ∂K}.

Definition 2 .2. Let X be a Banach space and K be a non-empty closed subset of X. Let T: K→X be a non-self mapping satisfying Rothe's boundary condition. According to Definition 2.1, let G be the disconnected graph associated with K. The Picard iteration is constructed as below:

Let $x_0 \in \partial K$. Define $x_1 = Tx_0$. For n≥ 2, $x_n = Tx_{n-1}$. If $Tx_{n-1} \notin K$, then define $x_n = (1-\lambda)x_{n-1} + \lambda Tx_{n-1}$.

In the Picard sequence, choose first integer m such that $x_m \in \partial K$. Define $y_1 = x_m$ and let $e_1 = (y_1, Ty_1)$. y_2 is the next term in the iterated sequence that belongs to ∂K. Let $e_2 = (y_2, Ty_2)$. Thus, we get a sequence of edges $e_1, e_2, e_3, \ldots..$ corresponding to the iterated sequence.

Lemma 2 .1. Let X be a Banach space and K be a non-empty closed subset of X. Let T: K→X be a non-self mapping satisfying Rothe's boundary condition. Let G be the disconnected graph associated with K. The iterated sequence

$\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in K if and only if the sequence $\{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero.

Proof. First, let us assume that the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in K, and we shall prove that the sequence $\{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero. Suppose that the sequence $\{l(e_j)\}_{j=1}^{\infty}$ is not convergent to zero. Then for $\in > 0$, there exists a $N \in I$,such that $|l(e_j) - 0| \nless \in \; for\ some\ j \geq N \Rightarrow l(e_j) \geq \in for\ some\ j \geq N$. Here, $e_j = (y_j, Ty_j)$, where $y_j = x_m \in \partial K$ for some m∈I. ⊠$l(e_j) = d(x_m, Tx_m)$. Hence, $d(x_m, Tx_m) \geq \in \; for\ some\ m \geq N$ implies, $d(x_m, x_{m+1}) \geq \in, \; for\ some\ m \geq N$. i.e., the sequence $\{x_n\}_{n=1}^{\infty}$ is not a Cauchy sequence in K, contradicting our assumption. Hence, the sequence $\{l(e_j)\}_{j=1}^{\infty}$ must converge to zero.

Conversely, assume that the sequence $\{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero. To prove that the iterated sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in K. Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is not a Cauchy sequence in K. Then for $\in > 0$ there exists a $N \in I$, such that $|x_m - x_n| \nless \in \; for\ some\ m, n \geq N$. In particular, $|x_m - x_{m+1}| \geq \in for\ some\ m \geq N$. Let $x_m \in \partial K$. Then, $x_{m+1} = Tx_m$. Hence, we have, $|x_m - Tx_m| \geq \in for\ some\ m \geq N$. Since $x_m \in \partial K$, we have, $x_m = y_j \; for\ some\ j \in I$ and $e_j = (y_j, Ty_j)$. i.e., $e_j = (x_m, Tx_m)$ and $l(e_j) = d(x_m, Tx_m)$ .This implies,$l(e_j) \geq \in for\ some\ j \geq N$. i.e., the sequence $\{l(e_j)\}_{j=1}^{\infty}$ is not convergent to zero, contradicting our assumption. Hence, the sequence $\{x_n\}_{n=1}^{\infty}$ must be a Cauchy sequence in K.

We now proceed to prove the Banach, Kannan, Chatterjea, and Bianchini fixed point theorems for non-self mappings using disconnected graphs.

3 Results and Discussions

Theorem 3.1: Let X be a Banach space and K be a non-empty closed subset of X. Let T: K→X be a Banach contraction such that property (L) holds, and let T satisfies Rothe's boundary condition. Let G be the disconnected graph associated with K. Then T has a unique fixed point.

Proof. Choose $x_0 \in \partial K$. The Picard sequence is constructed as given in Definition 2.2. Let G be the disconnected graph associated with K, defined as in Definition 2.1. We shall prove that the iterated sequence defined in Definition 2.2 is Cauchy.

Choose some n∈I such that $x_n \in \partial K$, then $x_n = y_j$ for some j∈I.

$l(e_j) = d(y_j, Ty_j) = d(x_n, Tx_n)$.

Using property (L), we have $d(x_n, Tx_n) \leq d(x_{n-1}, Tx_{n-1})$.

$l(e_j) \leq d(x_{n-1}, Tx_{n-1})$,

$= d(Tx_{n-2}, Tx_{n-1})$.

Since T is a Banach contraction, we have d(Tx,Ty)≤αd(x,y), where 0≤α<1.

Using this, we have:

$l(e_j) \leq \alpha d(x_{n-2}, x_{n-1})$.

Now we have two cases: $x_{n-2} \in K \quad (OR)\ x_{n-2} \in \partial K$.

Case (i).$x_{n-2} \in K$.

Then,

$l(e_j) \leq \alpha d(x_{n-2}, x_{n-1})$,

$= \alpha d(Tx_{n-3}, Tx_{n-2})$,

$\leq \alpha . \alpha d(x_{n-3}, x_{n-2})$,

$= \alpha^2 d(x_{n-3}, x_{n-2})$.

Now we discuss two sub-cases: $x_{n-3} \in \partial K \quad (OR)\ x_{n-i} \in K,\ i = 1,2,3 \ldots p$ for some p ∈ I.

Case (a). $x_{n-3} \in \partial K$.

This case is treated similarly to the one discussed in Case (ii), and hence we proceed to case (b).

Case (b). $x_{n-i} \in K,\ i = 1,2,3 \ldots p$.

Suppose there are p elements preceding x_n that belongs to K. Then we have,

$l(e_j) \leq \alpha^p d(x_{n-p-1}, x_{n-p})$.

In which case,$x_{n-p-1} \in \partial K$, and hence, $y_{j-1} = x_{n-p-1}$ and $l(e_{j-1}) = d(y_{j-1}, Ty_{j-1}) = d(x_{n-p-1}, Tx_{n-p-1}) = d(x_{n-p-1}, x_{n-p})$.

Hence, we have,

$l(e_j) \leq \alpha^p l(e_{j-1}) \rightarrow 0\ as\ p \rightarrow \infty$.

$\Rightarrow \{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero.

Case (ii). $x_{n-2} \in \partial K$.

If $x_{n-2} \in \partial K$, then $y_{j-1} = x_{n-2}$. $l(e_{j-1}) = d(y_{j-1}, Ty_{j-1}) = d(x_{n-2}, x_{n-1})$.

Hence, $l(e_j) \leq \propto l(e_{j-1})$.

(The edges $e_j\ and\ e_{j-1}$ are shown in Figure 1). After this stage, if there are p' elements preceding x_{n-2} that belongs to K, then we proceed as in Case (b), and we have

$l(e_j) \leq \propto . \propto^{p'} l(e_{j-2})$.

$\Rightarrow \quad l(e_j) \leq \alpha^{p'+1} l(e_{j-2}) \rightarrow 0\ as\ p' \rightarrow \infty$.

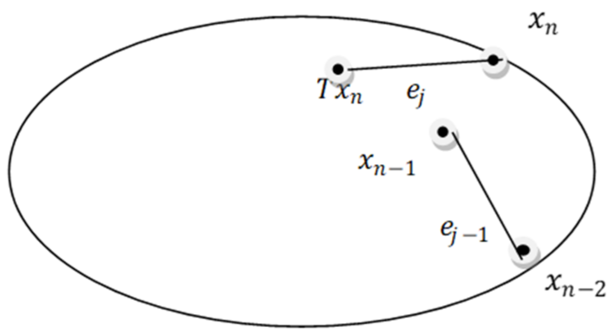

Fig 1. Edges defined for the boundary points of the Picard sequence

$\Rightarrow \{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero.

Hence, in both the cases, the sequence $\{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero. Hence, by Lemma 2.1, the iterated sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the closed subset K, and hence it is convergent.

Let $\lim_{n\to\infty} x_n = x^*$.

Now to prove that x^* is a fixed point of T.

The mapping T, being a contraction, is continuous. Hence, the sequence $\{Tx_n\}_{n=1}^{\infty}$ converges to Tx^*. But the sequence $\{Tx_n\}_{n=1}^{\infty}$ is a subsequence of the sequence $\{x_n\}_{n=1}^{\infty}$. Hence, the subsequence must have the same limit as the parent sequence. But the limit of a sequence is unique. Hence, we must have $Tx^* = x^*$. i.e. x^* is a fixed point of T.

To prove uniqueness, let, if possible, z^* be any other fixed point of T. Then $Tz^* = z^*$.

Since T is a contraction on X, we have,

$d(Tx^*, Tz^*) \le \alpha d(x^*, z^*),\ \ 0 \le \alpha < 1.$

$d(x^*, z^*) < d(x^*, z^*),$

which is a contradiction. Hence, the fixed point of T is unique.

Theorem 3.2. Let X be a Banach space and K be a non-empty closed subset of X. Let T: K→X be a mapping having property (L) and satisfying Rothe's boundary condition. Let G be the disconnected graph associated with K. Also, there exists a constant $\alpha \varepsilon \left(0, \frac{1}{2}\right)$ such that T satisfies the inequality

d(Tx,Ty)$\le\alpha$.[d(x,Tx)+d(y,Ty)] for all x, y $\in$ K,

then T has a unique fixed point.

Proof. Choose $x_0 \in \partial K$. The Picard sequence is constructed as given in Definition 2.2. Let G be the disconnected graph associated with K, defined as in Definition 2.1. We shall prove that the iterated sequence defined in Definition 2.2 is Cauchy.

Choose some n∈I such that, $x_n \in \partial K$. Then $x_n = y_j$ for some j∈I.

$l(e_j) = d(y_j, Ty_j) = d(x_n, Tx_n)$.

Using property (L), we have $d(x_n, Tx_n) \le d(x_{n-1}, Tx_{n-1})$.

$d(x_{n-1}, Tx_{n-1}) = d(Tx_{n-2}, Tx_{n-1}),$

$\le \alpha[d(x_{n-2}, Tx_{n-2}) + d(x_{n-1}, Tx_{n-1})]$.

$\Rightarrow \quad d(x_{n-1}, Tx_{n-1}) \le \frac{\alpha}{1-\alpha} d(x_{n-2}, Tx_{n-2})$.

$d(x_{n-1}, Tx_{n-1}) \le \alpha' d(x_{n-2}, Tx_{n-2})$, where $0 \le \alpha' < 1$.

Hence, we have $l(e_j) \le \alpha' d(x_{n-2}, Tx_{n-2})$ where $0 \le \alpha' < 1$.

We now consider two cases: $x_{n-2} \in K \quad (OR)\ x_{n-2} \in \partial K$.

Case (i). $x_{n-2} \in K$.

Then,

$d(x_{n-2}, Tx_{n-2}) = d(Tx_{n-3}, Tx_{n-2}),$

$\le \alpha[d(x_{n-3}, Tx_{n-3}) + d(x_{n-2}, Tx_{n-2})]$.

$\Rightarrow \quad d(x_{n-2}, Tx_{n-2}) \le \frac{\alpha}{1-\alpha} d(x_{n-3}, Tx_{n-3}),$

$d(x_{n-2}, Tx_{n-2}) \le \alpha' d(x_{n-3}, Tx_{n-3})$, where $0 \le \alpha' < 1$.

We discuss two sub cases: $x_{n-3} \in \partial K \quad (OR)\ x_{n-i} \in K,\ i = 1,2,3\ldots p$ for some p $\in$ I.

Case (a). $x_{n-3} \in \partial K$.

This case is treated similarly to the one discussed in Case (ii), and hence we proceed to case (b).

Case (b). $x_{n-i} \in K,\ i = 1,2,3 \ldots p$.

Suppose there are p elements preceding x_n that belongs to K. Then we have,

$l(e_j) \leq (\alpha')^p d(x_{n-p-1}, x_{n-p})$.

In which case, $x_{n-p-1} \in \partial K$,hence $y_{j-1} = x_{n-p-1}$ and $l\left(e_{j-1}\right) = d\left(y_{j-1}, Ty_{j-1}\right) = d\left(x_{n-p-1}, Tx_{n-p-1}\right) = d\left(x_{n-p-1}, x_{n-p}\right)$.

Hence,

$l\left(e_j\right) \leq (\alpha')^p\, l\left(e_{j-1}\right) \to 0\ as\ p \to \infty$.

$\Rightarrow \{l\left(e_j\right)\}_{j=1}^{\infty}$ is convergent to zero.

Case (ii) . $x_{n-2} \in \partial K$.

If $x_{n-2} \in \partial K$ then, $y_{j-1} = x_{n-2}$. $l\left(e_{j-1}\right) = d\left(y_{j-1}, Ty_{j-1}\right) = d(x_{n-2}, x_{n-1})$.

Hence, $l(e_j) \leq \alpha' l(e_{j-1})$.

After this stage, if there are p' elements preceding x_{n-2} that belongs to K, then we proceed as in Case (b), and we have

$l\left(e_j\right) \leq \alpha'.(\alpha')^{p'} l(e_{j-2})$.

$\Rightarrow \quad l\left(e_j\right) \leq (\alpha')^{p'+1} l\left(e_{j-2}\right) \to 0\ as\ p' \to \infty$.

$\{l\left(e_j\right)\}_{j=1}^{\infty}$ is convergent to zero.

Hence, we have in both cases, the sequence $\{l\left(e_j\right)\}_{j=1}^{\infty}$ is convergent to zero. Hence, by Lemma 2.1, the iterated sequence $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence in the closed subset K, and hence it is convergent.

Let $\lim_{n\to\infty} x_n = x^*$.

To prove that x^* is the fixed point of T.

$d\left(x^*, Tx^*\right) \leq d\left(x^*, x_n\right) + d(x_n, Tx^*)$, where $x_n \in K$,

$= d\left(x^*, x_n\right) + d\left(Tx_{n-1}, Tx^*\right)$,

$\leq d\left(x^*, x_n\right) + \alpha[d\left(x_{n-1}, Tx_{n-1}\right) + d\left(x^*, Tx^*\right)]$.

$\Rightarrow \quad d\left(x^*, Tx^*\right) \leq \frac{1}{1-\alpha}[d\left(x^*, x_n\right) + \alpha d\left(x_{n-1}, Tx_{n-1}\right)]$,

$\leq \frac{1}{1-\alpha}[d\left(x^*, x_n\right) + \alpha d\left(x_{n-1}, Tx_{n-1}\right)]$.

As n→∞, we have, $x_n \to x^*,\ and\ d(x_{n-1}, x_n) \to 0$.

Hence, $d\left(x^*, Tx^*\right) \to 0\ as\ n \to \infty$.

i.e. $Tx^* = x^*$. Hence, x^* is the fixed point of T.

Now, to prove that the fixed point of T is unique. If possible, let z^* be any other fixed point of T.

Then $d\left(x^*, z^*\right) = d(Tx^*, Tz^*) \leq \alpha[d\left(x^*, Tx^*\right) + d\left(z^*, Tz^*\right)]$.

Since x^*, z^* are the fixed points of T, the right side of the above inequality reduces to zero. Hence, we must have $x^* = z^*$.i.e the fixed point of T is unique.

Theorem 3.3. Let X be a Banach space and K be a non-empty closed subset of X. Let T: K→X be a mapping having property (L) and satisfying Rothe's boundary condition. Let G be the disconnected graph associated with K. Also, there exists a constant $\alpha \varepsilon \left(0, \frac{1}{2}\right)$ such that T satisfies the inequality

d(Tx,Ty)≤α.[d(x,Ty)+d(y,Tx)] for all x, y ∈ K,

then T has a unique fixed point.

Proof. Choose $x_0 \in \partial K$. The Picard sequence is constructed as given in Definition 2.2. Let G be the disconnected graph associated with K defined as in Definition 2.1. We shall prove that the iterated sequence so constructed is a Cauchy sequence in K.

Choose some n∈I such that $x_n \in \partial K$. Then $x_n = y_j$for some j∈I.

$l\left(e_j\right) = d\left(y_j, Ty_j\right) = d\left(x_n, Tx_n\right)$.

Using property (L), we have $d\left(x_n, Tx_n\right) \leq d\left(x_{n-1}, Tx_{n-1}\right)$.

$d\left(x_{n-1}, Tx_{n-1}\right) = d(Tx_{n-2}, Tx_{n-1})$,

$\leq \alpha[d\left(x_{n-2}, Tx_{n-1}\right) + d\left(x_{n-1}, Tx_{n-2}\right)]$,

$\leq \alpha[d\left(x_{n-2}, Tx_{n-1}\right) + d\left(x_{n-1}, x_{n-1}\right)]$,

$\leq \alpha d\left(x_{n-2}, Tx_{n-1}\right)$,

$\leq \alpha[d\left(x_{n-2}, x_{n-1}\right) + d\left(x_{n-1}, Tx_{n-1}\right)]$.

$\Rightarrow d(x_{n-1}, Tx_{n-1}) \leq \frac{\alpha}{1-\alpha} d\left(x_{n-2}, x_{n-1}\right)$.

i.e. $d\left(x_{n-1}, Tx_{n-1}\right) \leq \alpha' d\left(x_{n-2}, x_{n-1}\right)\ where\ \alpha' \in (0,1)$.

Hence, we have

$l(e_j) \leq \alpha' d\left(x_{n-2}, x_{n-1}\right) where\ \alpha' \in (0,1)$.

We now consider two cases: $x_{n-2} \in K \quad (OR)\ x_{n-2} \in \partial K$.

Case (i) . $x_{n-2} \in K$.

Then

$d(x_{n-2}, x_{n-1}) = d(Tx_{n-3}, Tx_{n-2})$,
$\leq \alpha [d(x_{n-3}, Tx_{n-2}) + d(x_{n-2}, Tx_{n-3})]$,
$\leq \alpha [d(x_{n-3}, Tx_{n-2}) + d(x_{n-2}, x_{n-2})]$,
$\leq \alpha [d(x_{n-3}, Tx_{n-2})]$,
$\leq \alpha [d(x_{n-3}, x_{n-2}) + d(x_{n-2}, Tx_{n-2})]$.
$\Rightarrow d(x_{n-2}, Tx_{n-2}) \leq \frac{\alpha}{1-\alpha} d(x_{n-3}, x_{n-2})$.

i.e. $d(x_{n-2}, Tx_{n-2}) \leq \alpha' d(x_{n-3}, x_{n-2})$ *where* $\alpha' \in (0,1)$.

We discuss two sub- cases: $x_{n-3} \in \partial K\ (OR)\ x_{n-i} \in K,\ i = 1,2,3\ldots p$ for some p ∈ I.

Case (a). $x_{n-3} \in \partial K$.

This case is treated similarly to the one discussed in Case (ii), and hence we proceed to case (b).

Case (b). $x_{n-i} \in K,\ i = 1,2,3\ldots p$.

Suppose there are p elements preceding x_n that belongs to K. Then we have,

$l(e_j) \leq \left(\alpha'\right)^p d(x_{n-p-1}, x_{n-p})$.

In which case, $x_{n-p-1} \in \partial K$, and hence, $y_{j-1} = x_{n-p-1}$ and $l(e_{j-1}) = d(y_{j-1}, Ty_{j-1}) = d(x_{n-p-1}, Tx_{n-p-1}) = d(x_{n-p-1}, x_{n-p})$.

Hence, we have,

$l(e_j) \leq (\alpha')^p l(e_{j-1}) \to 0$ *as* $p \to \infty$.

$\Rightarrow \{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero.

Case (ii). $x_{n-2} \in \partial K$.

If $x_{n-2} \in \partial K$, then $y_{j-1} = x_{n-2}$. $l(e_{j-1}) = d(y_{j-1}, Ty_{j-1}) = d(x_{n-2}, x_{n-1})$.

Hence, $l(e_j) \leq \alpha' l(e_{j-1})$.

After this stage, if there are p' elements preceding x_{n-2} that belongs to K, then we proceed as in Case (b), and we have

$l(e_j) \leq \alpha' .(\alpha')^{p'} l(e_{j-2})$.

$\Rightarrow \quad l(e_j) \leq (\alpha')^{p'+1} l(e_{j-2}) \to 0$ *as* $p' \to \infty$.

Hence, $\{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero.

In both the cases, the sequence $\{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero. Hence, by Lemma 2.1, the iterated sequence $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence in the closed subset K and hence it is convergent.

Let $\lim_{n\to\infty} x_n = x^*$.

To prove that x^* is the fixed point of T.

$d(x^*, Tx^*) \leq d(x^*, x_n) + d(x_n, Tx^*)$, where $x_n \in K$,
$\leq d(x^*, x_n) + d(Tx_{n-1}, Tx^*)$,
$\leq d(x^*, x_n) + \alpha [d(x_{n-1}, Tx^*) + d(x^*, Tx_{n-1})]$,
$\leq\ d(x^*, x_n) + \alpha [d(x_{n-1}, Tx^*) + d(x^*, x_n)]$,
$\leq d(x^*, x_n) + \alpha [d(x_{n-1}, x^*) + d(x^*, Tx^*) + d(x^*, x_n)]$.
$\Rightarrow d(x^*, Tx^*) \leq \frac{1}{1-\alpha}[(1+\alpha)\ d(x^*, x_n) + \alpha d(x_{n-1}, x^*)]$.

As n→∞, we have, $x_n \to x^*,\ x_{n-1} \to x^*$.

Hence, $d(x^*, Tx^*) \to 0$ *as* $n \to \infty$.

i.e. $Tx^* = x^*$. Hence, ,x^* is the fixed point of T. Now, to prove that the fixed point of T is unique. If possible, let z^* be any other fixed point of T.

Then

$d(x^*, z^*) = d(Tx^*, Tz^*) \leq \alpha [d(x^*, Tz^*) + d(z^*, Tx^*)]$,
$\leq \alpha [d(x^*, z^*) + d(z^*, x^*)]$,
$\Rightarrow d(x^*, z^*) \leq 2\alpha d(x^*, z^*)$.

i.e. $\alpha \geq \frac{1}{2}$, which is a contradiction.

Hence, we must have $x^* = z^*$.i.e the fixed point of T is unique.

Theorem 3.4. Let X be a Banach space and K be a non-empty closed subset of X. Let T: K→X be a mapping having property (L) and satisfying Rothe's boundary condition. Let G be the disconnected graph associated with K. Also, there exists a constant $\alpha \in (0,1)$ such that T satisfies the inequality d(Tx,Ty)$\leq\alpha$.max{d(x,Tx),d(y,Ty)} for all x, y $\in$ K,

then T has a unique fixed point.

Proof. Choose $x_0 \in \partial K$. The Picard sequence is constructed as given in Definition 2.2. Let G be the disconnected graph associated with K defined as in Definition 2.1. We shall prove that the iterated sequence so constructed is a Cauchy sequence in K.

Choose some n$\in$I such that $x_n \in \partial K$. Then $x_n = y_j$for some j$\in$I.

$l\left(e_j\right) = d\left(y_j, Ty_j\right) = d\left(x_n, Tx_n\right).$

Using property (L), we have $d\left(x_n, Tx_n\right) \leq d\left(x_{n-1}, Tx_{n-1}\right).$

$$d\left(x_{n-1,}Tx_{n-1}\right) = d(Tx_{n-2}, Tx_{n-1}) \tag{2}$$

$d\left(Tx_{n-2}, Tx_{n-1}\right) \leq \alpha.max\{d\left(x_{n-2}, Tx_{n-2}\right), d\left(x_{n-1}, Tx_{n-1}\right)\}.$

Case (i). $max\left\{d\left(x_{n-2}, Tx_{n-2}\right), d\left(x_{n-1}, Tx_{n-1}\right)\right\} = d\left(x_{n-1}, Tx_{n-1}\right).$

Then, from (Equation (2)) we have,

$d\left(x_{n-1,}Tx_{n-1}\right) \leq \alpha d(x_{n-1}, Tx_{n-1}).$

$\Rightarrow \qquad \alpha \geq 1$. This is not true.

Hence, this case is not possible.

Case (ii). $max\left\{d\left(x_{n-2}, Tx_{n-2}\right), d\left(x_{n-1}, Tx_{n-1}\right)\right\} = d(x_{n-2}, Tx_{n-2}).$

$l(e_j) \leq \alpha d(x_{n-2}, Tx_{n-2}).$

Now, we discuss two cases: either $x_{n-2} \in K \quad (OR)\ x_{n-2} \in \partial K.$

Case (a). $x_{n-2} \in K.$

$l(e_j) \leq \alpha d(x_{n-2}, Tx_{n-2}),$

$=\alpha d(Tx_{n-3}, Tx_{n-2}),$

$\leq \alpha.\alpha.max\{d\left(x_{n-3}, Tx_{n-3}\right), d\left(x_{n-2}, Tx_{n-2}\right)\}.$

Arguing as in Case (i), the maximum cannot be $d\left(x_{n-2}, Tx_{n-2}\right)$. Hence, the maximum must be $d\left(x_{n-3}, Tx_{n-3}\right)$.

$\Rightarrow l(e_j) \leq \alpha^2 d(x_{n-3}, Tx_{n-3}).$

If $x_{n-3} \in \partial K$, then we proceed as in Case (b), otherwise assume that there are p elements in K that precedes x_n. Then, we have

$l\left(e_j\right) \leq \alpha^{p-1} d\left(x_{n-p-1}, Tx_{n-p-1}\right).$

In this case, $x_{n-p-1} \in \partial K$, and hence $y_{j-1} = x_{n-p-1}\ and\ e_{j-1} = \left(x_{n-p-1}, Tx_{n-p-1}\right).$

Therefore,

$l\left(e_j\right) \leq \alpha^{p-1} l\left(e_{j-1}\right) \to 0 \ \ as\ p \to \infty.$

$\Rightarrow\{l\left(e_j\right)\}_{j=1}^{\infty}$ is convergent to zero.

Case (b). $x_{n-2} \in \partial K.$

$l(e_j) \leq \alpha d\left(x_{n-2}, Tx_{n-2}\right) = \alpha l(e_{j-1})$, where $e_{j-1} = (x_{n-2}, Tx_{n-2}).$

Hence,

$l(e_j) \leq \alpha d\left(x_{n-2}, Tx_{n-2}\right),$

$=\alpha d\left(Tx_{n-3}, Tx_{n-2}\right),$

$\leq \alpha.\alpha max\{d(x_{n-3}, Tx_{n-3}), d(x_{n-2}, Tx_{n-2})\}.$

Arguing as above the maximum cannot be $d(x_{n-2}, Tx_{n-2})$, and hence maximum must be $d(x_{n-3}, Tx_{n-3})$.

$\therefore \quad l(e_j) \leq \alpha^2 d(x_{n-3}, Tx_{n-3}),$

$=\alpha^2 d(Tx_{n-4}, Tx_{n-3}),$

$\leq \alpha^2.\alpha.max\{d\left(x_{n-4}, Tx_{n-4}\right), d\left(x_{n-3}, Tx_{n-3}\right)\},$

$= \alpha^3.max\{d\left(x_{n-4}, Tx_{n-4}\right), d\left(x_{n-3}, Tx_{n-3}\right)\}.$

Since the maximum should be $d\left(x_{n-4}, Tx_{n-4}\right)$, we have,

$l\left(e_j\right) \leq \alpha^3 d\left(x_{n-4}, Tx_{n-4}\right).$

If $x_{n-4} \in \partial K$, then we proceed as in this case, otherwise if there are p elements in K (including $x_{n-3}\ and\ x_{n-4}$) that precedes x_{n-2} in K, then we have

$l(e_j) \leq \alpha^{p+1} d(x_{n-p-2}, Tx_{n-p-2})$, where $x_{n-p-2} \in \partial K.$

Hence, $y_{j-2} = x_{n-p-2}\ and\ e_{j-2} = (x_{n-p-2}, Tx_{n-p-2}),$

$\therefore \quad l(e_j) \leq \alpha^{p+1} l\left(e_{j-2}\right) \to 0\ as\ p \to \infty.$

$\Rightarrow\{l\left(e_j\right)\}_{j=1}^{\infty}$ is convergent to zero.

In both the cases, the sequence $\{l(e_j)\}_{j=1}^{\infty}$ is convergent to zero. By Lemma 2.1, the iterated sequence $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence in the closed subset K and hence it is convergent.

Let $\lim_{n\to\infty} x_n = x^*$.

To prove that x^* is the fixed point of T.

$d(x^*, Tx^*) \le d(x^*, x_n) + d(x_n, Tx^*)$ where $x_n \in K$,

$\le\ d(x^*, x_n) + d(Tx_{n-1}, Tx^*)$,

$\le\ d(x^*, x_n) + \alpha.max\{d(x_{n-1}, Tx_{n-1}), d(x^*, Tx^*)\}$.

If $max\{d(x_{n-1}, Tx_{n-1}), d(x^*, Tx^*)\} =\ d(x_{n-1}, Tx_{n-1})$, then

$d(x^*, Tx^*) \le d(x^*, x_n) + d(x_{n-1}, Tx_{n-1})$.

$\Rightarrow d(x^*, Tx^*) \le d(x^*, x_n) + d(x_{n-1}, x_n)$.

$As\ n \to \infty\ x_n \to x^*\ and\ \ d(x_{n-1}, x_n) \to 0$.

Hence, we have

$d(x^*, Tx^*) \to 0\ as\ n \to \infty$.

$\therefore\quad Tx^* = x^*$. i.e. x^* is the fixed point of T.

If $max\left\{d(x_{n-1}, Tx_{n-1}),\ d(x^*, Tx^*)\right\} =\ d(x^*, Tx^*)$, then

$d(x^*, Tx^*) \le d(x^*, x_n) + \alpha d(x^*, Tx^*)$.

$\Rightarrow d(x^*, Tx^*) \le \frac{1}{1-\alpha} d(x^*, x_n)$.

$As\ n \to \infty,\ x_n \to x^*$ and we obtain $d(x^*, Tx^*) = 0$. Hence, $Tx^* = x^*$. i.e. x^* is the fixed point of T. Now, to prove the uniqueness of the fixed point. If possible, let z^* be any other fixed point of T.

Then $d(x^*, z^*) = d(Tx^*, Tz^*) \le \alpha.max\{d(x^*, Tx^*), d(z^*, Tz^*)\}$.

$\Rightarrow\quad d(x^*, z^*) \le \alpha.0$.

$\Rightarrow d(x^*, z^*) = 0$.

$\Rightarrow z^* = x^*$.

Hence, the fixed point of T is unique.

4 Conclusion

In this study, the fixed point theorems are approached using the edge sequence of disconnected graphs defined for the space under consideration. The fixed point theorems for non-self mappings by Banach, Kannan, Chatterjea, and Bianchini are proved using the new methodology, and it is observed that the complexity involved in the proofs is considerably reduced. The same procedure can be adopted and extended to other fixed point results of non-self mappings.

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