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RESEARCH ARTICLE

Lattice Ordered $\Gamma-$Semirings

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Abstract

Objectives: The main objective of this paper is to derive some of the results of lattice ordered $\Gamma-$semirings, distributive lattice, lattice ideals and $\Gamma-$morphisms. **Methods:** To establish the results, we use some conditions like commutativity, simple, multiplicative $\Gamma-$idempotent, additively idempotent, and finally, use the concept of lattice ideal in $\Gamma-$semirings. **Findings:** First we give some examples of lattice ordered $\Gamma-$semirings and then study some results regarding lattices, distributive lattices, commutative lattice ordered $\Gamma-$semirings and finally lattice ideals and $\Gamma-$morphisms. The unique feature of this study is that the concept of gamma is new for the study of lattices. **Novelty:** We consider a condition (c.f. Theorem 4.1.5) for an additively idempotent $\Gamma-$semiring due to which it becomes a distributive lattice ordered $\Gamma-$semiring. Again, in general, the sum of $k-$ideals of a $\Gamma-$semiring need not be $k-$ideal. Indeed, $2\,N$ and $3\,N$ are $k-$ ideals of N, N is a set of non-negative integers. Clearly, $2N+3N=N/\{1\}=I$(say) is not a $k-$ ideal, because $2\in I, 3=2+1\in I$ but $1\notin I$. However, this condition does not hold in the case of a lattice ordered $\Gamma-$semiring.

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1 Introduction

Semiring is a generalization of both an associative ring as well as of distributive lattices, it has many applications in different areas of idempotent analysis, physics, topological space, coding theory, fuzzy theory, computer science, graph theory, etc. The underlying semirings in idempotent analysis, syntactic semirings, Max-plus algebra and Kleene algebra are those whose additive reduct is a semilattice, that is, idempotent and commutative. Many researchers are investigating different types of semirings, including ternary semirings, complimented semirings, $\Gamma-$semirings, lattices in semirings and others. The study of semirings aims to extend the techniques derived from semigroup theory or ring theory and explore their practical uses. In 2015, Bhuniya and Mondal[(1)] studied the distributive lattice congruence on a semiring with a semilattice and proved many results regarding lattices.

In 1995, the idea of $\Gamma-$semiring was presented by[2] as a speculation of semiring as well as Γ-ring. Sharma and Gupta[3] introduced the concept of complementation of $\Gamma-$ semirings. Since the complimented elements play an important role in lattices. Therefore, another major source of inspiration for the theory of $\Gamma-$ semirings is lattice theory. So, Sharma[4] introduces the concept of lattices in Γ semirings. In 2021, P.Jipsen, O. Tuyt, and D. Valota[5] studied the structure of finite commutative idempotent involutive residuated lattices. In this paper, we generalizes the results of semirings[6] to Γ semirings. For further study of semirings, Γ $-$semirings and their generalization, one may refer to[2–4,6,7].

As a continuation of the paper "Lattices in Γ Semirings"[4] we here, consider and investigate some of the results of lattice ordered $\Gamma-$semiring, distributive lattice, lattice ideals and $\Gamma-$morphisms.

2 Preliminaries

One can refer to[3,4,7], for the definitions of $\Gamma-$semiring and their identity elements 0 and 1, simple, multiplicative $\Gamma-$ idempotent and additive idempotent. Now we include some necessary preliminaries for the sake of completeness. A lattice is a partially ordered set in which every two elements have a unique least upper bound and unique greatest lower bound. Let $(A, \leq)$ be a lattice. We define an algebraic system $(A, \wedge, \vee)$ where $\wedge$ and $\vee$ are two binary operations on A such that for a and b in A, $a \vee b$ is the least upper bound of a, b and $a \wedge b$ is the greatest lower bound of a and b. A lattice is a distributive lattice if the meet operation distributes over the join operation and the join operation distributes over the meet operation. That is, for any a, b and c, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. An element a in a lattice $(A, \leq)$ is called a universal lower bound if, for every element $b \in A$, we have $a \leq b$. An element a in a lattice $(A, \leq)$ is called a universal upper bound if, for every element $b \in A$, we have $b \leq a$. Let R be a $\Gamma-$ semiring. Then R is called partially ordered $\Gamma-$ semiring if and only if there exists a partial order relation $\leq$ on R satisfying the following conditions if $x \leq y$ and $z \geq 0$ then (i) $x + z \leq y + z$ (ii) $x\alpha z \leq y\alpha z$ (iii) $z\alpha x \leq z\alpha y$, for all $x, y, z \in R$ and $\alpha \in \Gamma$.

The following theorems are proved in[4].

Theorem 2.1. Let R be a $\Gamma-$ semiring. Then R is a bounded distributive lattice having unique minimal element 0 and unique maximal element 1 if and only if R is commutative, $\Gamma-$ idempotent and simple $\Gamma-$ semiring.

Theorem 2.2. Let R be a $\Gamma-$ semiring. A commutative $\Gamma-$ semiring is a bounded distributive lattice if and only if it is simple multiplicative $\Gamma-$ idempotent $\Gamma-$ semiring.

3 Methodology

Following[4,6], we will establish some results of lattice ordered $\Gamma-$semiring by using some conditions like commutativity, simple, multiplicative $\Gamma-$idempotent, additively idempotent, etc. Further, we find that the sum of $k-$ideals of a lattice ordered $\Gamma-$semiring is again a $k-$ideal, however, it does not hold in the case of the sum of $k-$ideals of a $\Gamma-$semiring.

4 Main results and discussions

4.1 Lattice- ordered $\Gamma-$semirings

We start this section by giving some examples of lattice- ordered $\Gamma-$semirings and then generalize some of the results of lattice-ordered semirings from[6] to $\Gamma-$ semirings.

Definition 4.1.1. Let R be a $\Gamma-$semiring. Then R is called lattice -ordered $\Gamma-$semiring if and only if it also has the structure of a lattice such that $x+y = x \vee y$ and $x\alpha y = x \wedge y$ for all $x, y \in R$ and $\alpha \in \Gamma$, where partial order here is the one induced naturally by the lattice structure on R. If R is a distributive lattice then R is distributive lattice ordered $\Gamma-$semiring. Any lattice ordered $\Gamma-$semiring is a partially ordered $\Gamma-$semiring.

A lattice- ordered $\Gamma-$semiring is an additively idempotent. Also, if x and y are elements of lattice ordered $\Gamma-$semiring and $\alpha \in \Gamma$satisfying $x\alpha y = x$ or $y\alpha x = x$,, $\alpha \in \Gamma$ then $x \leq y$. Therefore, if x is an element of a lattice ordered $\Gamma-$semiring R and $\alpha \in \Gamma$ then $x = x\alpha 1 \leq 1$.

Example 4.1.2. Any bounded distributive lattice R is a distributive lattice ordered $\Gamma-$semiring. If we define $x+y = x \vee y$ and $x\alpha y = x \wedge y$ for all $x, y \in R$ and $\alpha \in \Gamma$ then the set of all complemented elements of R is a sub $\Gamma-$semiring **(c.f.[3], theorem 3.13).**

Example 4.1.3. The $\Gamma-$semiring R of all ideals of R is lattice ordered $\Gamma-$semiring but in general, it is not a distributive lattice ordered $\Gamma-$semiring.

Example 4.1.4. Let X and Y be bounded distributive lattices. Then $R = X \times Y$ is again a bounded distributive lattice on which operations of join and meet are defined by $(x,y) \vee \left(x',y'\right) = (x \vee x',\ y \vee y')$ and $(x,y) \wedge \left(x',y'\right) = (x \wedge x',\ y \wedge y')$. Define an operation $*$ on a $\Gamma-$semiring R by $(x,y) * \left(x',y'\right) = (x \wedge x',\ y \vee y')$. Then $(R,\vee,*)$ is a commutative additively idempotent $\Gamma-$semiring but not a distributive lattice ordered $\Gamma-$semiring since$(x,y) * \left(x',y'\right) \geq (x,y) \wedge \left(x',y'\right)$.

We now consider a condition for an additively idempotent $\Gamma-$semiring which is a distributive lattice ordered $\Gamma-$semiring.

Theorem 4.1.5 . Let R be an additively idempotent $\Gamma-$semiring satisfying the condition that $x \in y\Gamma R \cap R\Gamma y$ whenever $x \leq y$ in R. Then $(R,+,\ \wedge)$ is a distributive lattice and multiplication distributes over $\wedge$ from either side. Further, if R is simple then it is a distributive lattice ordered $\Gamma-$semiring.

Proof. Let $r \in R$ then surely $r \leq r$ and so there exists an element r^* of R and $\alpha \in \Gamma$ satisfying $r^*\alpha r = r$. Again, if $r_1 \leq r$ in R then there is an element r_2 of R satisfying $r_1 = r\alpha r_2, \alpha \in \Gamma$and so $(r^*\alpha r)\alpha r_2 = r\alpha r_2 = r_1$. By hypothesis, we know that if $x,y \in R$then there exist elements x_1 ,x_2 , y_1 ,y_2 of R and $\alpha,\beta \in \Gamma$ such that $x = (x+y)\,\alpha x_1 = x_2\alpha(x+y)$ and $y = (x+y)\,\beta y_1 = y_2\beta(x+y)$. Now, $x\beta y_1 = x_2\alpha\,(x+y)\,\beta y_1 = x_2\alpha y$, while $y\alpha x_1 = y_2\beta\,(x+y)\,\alpha x_1 = y_2\beta x$. Then $y = x\beta y_1 + y\beta y_1 \geq x\beta y_1 = x_2\alpha y \leq x_2\alpha x + x_2\alpha y = x$ so $x\beta y_1 \leq x,\ \ x\beta y_1 \leq y$. Similarly , $y\alpha x_1 \leq x,\ \ y\alpha x_1 \leq y$. Therefore, $x\beta y_1 + y\alpha x_1 \leq x + x = x$ and similarly $x\beta y_1 + y\alpha x_1 \leq y$.Let $r \in R$such that $r \leq x, r \leq y$. Then there exists an element r_1 of R and $\alpha \in \Gamma$ such that $r = r_1\alpha(x+y)$ and hence $r = r_1\alpha x + r_1\alpha y = r_1\alpha\,(x+y)\,\alpha x_1 + r_1\alpha\,(x+y)\,\beta y_1 = r\alpha x_1 + r\beta y_1 \leq y\alpha x_1 + x\beta y_1 = x_2\alpha y + y_2\beta x$. Thus, $y\alpha x_1 + x\beta y_1 = x_2\alpha y + y_2\beta x$ is a well -defined infimum of x and y in R, which is independent of the choice of x_1 ,y_1 ,x_2 , y_2 and we will denote it by $x \wedge y$. If z is another element of R and $\alpha,\beta,\gamma \in \Gamma$ then $z\gamma x = (z\gamma x + z\gamma y)\,\alpha x_1$ and $z\gamma y = (z\gamma x + z\gamma y)\,\beta y_1$ and so $z\gamma x \wedge z\gamma y = (z\gamma x)\,\beta y_1 + (z\gamma y)\,\alpha x_1 = z\gamma(x\beta y_1 + y\alpha x_1) = z\gamma(x \wedge y)$. Similarly, $(x \wedge y)\,\gamma z = x\gamma z \wedge y\gamma z$. Thus, multiplication distributes over $\wedge$ from either side. Since $y = x\alpha x_1 + y\alpha x_1$, this implies that $y\alpha x_1 \leq y$ and so there exists an element t of R satisfying $y\alpha x_1 = t\alpha y$. Set $s = (x+y)^* \wedge t$. Then $s\alpha y = [(x+y)^* \wedge t]\alpha y = (x+y)^*\,\alpha y \wedge t\alpha y = y \wedge t\alpha y = y \wedge y\alpha x_1 = y\alpha x_1 \leq x$. Similarly, $s\beta x = x \wedge t\beta x \leq x$. Therefore, $y\alpha x_1 = s\alpha y = s\alpha\,(x+y)\,\beta y_1 = (s\alpha x + s\alpha y)\beta y_1 \leq x\beta y_1$. Similarly, $x\beta y_1 \leq y\alpha x_1$.Thus, $x \wedge y = x\beta y_1 = y\alpha x_1 = x_2\alpha y = y_2\beta x$. To complete the proof that R is a distributive lattice, we must show that if x, y and z are elements of R then $z \wedge (x+y) = (z \wedge x) + (z \wedge y)$. It is trivial that $z \wedge (x+y) \geq (z \wedge x) + (z \wedge y)$ .Now, we establish the reverse inequality. Since $z \leq z+x+y$, so by the hypothesis that there exists an element r of R and $\alpha \in \Gamma$such that $z = r\alpha(z+x+y)$. Let $r_1 = (z+x+y)^* \wedge r$. Then as above $z = r_1\alpha(z+x+y)$, $r_1\alpha x \leq x$ and $r_1\alpha y \leq y$. Therefore, $z + r_1\alpha x = r_1\alpha\,(z+x+y+x) = z$. Thus, $r_1\alpha x \leq z$ and $r_1\alpha y \leq z$. Hence, $z \wedge (x+y) = r_1\alpha\,(x+y) = r_1\alpha x + r_1\alpha y \leq (z \wedge x) + (z \wedge y)$. Finally, if R is simple then for $x,y \in R$and $\alpha \in \Gamma$, we have $x\alpha y \leq x$ and $x\alpha y \leq y$and so $x\alpha y \leq x \wedge y$. Hence, R a distributive lattice ordered $\Gamma-$semiring.

Theorem 4.1.6. Let x, y and z are elements of a lattice ordered $\Gamma-$semiring $R$and $\alpha \in \Gamma$then:

(i) $x + x\alpha y = x$

(ii) $x\alpha y + z = (x+z)\,\alpha y + z$

(iii) $x \leq y$ implies that $z\alpha x \leq z\alpha y$ and $x\alpha z \leq y\alpha z$

(iv) $x \leq y$ implies that $x\alpha x \leq x\alpha y \leq y\alpha y$

(v) $x\alpha y \wedge x\alpha z \geq x\alpha(y \wedge z)$ and $y\alpha x \wedge z\alpha x \geq (y \wedge z)\alpha x$

(vi) $(x \wedge y)\,\alpha\,(x+y) \leq y\alpha x + x\alpha y$

(vii) If $x+y = 1$ then $x \wedge y = x\alpha y + y\alpha x$

(viii) If $x+y = 1$ then $x\alpha z \leq y$ *or* $z\alpha x \leq y$ implies $z \leq y$

(ix) If $x+y = x+z = 1$ then $x + y\alpha z = x + (y \wedge z) = 1$.

Proof. Simple and straightforward.

Corollary 4.1.7. Let R be a commutative lattice ordered $\Gamma-$semiring and x, y and z are elements of R satisfying $x+y = x+z = 1$ then $x \wedge (y \wedge z) = x\alpha y \wedge x\alpha z$, $\alpha \in \Gamma$.

Proof. By Theorem 4.1.6(ix) and (vii) the result follows.

Theorem 4.1.8. Let x and $y$be elements of a lattice ordered $\Gamma-$semiring R satisfying $x+y=1$ then $(x\alpha)^{m-1}x + (y\alpha)^{n-1}y = 1$, for all positive integers m and $n$and $\alpha \in \Gamma$.

Proof. It is a direct consequence of Theorem 4.1.6(ix).

Theorem 4.1.9. Let R be a lattice ordered $\Gamma-$semiring. Then R is a multiplicatively $\Gamma-$ idempotent if and only if $x\alpha y = x \wedge y$ for all $x, y \in R$ *and* $\alpha \in \Gamma$.

Definition 4.1.10. An element x of a $\Gamma-$semiring R is a unit if and only if there exists an element y of R satisfying $x\alpha y = 1 = y\alpha x$ for all $x, y \in R$ *and* $\alpha \in \Gamma$. The element y of R is called the inverse of x in R. Let us denote the set of all elements of R having units by $U(\Gamma R)$. This set is non-empty since $1 \in U(\Gamma R)$ and is not all of R.

Theorem 4.1.11. Let R be a lattice ordered $\Gamma-$semiring. Then $R$is simple and positive with 1 as its only unit.

Definition 4.1.12. An element r of a $\Gamma-$semiring R is said to be semiprime if for any element x of R there exists $\alpha \in \Gamma$ such that $x\alpha x \leq r$ implies that $x \leq r$.

Theorem 4.1.13. Let r be a semiprime element of a lattice ordered $\Gamma-$semiring R and if x and y are elements of $R$and $\alpha \in \Gamma$then the following conditions are equivalent:

(i) $x\alpha y \leq r$

(ii) $y\alpha x \leq r$

(iii) $x \wedge y \leq r$

Proof. Straightforward.

Corollary 4.1.14. If y and z are semiprime elements of a lattice ordered $\Gamma-$semiring R then $y \wedge z$ is semiprime.

Let$x$ and y be elements of a partially ordered monoid $(M, \ *)$. We define the interval $[x,y] = \{z \in M \mid \ x \leq z \leq y\}$. (This set may be empty). We will denote the set of all such intervals by $int(M)$. Define the operation $[*]$ on $int(M)$ by $[x,y]\,[*]\,[h,k] = [x*h, \ y*k]$. It is easy to see that if $(M, \ *)$ is a partially ordered *monoid* with identity element e then $(int(M),[*])$ is a monoid with identity element $[e,e]$. In particular, we note that if R is a lattice ordered $\Gamma-$semiring then $(R,+)$ and (R,α) are partially ordered monoids. Further, if R is a lattice ordered $\Gamma-$semiring then $(int\,(R),[\vee],[\wedge])$ is a lattice and $(int\,(R),[+],[\alpha])$is a lattice ordered $\Gamma-$semiring.

Theorem 4.1.15. Let R be a distributive lattice-ordered $\Gamma-$semiring then in $int\,(R) = \{[x,y] \mid z \in R, \ x \leq z \leq y\}$ then

(i) $[x,y]\,[+]\,[z,t] = \{u+v \ \mid \ u \in [x,y], \ v \in [z,t]\}$

(ii) $[x,y]\,[\alpha]\,[z,t] \supseteq \{\, u\alpha v \ \mid \ u \in [x,y], \ v \in [z,t], \ \alpha \in \Gamma\}$.

Proof. (i) If $u \in [x,y]$ and $v \in [z,t]$ then $x+z \leq u+v \leq y+t$ and so $u+v \in [x,y]\,[+]\,[z,t]$. Conversely, if $w \in [x,y]\,[+]\,[z,t]$ then $x \leq w \wedge y \leq y$ and $z \leq w \wedge t \leq t$. Moreover, $(w \wedge y)+(w \wedge t) = w \wedge (y+t) = w$. Thus, we have equality.

(ii) $u \in [x,y]$ and $v \in [z,t]$ then $x\alpha z \leq x\alpha v \leq u\alpha v \leq y\alpha v \leq y\alpha t$ and so $u\alpha v \in [x,y]\,[\alpha]\,[z,t]$.

4.2 Lattice ideals in $\Gamma-$ semirings

In this section, we generalize some of the results of lattice ideals of semirings to $\Gamma-$ semirings and study $\Gamma-$morphisms of lattice ordered $\Gamma-$ semirings.

Definition 4.2.1. A subset X of a lattice L is called a lattice ideal if and only if $x \in X$ and $y \in L$ imply that $x \wedge y \in X$. In particular, if $x \in X$ and $y \leq x$ then $y \in X$. Thus, every subset X of L is contained in a unique smallest lattice ideal of L, namely $[X] = \{y \in L \mid y \leq x$ for some $x \in X\}$. If $x \in L$, then we write $[x]$ instead of $(\{x\})$.

Example 4.2.2. Let R be a lattice ordered $\Gamma-$semiring and let $e \neq x \in R$, e is the multiplicative identity. Set $[x] = \{r \in R \mid \ r \geq x\}$ and define an operation $*$ on $[x]$ by $r * r' = r\alpha r' + x$, $\alpha \in \Gamma$. Then by Theorem 4.1.6, clearly $([x],+,*)$ is a lattice ordered $\Gamma-$semiring with additive identity x and multiplicative identity e.

Definition 4.2.3. An ideal I of a $\Gamma-$ semiring R is called $k-$ideal if for $x, \ y \ \in \ R, \ x \ + \ y \ \in \ I$ and $y \ \in \ I$ implies that $x \ \in \ I$.

Definition 4.2.4. An ideal I of a $\Gamma-$ semiring R is called a strong ideal if and only if $x \ + \ y \ \in \ I$ implies that $x \ \in \ I$ and $y \ \in \ I$.

Theorem 4.2.5 . Let R be a lattice ordered $\Gamma-$semiring and I be an ideal of R. Then the following conditions are equivalent:

(i) I is a lattice ideal

(ii) I is a strong ideal

(iii) I is a $k-$ideal.

Proof. (i) Implies (ii). Let x and y be elements of R such that $x+y \in I$. Then $x = x \wedge (x \vee y) = x \wedge (x+y)$ and so by (i), $x \in I$. Similarly, $y \in I$ and so I is a strong ideal of R.

(ii) Implies (iii). This is trivial.

(ii) Implies (i). Let $x \in I$ and $r \in R$. Then $x = x \vee (x \wedge r) = x + (x \wedge r) \in I$ and so by (iii) $x \wedge r \in I$. Thus, I is lattice ideal of R.

The sum of $k-$ideals of a $\Gamma-$semiring need not be a $k-$ideal. Indeed, 2 N and 3 N are $k-$ ideals of N, N is a set of non-negative integers. But $2\,N \ + \ 3N \ = \ N/\{1\} \ = \ I$(say) is not a $k-$ ideal, because $2 \ \in \ I$ and $3 \ = \ 2 \ + \ 1 \ \in \ I$ but $1 \ \notin \ I$. However, the condition does not hold in case of a lattice ordered $\Gamma-$semiring.

Corollary 4.2.6. If $\{I_k | k \in \Omega\}$ is a family of $k-$ideals of a lattice ordered $\Gamma-$semiring R then $\sum_{j\in\Omega} I_j$ is a $k-$ideal.

Proof. Let $x \in \sum_{j\in\Omega} I_j$ and $y \in R$. Then there exists a finite subset Λ of Ω and elements $x_k \in I_k$ for all $k \in \Lambda$ such that $x = \sum_{k\in\Lambda} x_k$. Then by Theorem 4.2.5, $x \wedge y = (\sum_{k\in\Lambda} x_k) \wedge y = \sum_{k\in\Lambda} (x_k \wedge y) \in \sum_{j\in\Omega} I_j$ and so $\sum_{j\in\Omega} I_j$ is a lattice ideal of R and hence by Theorem 4.2.5, $\sum_{j\in\Omega} I_j$is a $k-$ideal of R.

Theorem 4.2.7. Let R be a lattice ordered $\Gamma-$semiring and I an ideal of R which is also a lattice ideal of R.If x and y are elements of R and $\alpha \in \Gamma$ satisfying $x\alpha y \in I$ then $(x)\Gamma(y) \subseteq I$.

Proof. Let $r, r_1,$ and $r_2$be elements of R and $\alpha,\beta,\gamma \in \Gamma$ then $r\alpha x\beta r_1 \leq x$ and $y\gamma r_2 \leq y$.This implies that $(r\alpha x\beta r_1)\delta(y\gamma r_2) \leq x\delta y$, $\alpha,\beta,\gamma,\delta \in \Gamma$. Thus, $(r\alpha x\beta r_1)\delta(y\gamma r_2) \in I$. Since every element of $(x)\Gamma(y)$ is a finite sum of elements of R of this form, it follows that $(x)\Gamma(y) \subseteq I$.

Theorem 4.2.8. Let R be a lattice- ordered $\Gamma-$semiring. If I is an ideal of R which is also a lattice ideal. Then I is prime if and only if x and y are elements of R and $\alpha \in \Gamma$ satisfying $x\alpha y \in I$ implies that either $x \in I$ or $y \in I$.

Proof. Let I be prime and $x, y \in R$ and $\alpha \in \Gamma$ satisfying $x\alpha y \in I$. By Theorem 4.2.7, $(x)\Gamma(y) \subseteq I$. Thus, either $(x) \subseteq I$ or $(y) \subseteq I$. This implies that either $x \in I$ or $y \in I$. Converse follows from **(c.f.,[8], Proposition 3.4).**

Let R and Sbe two lattice- ordered $\Gamma-$semirings. If $f : R \to S$ is a $\Gamma-$morphism of $\Gamma-$semirings R and S then f is a $\Gamma-$morphism between the semigroups $(R, +)$ and $(S, +)$ and so is order-preserving. Further, if $r \geq r_1$ in R then $r_1 + r = r_1 \vee r = r$. Therefore, $f(r_1) \vee f(r) = f(r_1) + f(r) = f(r_1 + r) = f(r)$. Thus, $f(r) \geq f(r_1)$.

Theorem 4.2.9. Let R and S be lattice- ordered $\Gamma-$semiring and $f : R \to S$ be a $\Gamma-$morphism of $\Gamma-$semirings . Let $g : R \to S$ be a $\Gamma-$morphism of lattices. Then

(i) $f(r \wedge r_1) \leq f(r) \wedge f(r_1)$ for all $r, r_1 \in R$;

(ii) $g(r + r_1) \geq g(r) + g(r_1)$ for all $r, r_1 \in R$

Moreover, if either f or g is bijective then the corresponding inequality becomes an equality.

Proof. Since $\Gamma-$morphisms of $\Gamma-$semirings between lattice -ordered $\Gamma-$semirings are order-preserving, therefore we have $f(r \wedge r_1) \leq f(r)$ and $f(r \wedge r_1) \leq f(r_1)$. Furthermore, if f is bijective then it has an inverse. Therefore, $f^{-1}(f(r) + f(r_1)) = f^{-1}(f(r + r_1)) = r + r_1 = f^{-1}(f(r)) + f^{-1}(f(r_1))$ and so f^{-1} is a $\Gamma-$morphism between the semigroups $(S, +)$ and $(R, +)$. This implies that f^{-1} is an order preserving and so $f^{-1}((f(r) \wedge f(r_1)) \leq f^{-1}(f(r) \wedge f^{-1} f(r_1)) = r \wedge r_1$. This implies that $f(r) \wedge f(r_1) \leq f(r \wedge r_1)$, and thus we have equality. The proof of (ii) is similar.

Definition 4.2.10. Let Rbe a partially ordered $\Gamma-$semiring. Then a function $h : R \to R$ is called a middle function if and only if the following conditions are satisfied.

(i) If $x \leq y$ in R then $h(x) \leq h(y)$

(ii) If $x \in R$ then $h(h(x)) = h(x) \geq x$

(iii) If $x,\ y \in R$ then $h(x\alpha y) \leq h(x)\alpha h(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$.

If R is a lattice ordered $\Gamma-$semiring and $h : R \to R$ is a middle function then $h(x\alpha y) \leq h(x) \wedge h(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$, since h is order preserving. The following result gives necessary and sufficient conditions for equality.

Theorem 4.2.11. Let R be a lattice ordered $\Gamma-$semiring and $h : R \to R$ is middle function then $h(x\alpha y) = h(x) \wedge h(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$ if and only if $h(x\alpha y) = h(y\alpha x)$and $h(x\alpha x) = h(x)$ for all $x,\ y \in R$and $\alpha \in \Gamma$.

Proof. Let $h(x\alpha y) = h(x) \wedge h(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$ then the result follows. Conversely, assume that $h(x\alpha y) = h(y\alpha x)$ and $h(x\alpha x) = h(x)$ for all $x,\ y \in R$ and $\alpha \in \Gamma$. Then $x \wedge y \leq h(x) \wedge h(y)$. Now, $h(x \wedge y) \leq h(h(x) \wedge h(y)) = h([h(x) \wedge h(y)]\alpha[h(x) \wedge h(y)]) \leq h(h(x\alpha y)) \leq h(h(x)\alpha h(y)) = h(x\alpha y)$,$\alpha \in \Gamma$. The reverse inequality, that is $h(x\alpha y) \leq h(x \wedge y)$ is always true, and so we have equality.

5 Conclusion

In this paper, we study different results in lattices, lattice ideals and $\Gamma-$morphisms of lattice ordered $\Gamma-$ semirings by using some of the conditions such as simple, multiplicative $\Gamma-$ idempotent, additive idempotent, distributive lattices, semiprimeness and units in $\Gamma-$ semirings. We consider a condition $x \in y\Gamma R \cap R\Gamma y$ whenever $x \leq y$ in R in theorem 4.1.5 and find, that $(R, +,\ \wedge)$ is a distributive lattice and multiplication distributes over $\wedge$ from either side. Finally, we prove that the sum of $k-$ideals of a lattice ordered $\Gamma-$semiring is a $k-$ideal. However, in general, this condition does not hold for $\Gamma-$semirings. Since the concept of gamma is new in the theory of semirings so the ideas described in this article for lattices in $\Gamma-$ semirings have a lot of potential for nourishing and therefore, this article is very useful as it invites the researchers to explore more in different structure of lattices in $\Gamma-$semirings.

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