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RESEARCH ARTICLE

# Numerical Solution of Subdiffusion Bioheat equation with Single Phase lag

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## Abstract

**Objectives:** The aim of this study is to investigate thermal beheviour of living tissues, for this we use subdiffusion bioheat equation with single phase lag model. **Methods:** The Crank-Nicolson finite difference scheme is used for subdiffusion bioheat equation with single phase lag by using Caputo fractional derivative. A Python program is used to calculate a numerical solution, which is then visually depicted through graphical representation. **Finding:** We discuss significance of heat flux time relaxation parameter and fractional parameter. To explore our findings, we conducted comparative analysis of subdiffusion bioheat equation with single phase lag model with classical bioheat model and fractional bioheat model. Moreover, We discussed stability and convergence of developed scheme. **Novelty:** The bio heat equation can be effectively addressed using the Crank-Nicolson finite difference method of fractional-order. Developed Python program provides effective tool for getting numerical solution of fractional differential equation. Single phase lag model is useful for short time heating process.

**Keywords:** Fractional Differential Equation; Caputo fractional derivatives; Subdiffusion bioheat equation; Single Phase Lag; Python etc

## 1 Introduction

In recent times, numerous researchers have expressed keen interest in investigating thermal phenomena and temperature dynamics within living tissues. This area of study holds significant relevance in contemporary clinical treatments and medical applications, including thermal disease diagnostics, cancer hyperthermia, cryosurgery, and more. The prevalent issue of skin burns resulting from heat sources in both daily life and industrial settings has underscored the importance of exploring bio heat transfer in living tissues, making it a compelling and actively researched topic. Establishment of thermal protections for various purposes, evaluation of skin burns and design of thermal treatment clinical equipment's can be assist by study of biothermal properties of living tissue. In this direction, Harry H. Penne's first formulated a mathematical model for the temperature in the resting human forearm in 1948 [(1)], which is now known as

the Penne's Bio heat model, given as below:

$$\rho\, C\frac{\partial T\,(x,t)}{\partial t} = k\frac{\partial^2 T\,(x,t)}{\partial x^2} + Q,\;\; x \in (0,L]\,,\; t \in (0,T] \tag{1}$$

Where $T$ =temperature of tissue, $t$ =time parameter, $\rho$ =density of tissue, $C$ =specific heat of tissue, $x$ =spatial parameter, $k$ =thermal conductivity and $Q$ represents the volumetric heat generated through metabolism and blood perfusion as,

$$Q = G_B C_B\,(T_B - T) + Q_m$$

Where $T_B$ =temperature of artillery, $G_B$ =blood perfusion rate, $Q_m$ =metabolic heat source, $C_B$ =specific heat of blood.

After Pennes, many researchers like Wulff, Klinger, Weinbaum-Jiji and Chen-Holmes developed various bioheat models which are tailored versions of Pennes model[(1)]. Amongst all these model, Pennes model is most simplest and realistic model, due to this reason our study is also based on this model. All this models are based on Fourier law of heat conduction, given by

$$q = -k\frac{\partial\, T}{\partial\, x} \tag{2}$$

where $k$ is thermal conductivity. Note that Fourier law consider an infinite speed of heat flux, which is applicable in long time heating process such as hyperthermia cancer therapy, microwave, cryosurgical ablation and radiofrequency. Nevertheless, during rapid heating processes, particularly in actual systems, the propagation of heat flux occurs at a finite speed. This gives rise to the development of thermal wave bio heat models utilizing the Maxwell- Cattaneo[(2)] approach, incorporating a time lag in heat flux, commonly referred to as the single-phase lag approach. Hongyue Zhou, Pu Li, Yuming Fang[(3)], O. Nikan, Z. Avazzadeh, J.A. Tenreiro Machado[(4)], P. Goudarzi, A. Azimi[(5)] discussed this approach in literature. According to this approach, the extension of the Fourier law is modelled with a single time phase lag in the flux is

$$q\,(x,t+\tau) = q\,(x,t) + \tau\frac{\partial\, q}{\partial\, t} = -k\frac{\partial\, T}{\partial\, x} \tag{3}$$

Here, $\tau$ (>0) represents the parameter for heat flux relaxation time. In biological tissues, these relaxation times typically fall within the range of 10 to 30 seconds[(1)]. In accordance with this hypothesis, the temperature gradient at a specific point $x$ and time $t$ results in a heat flux vector at the same point $x$ but at a later time $t+\tau$.

Last few years, researchers modified many physical models by using fractional calculus, for the purpose of study of effect of fractional parameter $\alpha$. Applications of fractional differential equation have been found in many areas, like engineering, physics, economics and biology etc. Caputo time fractional derivative of order $\alpha$ have been played important role in modelling of fractional differential equations[(6)], as it behave like generalization of integer order derivative. In this scenario, fractional bioheat model is one of the interesting topic of research. In the year 2019, Dhaigude and Takale[(7)] developed fractional bioheat transfer model with assumption that the thermal diffusivity coefficient varies with respect to spatial coordinate, given as

$$\frac{\partial^\alpha \theta}{\partial t^\alpha} + a\theta - b\frac{\partial}{\partial x}\left[k(x)\frac{\partial \theta}{\partial x}\right] = 0 \tag{4}$$

where $0 < \alpha \leq 1$ and $x \in [0,L]$, $t \in [0,T]$

$$\textit{Initial condition}:\;\; \theta\,(x,0) = 0, 0 < x < L \tag{5}$$

$$\textit{Boundary conditions}:\;\; \theta\,(0,t) = \theta_0, \theta_x\,(L,t) = 0,\;\; t \geq 0 \tag{6}$$

by using

$$\theta = T - T_B - \frac{Q_m}{G_B C_B},\;\; a = \frac{1}{\rho\, C} G_B C_B,\;\; b = \frac{1}{\rho\, C} > 0$$

in PBT model (1). Where $\frac{\partial^\alpha \theta}{\partial t^\alpha}$ is Caputo fractional derivative[(4)] of order $\alpha$, such that $0 < \alpha \leq 1$ given as

$${}_0^C D_t^\alpha f(t) = {}_0\, I_t^{1-\alpha} D^1 f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} f'(s) ds \tag{7}$$

We used above model for comparison purpose and test relaibility of solution of fractional single phase lag model in later section by assuming $k(x) = 0.5$. Furthermore, fractional Pennes bioheat models is developed by Damor in the year 2013 and Ferras in the year 2015. In 2014, Ezzat developed new fractional Pennes bioheat model with single phase lag approach discussed in (3). They applied fractional Taylor series expansion on Catteneo approach as

$$q(x,t+\tau) = q + \frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha q}{\partial t^\alpha} = -k\frac{\partial T}{\partial x} \tag{8}$$

where $\frac{\partial^\alpha q}{\partial t^\alpha}$ is Caputo time fractional derivative. Finally, fractional bioheat equation with single phase lag developed is given as

$$\rho C\frac{\partial}{\partial t}\left[T + \frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha T}{\partial t^\alpha}\right] = k\frac{\partial^2 T}{\partial x^2} + \left[Q + \frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha Q}{\partial t^\alpha}\right] \tag{9}$$

where $Q = G_B C_B(T_B - T) + Q_m$, $0 < \alpha \leq 1$ and $\tau$ is heat flux relaxation time parameter. For $\tau = 0$, above equation transforms to classical Pennes bioheat equation (1).

Reserchers used Laplace Fourier transform technique[2], finite difference method[7],[4], and Finite element/volume method for solving classical and Fractional bio heat equation. In the year 2019, Patil applied finite difference method to find solution of bio-heat equation[8]. Urszula Siedlecka, solved of the time-fractional single-phase-lag heat model in Laplace transform domain which is restricted to certain initial conditions[9]. Urszula Siedlecka and Mariusz Ciesielski find solution of Catttaneo heat transfer equation by using Laplace transform technique by using Dirac delta function as initial condition.[2]. Ibrahim Abbas use ADI Finite difference scheme to obtain solution of 2-D single Phase lag bio heat model[10]. Zhang discussed solution of three phase lag model by using variable separation method[11]. P. Goudarzi discussed super-diffusion fractional single phase lag model using a finite volume method[5]. Yanli Qiao, Xiaoping Wang, Haitao Qi, Huanying Xu discussed fractional dual phase lag heat conduction model by using Laplace-Fourier technique and explicit finite difference method[9]. O. Nikan, Z. Avazzadeh, J.A. Tenreiro Machado discussed time fractional Catttaneo heat equation by using local mesh less RBF-PU method[4].

However as per the authors' knowledge, there are limited references on the fractional single-phase lag bio heat model, making them quite scarce. Additionally, despite numerous proposed analytical approaches, solutions to fractional differential equations often manifest as intricate series with special functions, posing challenges in numerical computation and application. In this paper we focused on use of Crank-Nicolson finite difference method which is unconditionally stable. It allows to increase time step size that can lead to significant decreases total computational time for numerical problems. Now days Python program is one of the powerful tool in research because it has many inbuilt libraries like NumPy, SciPy etc. which is helpful for numerical and graphical computation, linear algebra and data analysis. We also develop subdiffusive approach by considering fractional parameter $\alpha$ as $0 < \alpha < 1$. It is important to note that time fractional derivative introduce subdiffusion without altering the properties of material density and specific heat[12],[13].

## 2 Methodolgy

### 2.1 Finite difference scheme

We consider one dimensional form of fractional bioheat equation with single phase lag by using subdiffusive approach is obtained by substituting $\theta = T - T_0$ in equation (9), where $T_0$ is initial steady state temperature and considering $0 < \alpha < 1$ which is given as

$$k\frac{\partial^2\theta}{\partial x^2} = \left(G_B C_B + \rho C\frac{\partial}{\partial t}\right)\left[\theta + \frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha\theta}{\partial t^\alpha}\right] \tag{10}$$

where $x \in [0,L]$, $t \in [0,T]$

$$\textit{Initial condition}: \ \theta(x,0) = 0, \theta_t(x,0) = 0, 0 < x < L \tag{11}$$

$$\textit{Boundary conditions}: \ \theta(0,t) = \theta_0, \theta_x(L,t) = 0, \ \ t \geq 0 \tag{12}$$

Above equation can be rewrite as

$$\frac{\partial}{\partial t}[\frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha\theta}{\partial t^\alpha}] + \frac{G_B C_B}{\rho\ C}\frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha\theta}{\partial t^\alpha} = \frac{k}{\rho\ C}\frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial t} - \mu\ \theta \tag{13}$$

To formulate the Crank- Nicolson finite difference approach for subdiffusion bioheat equation with single phase lag(13), we define $x_i = i\Delta x,\ i = 0,1,\ldots,M$ and $t_n = n\Delta t,\ n = 0,1,\ldots,N$; where $\Delta x = \frac{L}{M}$, a space step size and $\Delta t = \frac{T}{N}$, a time step size. Consider $\theta(x_i,t_n)$is the exact solution and $\theta_i^n$ is the numerical solution at mesh point $(x_i,t_n)$where, $i = 0,1,\ldots,M$ and $n = 0,1,\ldots,N$.

we discretize derivatives by using forward difference in time and central diffrence in space as follows[13],

$$\left(\frac{\partial\theta}{\partial x}\right)_n^i = \frac{\theta_{i+1}^n - \theta_{i-1}^n}{2(\Delta x)} \tag{14}$$

$$(\frac{\partial^2\theta}{\partial x^2})_n^i = \frac{\theta_{i-1}^{n+1} - 2\theta_i^{n+1} + \theta_{i+1}^{n+1} + \theta_{i-1}^n - 2\theta_i^n + \theta_{i+1}^n}{2(\Delta x)^2} \tag{15}$$

$$(\frac{\partial\theta}{\partial t})_n^i = \frac{\theta_i^{n+1} - \theta_i^n}{(\Delta t)} \tag{16}$$

$$(\frac{\partial^\alpha\theta}{\partial t^\alpha})_n^i = \frac{1}{(\Delta t)^\alpha\Gamma(2-\alpha)}\sum_{j=0}^{n}[\theta_i^{n-j+1} - \theta_i^{n-j}]d_j \tag{17}$$

The term $\frac{\partial}{\partial t}[\frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha\theta}{\partial t^\alpha}]$ can be discretized as below

$$\frac{\partial}{\partial t}[\frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha\theta}{\partial t^\alpha}]_n^i = \frac{\tau^\alpha}{\alpha!}\frac{1}{(\Delta t)^{\alpha+1}\Gamma(2-\alpha)}\sum_{j=0}^{n}[\theta_i^{n-j+1} - 2\theta_i^{n-j} + \theta_i^{n-j-1}]d_j \tag{18}$$

where

$$d_j = (j+1)^{1-\alpha} - (j)^{1-\alpha}, j = 0,1,2,\ldots,n \tag{19}$$

Thus by using (15)-(18) in (13), we get

$$\frac{\tau^\alpha}{\alpha!}\sum_{j=0}^{n}\left[\frac{\theta_i^{n-j+1} - 2\theta_i^{n-j} + \theta_i^{n-j-1}}{(\Delta t)^{\alpha+1}\Gamma(2-\alpha)}\right]d_j + \frac{G_BC_B}{\rho C}\frac{\tau^\alpha}{\alpha!}\sum_{j=0}^{n}\left[\frac{\theta_i^{n-j+1} - \theta_i^{n-j}}{(\Delta t)^\alpha\Gamma(2-\alpha)}\right]d_j$$
$$= \frac{k}{2\rho C(\Delta x)^2}\left[\theta_{i-1}^{n+1} - 2\theta_i^{n+1} + \theta_{i+1}^{n+1} + \theta_{i-1}^n - 2\theta_i^n + \theta_{i+1}^n\right] - \left[\frac{\theta_i^{n+1} - \theta_i^n}{\Delta t}\right]$$
$$-\frac{G_BC_B}{\rho C}\left[\frac{\theta_i^{n+1} + \theta_i^n}{2}\right]$$

implies

$$r_2\sum_{j=0}^{n}[\theta_i^{n-j+1} - 2\theta_i^{n-j} + \theta_i^{n-j-1}]d_j + 2r_1r_2\sum_{j=0}^{n}[\theta_i^{n-j+1} - \theta_i^{n-j}]d_j$$

$$= r[\theta_{i-1}^{n+1} - 2\theta_i^{n+1} + \theta_{i+1}^{n+1} + \theta_{i-1}^n - 2\theta_i^n + \theta_{i+1}^n] - [\theta_i^{n+1} - \theta_i^n] - r_1[\theta_i^{n+1} + \theta_i^n]$$

where $r = \frac{k\,\Delta t}{2\rho\,C\,(\Delta x)^2}, r_1 = \frac{G_BC_B\Delta t}{2\,\rho\,C}, r_2 = \frac{\tau^\alpha}{\alpha!(\Delta t)^\alpha\Gamma(2-\alpha)}$

We can rewrite as

$$-r\theta_{i-1}^{n+1} + [r_2(1+2r_1) + 2r + 1 + r_1]\theta_i^{n+1} - r\theta_{i+1}^{n+1}$$
$$= r\theta_{i-1}^n + [r_2(2+2r_1) - 2r + 1 - r_1]\theta_i^n + r\theta_{i+1}^n - r_2\theta_i^{n-1}$$
$$-r_2\sum_{j=1}^{n}\left[(1+2r_1)\theta_i^{n-j+1} - (2+2r_1)\theta_i^{n-j} + \theta_i^{n-j-1}\right]d_j$$

We can approximate initial condition (11) as

$$\theta_i^0 = 0, and\ \theta_i^{-1} = \theta_i^0\ i = 1,2,\ldots,M.$$

and boundary condition(12) are approximated as

$$\theta_0^n = \theta_0, \theta_{M+1}^n = \theta_{M-1}^n\ n = 0,1,2,\ldots,N.$$

Finally, discretized form of equation (13) and hence (10)-(12) is as follows,

$$-r\ \theta_{i-1}^1 + \pi\ \theta_i^1 - r\ \theta_{i+1}^1 = r\ \theta_{i-1}^0 + \Lambda\theta_i^0 + r\ \theta_{i+1}^0 \tag{20}$$

$for\ n = 0$

$$-r\ \theta_{i-1}^2 + \pi\ \theta_i^2 - r\ \theta_{i+1}^2 = r\ \theta_{i-1}^1 + \Lambda_1\theta_i^1 + r\ \theta_{i+1}^1 - [r_2 - r_2(1+2r_1)d_1]\theta_i^0 \tag{21}$$

$for\ n = 1$

$$-r\ \theta_{i-1}^{n+1} + \pi\ \theta_i^{n+1} - r\ \theta_{i+1}^{n+1}$$

$$= r\ \theta_{i-1}^n + \Lambda_1\theta_i^n + r\ \theta_{i+1}^n - H\ \theta_i^{n-1} - r_2\sum_{j=2}^{n-1}[d_{j-1} - (2+2r_1)d_j + (1+2r_1)d_{j+1}]\theta_i^{n-j}$$
$$-[r_2 d_{n-1} - r_2\,(1+2r_1)\,d_n]\,\theta^0 \tag{22}$$

$for\ n \geq 2$
where

$$\pi = [r_2\,(1+2r_1) + 2r + 1 + r_1]\,, \Lambda = [r_2\,(1+2r_1) - 2r + 1 - r_1]$$

$$\Lambda_1 = [\Lambda + r_2 - r_2(1+2r_1)d_1], \quad H = [r_2 - r_2(2+2r_1)d_1 + r_2(1+2r_1)d_2]$$

$$initial\ condition: \theta_i^0 = 0, \theta_i^{-1} = \theta_i^0\ \ i = 1,2,\ldots,M. \tag{23}$$

$$boundary\ \ condition: \theta_0^n = \theta_0, \theta_{M+1}^n = \theta_{M-1}^n\ n = 0,1,2,\ldots,N. \tag{24}$$

Therefore, matrix form of discretized scheme (20)-(24) is as follows

$$A\ \theta^1 = B\ \theta^0\ + C, for\ n = 0 \tag{25}$$

$$A\ \theta^2 = F\ \theta^1\ - [r_2 - r_2(1+2r_1)d_1]\theta^0 + C, for\ n = 1 \tag{26}$$

$$A\ \theta^{n+1} = F\ \theta^n - H\ \theta^{n-1}\ - r_2\sum_{j=2}^{n-1}\left(d_{j-1} - (2+2r_1)\,d_j + (1+2r_1)\,d_{j+1}\right]\theta_i^{n-j}$$
$$-[r_2 d_{n-1} - r_2\,(1+2r_1)\,d_n]\,\theta^0 + C$$

$$\text{for } n \geq 2 \tag{27}$$

$$initial\ condition: \theta_i^0 = 0, \theta_i^{-1} = \theta_i^0 \;\; i = 1,2,\ldots,M \tag{28}$$

$$boundary\ condition: \theta_0^n = \theta_0, \theta_{M+1}^n = \theta_{M-1}^n \; n = 0,1,2,\ldots,N. \tag{29}$$

where $\theta^n = (\theta_1^n, \theta_2^n, \ldots, \theta_M^n)^T$, $C = (2r_1\theta_0, 0, 0, \ldots, 0)^T$ are the constant matrix, and

$$A = \begin{pmatrix} \pi & -r & & & & & \\ -r & \pi & -r & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -r & \pi & -r & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -r & \pi & -r \\ & & & & & -2\,r & \pi \end{pmatrix}, B = \begin{pmatrix} \Lambda & r & & & & & \\ r & \Lambda & r & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & r & \Lambda & r & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & r & \Lambda & r \\ & & & & & 2\,r & \Lambda \end{pmatrix}$$

$$F = \begin{pmatrix} \Lambda_1 & r & & & & & \\ r & \Lambda_1 & r & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & r & \Lambda_1 & r & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & r & \Lambda_1 & r \\ & & & & & 2\,r & \Lambda_1 \end{pmatrix}$$

are tri-diagonal matrices.

## 2.2 Stability

**Lemma 1** If

$$A = \begin{pmatrix} a_1 & a_2 & & & & & \\ a_3 & a_1 & a_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & a_3 & a_1 & a_2 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & a_3 & a_1 & a_2 \\ & & & & & a_3 & a_1 \end{pmatrix}$$

is tri-diagonal matrix of order $M$ with $a_1, a_2$ and $a_3$are may be real or complex numbers , then its eigenvalues are[13]

$$\lambda_i = a_1 + 2\sqrt{a_2 a_3}cos(\frac{i\pi}{M+1})$$

**Lemma 2** If $\lambda_i\ (A)$, $i = 1,2,\ldots,M$ denotes eigen values of matrix $A$, then
i) $\lambda_i(A) \geq 1$, ii)$\| A^{-1} \|_2 \leq 1$, where $\| \cdot \|_2$ is second norm of matrix[14].
Proof:Let $\lambda$ is eigen value of squre matrix $A$
Every eigenvalue $\lambda$ should lies in at least one of the following circle

$$|\lambda - a_{ij}| \;\leq \sum_{i=1, i\neq j}^{M} a_{ij}, \;\; j = 1,2,3, \ldots M \tag{30}$$

as per Gerschgorin's circle theorem.
Thus every eigen value of square matrix $A$ satisfy at least one of the following inequality

$$|\lambda| \;\leq \sum_{i=1}^{M} |a_{ij}| \tag{31}$$

$$|\lambda| \geq |a_{ij}| - \sum_{i=1, i\neq j}^{M} |a_{ij}| \tag{32}$$

using inequality (32) to for the matrix $A$, then

$$|\lambda_1(A)| \geq |(\pi) - r| = \pi - r \geq 1$$

$$|\lambda_2(A)| \geq |(\pi) - r - r| = \pi - 2r \geq 1$$

$$|\lambda_3(A)| \geq |(\pi) - r - r| = \pi - 2r \geq 1$$

$$\vdots$$

$$|\lambda_M(A)| \geq |(\pi) - 2r,| = \pi - 2r \geq 1$$

Thus each eigen values of matrix $A$ are always greater than or equal to 1

Now, to prove that second norm of inverse of matrix $A$ are always less than or equal to 1, We have,

$$\| A \|_2 = max_{1 \leq j \leq n} |\lambda_j(A)|$$

Therefore, from discussion, we get $\| A \|_2 \geq 1$, this implies that $\| A^{-1} \|_2 \leq 1$

That is , second norm of inverse of matrix $A$ are always less than or equal to 1

Hence proved.

**Lemma 3** The discretized Crank-Nicolson finite difference method of fractional order (20)-(24) is solvable unconditionally for every time step[13].

Proof: We can observed that in first row of matrix $A$ is diagonally dominant since $\pi > |-r|$, also in last row $\pi > |-2r|$, thus last row is also diagonally dominant.

Now, in remaining rows rows $\pi > |-r| + |-r|$, implies diagonal dominance of each remaining row.

Hence matrix $A$ is strictly diagonally dominant that implies matrix $A$ is invertible.

Which is sufficient to prove solvabilty of developed scheme.

**Lemma 4** If eigenvalues of matrix $B$ are $\lambda_i(B)$ and eigenvalues of matrix $F$ are $\lambda_i(F), i = 1, 2, \ldots, M$, then

i)$|\lambda_i(B)| \leq 1, |\lambda_i(F)| \leq 1, \quad i = 1, 2, \ldots, M$

ii)$\| B_2 \|_2 \leq 1, \| F_2 \|_2 \leq 1, \quad i = 1, 2, \ldots, M$

**Theorem 1** The solution of Crank-Nicolson finite difference method of fractional order (20)-(24) for subdiffusion bio heat equation with single phase lag (10)-(12) is unconditionally stable.

Proof: To prove, unconditional stability of develeped scheme, we have to prove that

$$\| \theta^n \|_2 \leq K \| \theta^0 \|_2$$

where $K$ is any positive number which is not depending on $x$ or $t$.

For $n = 1$

$$\theta^1 = A^{-1} B\theta^0 + A^{-1} C$$

$$\therefore \| \theta^n \|_2 \leq \| A^{-1} \|_2 \| B \|_2 \| \theta^0 \|_2 + \| A^{-1} \|_2 \| C \|_2$$

$$\leq \| \theta^0 \|_2 + \| C \|_2$$

$$\leq \| \theta^0 \|_2 + K_1 \| \theta^0 \|_2, where \| C \|_2 = K_1$$

$$= K \| \theta^0 \|_2$$

Hence result is hold for $n = 1$.

For $n \leq m$, let us assume that $\| \theta^m \|_2 \leq K \| \theta^0 \|_2$

Now, for $n = m+1$

$$\theta^{m+1} = A^{-1}F\theta^m - A^{-1}H\theta^{m-1} - A^{-1}r - 2\sum_{j=2}^{m-1}\left[d_{j-1} - (2+2r_1)d_j + (1+2r_1)d_{j+1}\right]\theta_i^{m-j}$$
$$-A^{-1}[r_2 d_{n-1} - r_2(1+2r_1)d_n]\theta^0 + A^{-1}C$$

$$\therefore \| \theta^{m+1} \|_2 \leq \| \theta^m \|_2 - H \| \theta^{m-1} \|_2 - r_2 \sum_{j=2}^{m-1}[d_{j-1} - (2+2r_1)d_j + (1+2r_1)d_{j+1}] \| \theta^{m-j} \|_2$$

$$-[r_2 d_{n-1} - r_2(1+2r_1)d_n] \| \theta^0 \|_2 - \| C \|_2$$
$$= \|\theta^m\|_2 - H\left\|\theta^{m-1}\right\|_2 - r_2\left[(d_1 - (2+2r_1)d_2 + (1+2r_1)d_3)\left\|\theta^{m-2}\right\|_2\right.$$
$$+ (d_2 - (2+2r_1)d_3 + (1+2r_1)d_4)\left\|\theta^{m-3}\right\|_2$$
$$+ \ldots + (d_{n-2} - (2+2r_1)d_{n-1} + (1+2r_1)d_n)\left\|\theta^1\right\|_2\left.\right]$$
$$-[r_2 d_{n-1} - r_2(1+2r_1)d_n]\|\theta^\circ\|_2 - \|C\|_2$$
$$= K_2\left\|\theta^0\right\|_2 - HK_3\left\|\theta^0\right\|_2 - r_2[(d_1 - (2+2r_1)d_2 + (1+2r_1)d_3)$$
$$+ (d_2 - (2+2r_1)d_3 + (1+2r_1)d_4)$$
$$+ \ldots\ldots + (d_{n-2} - (2+2r_1)d_{n-1} + (1+2r_1)d_n)]K_4\|\theta^\circ\|_2$$
$$-[r_2 d_{n-1} - r_2(1+2r_1)d_n]K_4\|\theta^\circ\|_2 - \|C\|_2$$
$$\leq \left[K_2 - HK_3 - r_2[d_1 - (1+2r_1)d_2]\right]\left\|\theta^0\right\|_2 - K_1\left\|\theta^0\right\|_2$$
$$\leq K\|\theta^\circ\|_2$$

Hence, by induction, for all $n$, we have $\| \theta^n \|_2 \leq K \| \theta^0 \|_2$.where $K$ is any positive number which is not depending on $x$ or $t$. This implies that unconditional stability of developed scheme.

Hence proved.

## 2.3 Convergence

Let $\Omega : [0,L] \times [0,T]$, is region .

Consider, exact solution of the subdiffusion bioheat equation with single phase lag (10)-(12) at time level $t_n$ given by vector, $\overline{\theta}^n = (\overline{\theta}(x_0,t_n), \overline{\theta}(x_1,t_n), \overline{\theta}(x_2,t_n), \ldots, \overline{\theta}(x_M,t_n))^T$, of size M+1.

Let truncation error, representing by vector $\tau^n = (\tau_1^n, \tau_2^n, \tau_3^n, \ldots, \tau_M^n)^T$ at time level $t_n$. Since $\overline{\theta}^n$ is the exact solution of the equation (10)-(12) , we have

$$A\,\overline{\theta}^1 = B\,\overline{\theta}^0 + C, for\ n = 0 \tag{33}$$

$$A\,\overline{\theta}^2 = F\,\overline{\theta}^1 - [r_2 - r_2(1+2r_1)d_1]\theta^0 + C, for\ n = 1 \tag{34}$$

$$A\bar{\theta}^{n+1} = F\bar{\theta}^n - H\theta^{n-1} - r_2\sum_{j=2}^{n-1}\left[d_{j-1} - (2+2r_1)d_j + (1+2r_1)d_{j+1}\right]\theta_i^{n-j}$$
$$-[r_2 d_{n-1} - r_2(1+2r_1)d_n]\theta^0 + C \quad \text{for } n \geq 2 \tag{35}$$

**Lemma 5** Following conditions are satisfied by the coefficient $d_j,\ j=0,1,2,3,\ldots$

i) $d_j>0$ , and $d_0=1$ ii) $d_j>d_{j+1}$ and $d_j\to 0$ as $j\to\infty$

**Theorem 2** The Crank-Nicolson finite difference method of fractional order (20)-(24) for subdiffusion bio heat equation with single phase lag (10)-(12) is unconditionally convergent.

Proof: We set,the error vector in the solution at time level $t_n$ as $E^n=\overline{\theta}^n-\theta^n=(e_1^n,e_2^n,e_3^n,\ \ldots,e_M^n)^T$. Furthermore, we assume that

$$|e_l^n|=\max_{1\le i\le M}|e_i^n|=\|E^n\|_\infty$$

$$|\tau_l^n|=\max_{1\le i\le M}|\tau_i^n|, for\ l=1,2,3,\ldots$$

Then, using equation (20), we obtain

$$\left|e_l^1\right|=\left|-re_{i-1}^1+\pi e_i^1-re_{i+1}^1\right|$$

$$\le r\left|e_{i-1}^0\right|+\Lambda|e_i^0|+r\left|e_{i+1}^0\right|+|\tau_l^1|$$

$$\le (r+\Lambda+r)\max_{1\le i\le M}|e_i^0|+\max_{1\le i\le M}|\tau_i^1|$$

$$\le \left\|E^0\right\|_\infty+\max_{1\le i\le M}\left|\tau_i^1\right|$$

$$\therefore \left\|E^1\right\|_\infty\le\left\|E^0\right\|_\infty+\max_{1\le i\le M}\left|\tau_i^1\right|$$

Also, from (21), we get

$$\left|e_l^2\right|=\left|-re_{i-1}^2+\pi e_i^2-re_{i+1}^2\right|$$

$$\le r\left|e_{i-1}^1\right|+\Lambda_1\left|e_i^1\right|+r\left|e_{i+1}^1\right|-\left(r_2-r_2\left(1+2r_1\right)d_1\right)\left|e_i^0\right|+\left|\tau_l^1\right|$$

$$\le (r+\Lambda_1+r)\left|e_i^1\right|-\left(r_2-r_2\left(1+2r_1\right)d_1\right)\left|e_i^1\right|+\max_{1\le i\le M}\left|\tau_i^1\right|$$

$$\therefore \left\|E^2\right\|_\infty\le\left\|E^1\right\|_\infty+\max_{1\le i\le M}\left|\tau_i^1\right|$$

From equation (21), we get

$$\left|e_l^{m+1}\right|=\left|-re_{i-1}^{m+1}+\pi e_i^{m+1}-re_{i+1}^{m+1}\right|$$

$$\le r,\left|e_{i-1}^m\right|+\Lambda_1\left|e_i^m\right|+r\left|e_{i+1}^m\right|-H\left|e_i^{m-1}\right|-r_2\textstyle\sum_{j=2}^{m-1}\left[d_{j-1}-\left(2+2r_1\right)d_j+\left(1+2r_1\right)d_{j+1}\right]\left|e_i^{m-j}\right|$$

$$-\left[r_2d_{n-1}-r_2\left(1+2r_1\right)d_n\right]\left|e_i^0\right|+\left|\tau_l^{m+1}\right|$$

$$\le r,\left|e_{i-1}^m\right|+\Lambda_1\left|e_i^m\right|+r\left|e_{i+1}^m\right|-H\left|e_i^{m-1}\right|-r_2\left[d_1-\left(2+2r_1\right)d_2+\left(1+2r_1\right)d_3\right]\left|e_i^{m-2}\right|$$

$$-r_2[d_2-(2+2r_1)d_3+(1+2r_1)d_4]|e_i^{m-3}|-\cdots-r_2[d_{m-2}-(2+2r_1)d_{m-1}+(1+2r_1)d_m]|e_i^1|$$

$$-[r_2d_{n-1}-r_2(1+2r_1)d_n]|e_i^0|+|\tau_l^{m+1}|$$

$$\le (r+\Lambda_1+r)|e_i^m|\ -H\left|e_i^m\right|-r_2[(d_1-(2+2r_1)\,d_2+(1+2r_1)\,d_3)$$

$$+(d_2-(2+2r_1)d_3+(1+2r_1)d_4)+\cdots+(d_{m-2}-(2+2r_1)d_{m-1}+(1+2r_1)d_m)]|e_i^m|$$

$$-[r_2d_{n-1}-r_2(1+2r_1)d_n]|e_i^m|+\max_{1\le i\le M}|\tau_i^{m+1}|$$

$$=\|E^m\|_\infty+\max_{1\le i\le M}|\tau_i^{m+1}|$$

This is true for every $m$, therefore we have

$$\| E^{m+1} \|_\infty \le \| E^m \|_\infty + \max_{1 \le i \le M} |\tau_i^{m+1}|$$

Hence, by induction, we get

$$\| E^{n+1} \|_\infty \le \| E^n \|_\infty + \max_{1 \le i \le M} |\tau_i^{n+1}|$$

$$As \| E^0 \|_\infty = 0 \; implies \| E^n \|_\infty = 0, Therefore \| E^n \|_\infty \le \max_{1 \le i \le M} |\tau_i^{n+1}|.$$

$Since \max_{1 \le i \le M} |\tau_i^{n+1}| \; \tau_i^{n+1} \to 0 as (\Delta x, \Delta t) \to (0,0), implies that \| E^n \|_\infty \to 0$ uniformly on $\Omega$ an $(\Delta x, \Delta t) \to (0,0)$.

Therefore, this shows that for any $x$ and $t$, as $(\Delta x, \Delta t) \to (0,0)$, the vector $\theta^n$ converges to $\overline{\theta}^n$. Hence, this complete the proof.

## 3 Results and Discussion

We developed Python program to find tempeature distribution $\theta$ by using values as given[12] :$\rho = 1050 \; \frac{kg}{m^3}$, $C = 3770 \; \frac{J}{kg^0C}$ , $C_B = 3770 \; \frac{J}{kg^0C}$ , $k = 0.5 \frac{W}{^0C}$ , $L = 0.01208 \; mm$, $\theta_0 = 37^0C$, $\tau = 20 \; S$ , $G_B = 0.5 \; \frac{kg}{m^3}$.

### 3.1 Results are compared with anlytical and numerical calculations as shown in figure.

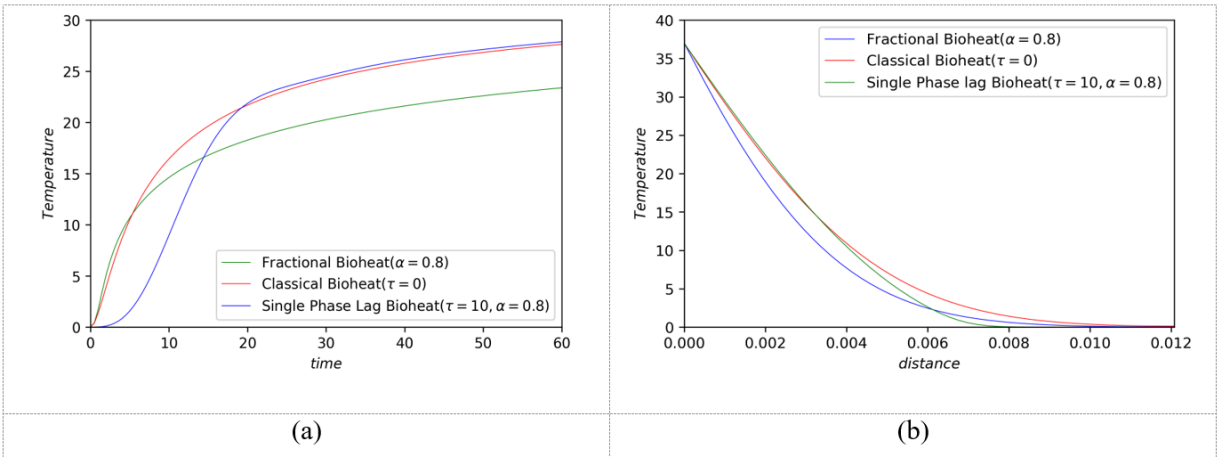


**Fig 1.** (a)Temporal evolution of temperature distribution in the skin under instatneous surface heating for classical , fractional and single phase lag theory for $x = 0.001208mm$ (b) Temperature distribution across the tissue depth from the skin subjected to the instantaneous surface heating for classical, fractional and single phase lag theory for $t = 60sec$

In Figure Figure 1 (a), the graph represents the temporal evolution of temperature elevation. Notably, it is observed that in the case of the classical and fractional models, the temperature elevation remains approximately same. However, in the

single phase lag model, the temperature elevation is lower than that of the classical and fractional models until reaching a time of 20*seconds*, after which it matches the temperature trend seen in the classical model. It is observed that elevated view of temperature is delayed by few seconds as compared with classical and fractional bioheat model.

In Figure 1 (b), the variation in temperature with tissue depth from the skin surface is depicted for the classical ($\tau = 0$), fractional ($\alpha = 0.8$), and single phase lag ($\tau = 10, \alpha = 0.8$) theories. Under the influence of instateneous surface heating on the skin the temperature within the body tissue gradually diminishes. Remarkably, it is observed that the tissue temperature diminishes rapidly in the single phase lag model (at $x = 0.007mm$), in contrast to the classical (at $x = 0.0010mm$) and fractional (at $x = 0.009mm$) models.

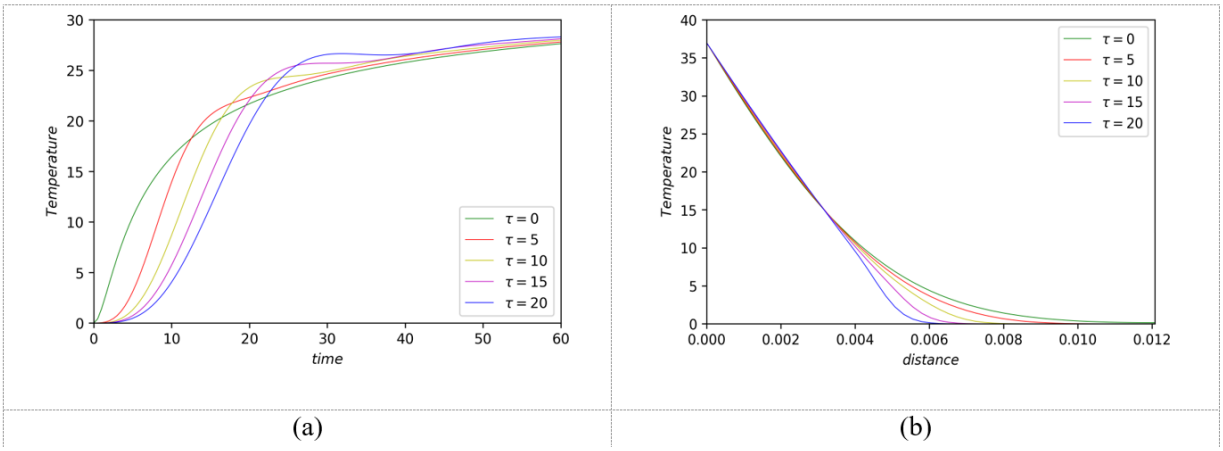


**Fig 2.** (a) Temperature distribution verses time for variuos values of $\tau$. (b) Temperature distribution verses distance for variuos values of $\tau$.

Figure 2 (a) and Figure 2 (b) illustrate the impact of the heat flux relaxation time parameter, denoted as $\tau$, on the temperature distribution within skin tissue, with a focus on time and distance, respectively. In Figure 2 (a), it was observed that an increase in the value of $\tau$ results in a decreasing trend in temperature elevation. Meanwhile, Figure 2 (b) reveals that a grater value of $\tau$ causes a more rapid decrese in body temperature.

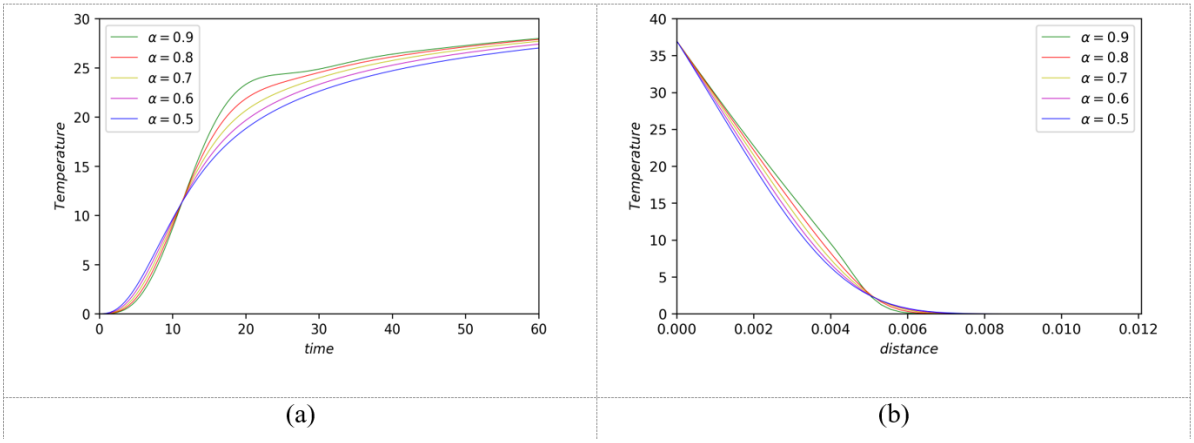


**Fig 3.** Temperature distribution verses time for 3 various values of $\alpha$.(b) Temperature distribution verses distance for values $\alpha$.

Figure 3 (a) and 3(b) illustrate how the subdiffusion parameter, denoted as $\alpha$ (where $0 < \alpha < 1$), affects the temperature distribution in the skin under instantaneous surface heating with respect to both time and distance, respectively. Figure 3 (a) reveals that as $\alpha$ approaches 1, there is a noticeable delay in the time required for the skin's surface to heat up. In Figure 3Figure 3 (b) we can observe that as the value of $\alpha$ rises, the temperature experiences a more rapid decrease.

## 4 Conclusion

We have effectively implemented the Crank-Nicolson finite difference scheme to address the subdiffusion bioheat equation with a single phase lag and this is achieved through the discretization of the Caputo time fractional derivative. Also, we succeesfully proved the convergence and unconditional stability of the developed scheme. We successfully developed a Python code for the Crank-Nicolson finite difference method of fractional order and visualized solutions graphically. The temperature distribution within living tissue is notably influenced by the subdiffusion parameter, denoted as $\alpha$ (where $0 < \alpha < 1$), and the heat flux relaxation time parameter, $\tau$. In the framework of single phase lag theory, we observed a temporal delay in temperature elevation compared to both classical and fractional theories. Additionally, temperature delay occurs at a faster rate in the single phase lag model as opposed to the classical and fractional bioheat models. Consequently, the single phase lag model proves to be well-suited for finite domains, such as small-length tissue, while the classical and fractional bioheat models are more appropriate for infinite domains.

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